ON REAY'S RELAXED TVERBERG CONJECTURE AND GENERALIZATIONS OF CONWAY'S THRACKLE CONJECTURE

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ABSTRACT. Reay's relaxed Tverberg conjecture and Conway's thrackle conjecture are open problems about the geometry of pairwise intersections. Reay asked for the minimum number of points in Euclidean *d*-space that guarantees any such point set admits a partition into r parts, any kof whose convex hulls intersect. Here we give new and improved lower bounds for this number, which Reay conjectured to be independent of k. We prove a colored version of Reay's conjecture for k sufficiently large, but nevertheless k independent of dimension d. Requiring convex hulls to intersect pairwise severely restricts combinatorics. This is a higher-dimensional analog of Conway's thrackle conjecture or its linear special case. We thus study convex-geometric and higher-dimensional analogs of the thrackle conjecture alongside Reay's problem and conjecture (and prove in two special cases) that the number of convex sets in the plane is bounded by the total number of vertices they involve whenever there exists a transversal set for their pairwise intersections. We thus isolate a geometric property that leads to bounds as in the thrackle conjecture. We also establish tight bounds for the number of facets of higher-dimensional analogs of linear thrackles and conjecture their continuous generalizations.

1. INTRODUCTION

Given a finite point set in \mathbb{R}^d the intersection pattern of convex hulls determined by subsets of those points is the focus of *Tverberg-type theory*. The namesake of the area, Helge Tverberg, established in 1966 that for any (r-1)(d+1) + 1 points in \mathbb{R}^d there exists a partition into r parts X_1, \ldots, X_r such that conv $X_1 \cap \cdots \cap \operatorname{conv} X_r \neq \emptyset$, and this number of points is optimal in general [19]. Since then a multitude of extensions and variants of this result have been proven; see for instance the recent survey article [2].

Many seemingly simple questions of Tverberg-type remain open — among them a conjecture of Reay [17]: for any $r \ge 2$ and $d \ge 1$ there are (r-1)(d+1) points in \mathbb{R}^d such that for any partition of them into r parts, two of them have disjoint convex hulls. This would imply that there is no relaxation of Tverberg's theorem, where fewer than (r-1)(d+1) + 1 points can be partitioned into r sets of pairwise intersecting convex hulls. More generally, this problem has been studied for k-fold intersections among the r convex hulls instead of only pairwise intersections. This was done already by Reay and later by Perles and Sigron [15].

Reay's problem seeks to understand the pairwise intersection pattern of disjoint faces in a simplicial complex K when affinely mapped to Euclidean space. Conversely, if we are given that all facets have nonempty pairwise intersections, how does this restrict the possible combinatorics of K? In the special case of graphs this would be answered by Conway's thrackle conjecture: a *thrackle* is a graph that can be drawn in the plane in such a way that any pair of edges intersects precisely once, either at a common vertex or a transverse intersection point. Conway conjectured that in any thrackle the number of edges is at most the number of vertices. This has remained open but is simple to prove if all edges are required to be straight line segments, that is convex; see Erdős [8]. It is an open question whether one needs to distinguish between the affine and continuous theory for thrackles; this distinction is significant for Tverberg-type results [4, 10, 14]. Not wanting to restrict our attention to 1-dimensional objects, we set

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out to find convex-geometric and higher-dimensional analogs of Conway's thrackle conjecture as a true counterpart of Reay's problem.

Our contributions. Denote by T(d, r, k) the minimum number n such that any n points a_1, \ldots, a_n in \mathbb{R}^d (not necessarily distinct) admit a partition of the indices $\{1, \ldots, n\}$ into r pairwise disjoint sets I_1, \ldots, I_r such that any size k subfamily of $\{\operatorname{conv}(a_i)_{i \in I_1}, \ldots, \operatorname{conv}(a_r)_{i \in I_r}\}$ has nonempty intersection. In Section 2 we give new and improved lower bounds for the numbers T(d, r, k). We show that $T(d+1, r, k) \geq T(d, r, k) + k - 1$, see Theorem 2.2, and $T(d, r, k) \geq r(\frac{k-1}{k} \cdot d + 1)$, see Theorem 2.4.

Perles and Sigron [15] showed that T(d, r, k) = (r-1)(d+1)+1 for specific values of k; see Theorem 2.1 for details. However, in those cases k grows linearly with the dimension d, and in fact Perles and Sigron do not believe that T(r, d, k) = (r - 1)(d + 1) + 1 in general. In contrast, Theorem 3.2 establishes a colorful analog of Reay's conjecture for any dimension d and a constant k.

Given a collection $C_1, \ldots, C_m \subseteq \mathbb{R}^2$ of convex polygons on a total number of n vertices such that any two polygons have nonempty intersection, it is simple to see that the naive extension $m \leq n$ of the linear case of the thrackle conjecture cannot hold in general. Here we isolate a feature of the pairwise intersection pattern of convex sets that allows us to prove an extension of the linear thrackle conjecture: we establish the bound $m \leq n$ if the full-dimensional C_i are vertex-disjoint from one another and there is a *transversal* set W that contains all vertices and possibly more points such that $|C_i \cap C_j \cap W| = 1$ for all $i \neq j$; see Theorem 4.3. We further conjecture that it is superfluous to require the full-dimensional C_i to be vertexdisjoint; see Conjecture 4.1. It is a purely combinatorial statement about pairwise intersection patterns of arbitrary sets C_1, \ldots, C_m (not even necessarily contained in any \mathbb{R}^d), that if there is a transversal of pairwise intersections W, that is $|C_i \cap C_j \cap W| = 1$ for all $i \neq j$, then $m \leq |W|$; see Theorem 4.4.

We present higher-dimensional generalizations of the linear thrackle conjecture in Section 5 and conjecture their continuous analogs.

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2. Lower bounds for Reay's relaxed Tverberg conjecture

Recall that T(d, r, k) denotes the minimum number n such that any n points a_1, \ldots, a_n in \mathbb{R}^d (not necessarily distinct) admit a partition of the indices $\{1, \ldots, n\}$ into r pairwise disjoint sets I_1, \ldots, I_r such that any size k subfamily of $\{\operatorname{conv}(a_i)_{i \in I_1}, \ldots, \operatorname{conv}(a_r)_{i \in I_r}\}$ has nonempty intersection. By Tverberg's theorem $T(d, r, k) \leq (r-1)(d+1) + 1$, and since that theorem is tight we have the equality T(d, r, r) = (r-1)(d+1) + 1. Reav conjectured that in fact this bound is tight even for smaller k, that is, T(d, r, k) = (r-1)(d+1) + 1 for all $2 \leq k \leq r$. Reay's conjecture is known to be true in some cases, and there are a few general lower bounds for the number T(d, r, k). We collect these results here:

Theorem 2.1. We have the following lower bounds for T(d, r, k):

- (i) Let $2 \le k \le d$, $k \le r$, and $d \ge 2$. Then $T(d, r, k) \ge (r-1)k$, T(2, r, 2) = 3r 2, and $T(d, d+1, d) \ge (r-1)(d+1)$. Also, for $r \ge 3$, we have $T(3, r, 2) \ge 3r$; see Reay [17].
- (ii) Let $d + 1 \le 2k 1$ or $k < r < \frac{d+1}{d+1-k}k$. Then T(d, r, k) = (r-1)(d+1) + 1. Also T(3, 4, 2) = 13 and T(5, 3, 2) = 13; see Perles and Sigron [15].
- (iii) We have that $T(d, r, 2) \ge r(\lfloor \frac{d}{2} \rfloor + 1)$; see Ziegler [1].

We observe that the general lower bounds for T(d, r, k) that can be found in the (traditional) literature do not even depend on d. The best lower bounds for pairwise intersections k = 2 seem to follow from Ziegler's reply to a mathoverflow post by Roland Bacher. Ziegler puts points in cyclic position. We extend his reasoning to larger k > 2 by putting points in strong general position; see Theorem 2.4.

It is simple to see that Tverberg's theorem is tight. For example any sufficiently generic point set will show the tightness. Alternatively, this can also be verified by an induction on dimension; see de Longueville [6]. We will use similar arguments to establish general lower bounds for T(d, r, k).

Theorem 2.2. Let $d \ge 2$ and $2 \le k \le r$ be integers. Then $T(d+1, r, k) \ge T(d, r, k) + k - 1$ and in particular $T(d, r, k) \ge 3r - 2 + (k - 1)(d - 2)$.

Proof. Let $X \subseteq \mathbb{R}^d$ be a set of T(d, r, k) - 1 points such that for any partition X_1, \ldots, X_r of X into r parts there are k sets whose convex hulls avoid a common point of intersection. We will explicitly construct a set $Y \subseteq \mathbb{R}^{d+1}$ of T(d, r, k) + k - 2 points with the same property. To this end place \mathbb{R}^d as the hyperplane $\mathbb{R}^d \times \{0\}$ into \mathbb{R}^{d+1} . Let Y consist of the points in X and k - 1 additional points strictly on the positive side of $\mathbb{R}^d \times \{0\}$.

Suppose Y had a partition into r sets Y_1, \ldots, Y_r such that for every k of these sets their convex hulls intersect. We claim that $Y_1 \cap X, \ldots, Y_r \cap X$ is a partition of X with the same property: for any k of the Y_i , say Y_1, \ldots, Y_k , at least one Y_j is entirely contained in X and thus there is a point of intersection among their convex hulls in $\mathbb{R}^d \times \{0\}$. But this is only possible if $\operatorname{conv}(Y_1 \cap X) \cap \cdots \cap \operatorname{conv}(Y_k \cap X) \neq \emptyset$. Thus $Y_1 \cap X, \ldots, Y_r \cap X$ is a partition of X such that any k of these sets have intersecting convex hulls — a contradiction.

The bound $T(d, r, k) \ge 3r - 2 + (k - 1)(d - 2)$ now follows inductively starting from T(2, r, k) = 3r - 2 given by Theorem 2.1.

Remark 2.3. Theorem 2.2 recovers the tightness of Tverberg's theorem for k = r.

A point set $X \subset \mathbb{R}^d$ is said to be in *strong general position* if for any $r \geq 2$ and any disjoint subsets X_1, \ldots, X_r of X the codimension of $\bigcap_i \operatorname{aff}(X_i)$ is equal to the sum of the codimensions of $\operatorname{aff}(X_i)$ or $\bigcap_i \operatorname{aff}(X_i)$ is empty; see Reay [18], Doignon and Valette [7], and Perles and Sigron [16].

Theorem 2.4. Let $d \ge 1$ and $2 \le k \le r$ be integers. Then $T(d, r, k) \ge r(\frac{k-1}{k} \cdot d + 1)$.

Proof. Any point set $X \subseteq \mathbb{R}^d$ in strong general position admitting a partition into r parts X_1, \ldots, X_r such that any k of the sets have intersecting convex hulls has at least $r(\frac{k-1}{k} \cdot d + 1)$ points. Denote the dimension of $\operatorname{aff}(X_i)$ by d_i . Suppose $\sum_{1 \leq i \leq r} d_i < r \cdot \frac{k-1}{k} d$. Then we can find indices $i_1 \ldots, i_k$ such that $\sum_{1 \leq j \leq k} d_{i_j} < (k-1)d$. But we know $\bigcap_{i_1 < \cdots < i_k} \operatorname{conv}(X_i) \neq \emptyset$ which implies $\sum_{1 \leq j \leq k} d - d_{i_j} \leq d$, a contradiction. So we have $\sum_{1 \leq i \leq r} d_i \geq r \cdot \frac{k-1}{k}d$, and $|X_i| \geq d_i + 1$ implies $\sum_{1 \leq i \leq r} |X_i| \geq r(\frac{k-1}{k} \cdot d + 1)$, as desired.

Remark 2.5. Note that r = k here also recovers the tightness of Tverberg's theorem. This bound can be rewritten $(r-1)(d+1) + 1 - (r-k) \cdot d/k$, and for k = r-1 and $k \ge d+1$ the bound recovers T(d, r, k) = (r-1)(d+1) + 1, which follows alternately from Helly's Theorem. This bound is better than the bound in Theorem 2.2 for d or k sufficiently large.

3. Proof of a colored version of Reay's conjecture

Reay's conjecture is known to be true only for k-fold intersections, where k grows linearly with d. Here we present a variant of Reay's conjecture that turns out to be true for $k > \lceil \frac{r}{2} \rceil$ in any dimension d. We view this as further evidence that the conjecture is true. Our variant is a k-fold analog of the following conjecture which is open in general:

Conjecture 3.1 (Bárány–Larman conjecture). Given sets $C_0, \ldots, C_d \subseteq \mathbb{R}^d$ of cardinality r, there are pairwise disjoint sets $X_1, \ldots, X_r \subseteq \bigcup C_i$ such that $|X_i \cap C_j| \leq 1$ for every i and j and $\operatorname{conv}(X_1) \cap \cdots \cap \operatorname{conv}(X_r) \neq \emptyset$.

Bárány and Larman [3] proved that this conjecture holds in the plane. Lovász observed that the case r = 2 is an immediate consequence of the Borsuk–Ulam theorem; this was remarked on in [3]. More generally, the truth of this conjecture was established for r + 1 a prime by Blagojević, Matschke, and Ziegler [5]. Here we show that in general one cannot even delete a single point and still find sets X_1, \ldots, X_r as in Conjecture 3.1 such that the convex hulls of any $k > \left\lceil \frac{r}{2} \right\rceil$ of them intersect.

Theorem 3.2. Let $d \ge 1$, $r \ge 2$ and $k > \lceil \frac{r}{2} \rceil$ be integers. There are point sets $C_1, \ldots, C_d \subseteq \mathbb{R}^d$ of cardinality r, and C_0 of cardinality r-1, such that for any r pairwise disjoint sets $X_1, \ldots, X_r \subseteq \bigcup C_i$ with $|X_i \cap C_j| \le 1$ for every i and j, the convex hulls of some k of them have empty intersection.

Proof. We construct the point set $\bigcup C_i$ by induction over dimension. The theorem holds for any set $C_0 \subseteq \mathbb{R}^0$ of cardinality r-1, since any partition of C_0 into r parts must include the empty set. Having inductively constructed $C_0, \ldots, C_d \subseteq \mathbb{R}^d$ as in the statement of the theorem, we place \mathbb{R}^d as the hyperplane $\mathbb{R}^d \times \{0\}$ in \mathbb{R}^{d+1} and add point set C_{d+1} : place $\lceil \frac{r}{2} \rceil$ points of C_{d+1} above $\mathbb{R}^d \times \{0\}$ and $\lfloor \frac{r}{2} \rfloor$ points below. For any r pairwise disjoint sets in $\bigcup C_i$ any intersection among the convex hulls of k of them must already occur in $\mathbb{R}^d \times \{0\}$ since no convex hull can contain two points of C_{d+1} , and this finishes the induction.

In particular, for k = r this shows that the Bárány–Larman conjecture is tight in the sense that not even a single point may be deleted in general.

4. Convex generalizations of Conway's thrackle conjecture

Recall that a *thrackle* is a graph that can be drawn in the plane such that any pair of edges intersects precisely once, either at a common vertex or at a point of transverse intersection. Conway conjectured that in any thrackle the number of edges does not exceed the number of vertices. This is simple to prove if all edges are straight line segments, see Erdős [8] for a short proof of this *linear thrackle conjecture*, but has remained open in general. Lovász, Pach, and Szegedy [13] proved that any thrackle on *n* vertices has at most 2n - 3 edges. This bound was improved to roughly 1.428*n* by Fulek and Pach [11].

Here we are interested in convex-geometric generalizations of the linear thrackle conjecture, where we replace straight edges by more general convex sets. The naive conjecture that if C_1, \ldots, C_m are convex polygons in the plane on a total number of n vertices with pairwise nonempty intersections, then $m \leq n$ is wrong: consider the vertices of a regular 7-gon and the twenty-one triangles containing precisely one edge of the 7-gon.

If, however, the pairwise intersections admit a transversal set W as explained below, then we conjecture that the number of convex sets is bounded by the total number of vertices:

Conjecture 4.1. Let $W \subseteq \mathbb{R}^2$ be a finite set of points, $V \subseteq W$ a set of n points, C_1, \ldots, C_m distinct convex hulls of subsets of V and $|C_i \cap C_j \cap W| = 1$ for all $i \neq j$. Then $m \leq n$.

A system of convex sets as in Conjecture 4.1 is a *thrackle of convex sets*. If all the C_i have two elements, that is, they are edges, then this reduces to the linear case of Conway's thrackle conjecture. Here the transversal set W consists of all vertices and intersection points. Theorem 4.3 is special case of this conjecture, which is properly stronger than the linear case of the thrackle conjecture.

Example 4.2. Tight examples for Conjecture 4.1 can be obtained from finite projective planes; see Chapter 19 of van Lint and Wilson [20] for an introduction to combinatorial designs. A projective plane is an incidence relation among an abstract set of points and an abstract set of lines such that any two distinct points are incident to exactly one line, any two distinct lines are incident to exactly one point, and there are four points such that no line is incident with three of them. In a finite projective plane the number of points is equal to the number of lines. Finite projective planes on $q^2 + q + 1$ points, with the order q a power of a prime, are simple to construct, while it is unknown whether projective planes of order that is not a prime power exist. Given a projective plane with n points and n lines, consider a convex n-gon in the plane with vertices in bijection with points, and let C_1, \ldots, C_n be those convex sets

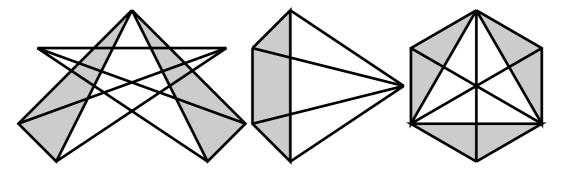


FIGURE 1. Other examples of tight thrackles for Conjecture 4.1

that are determined by lines of the projective plane. Then any two distinct sets intersect at a common vertex and no other vertices. Thus the set of vertices is a transversal set in the sense of Conjecture 4.1 and the number of convex sets is equal to the total number of vertices.

Theorem 4.3. Conjecture 4.1 holds in the case that the vertex sets of C_i, C_j are disjoint whenever C_i, C_j are both 2-dimensional.

Proof. Each vertex is incident to at most one 2-dimensional set. Therefore, the neighborhood of a given vertex consists of some rays along with at most one wedge, which represents a 2-dimensional convex set. We describe a surjection from a subset of the vertices onto the set of convex sets. Each vertex selects at

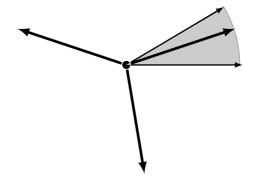


FIGURE 2. Example configuration about some vertex

most one incident set C_i :

- Case 1: If there are no wedges around v, then if the measure of the clockwise angle from some ray to every other ray around v is in $(0, \pi)$, that ray is selected. Otherwise, no ray is selected.
- Case 2: If the wedge around v contains some ray internally, then the wedge is removed from consideration and a ray is chosen as in Case 1.
- Case 3: If the wedge around v contains no ray internally, then the wedge is replaced with its counterclockwisemost representative ray, and then a ray is selected as in Case 1.

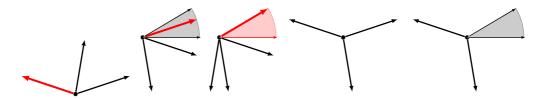


FIGURE 3. Examples of the described selection about a vertex; in the last two cases, no ray or wedge is chosen.

Every convex set is chosen by one of its vertices. First, we observe that this holds for all edges. Indeed, suppose this is not the case for some edge $\overline{v_i v_j}$. Since ray $\overline{v_i v_j}$ was not chosen, some convex set containing v_i as a vertex lies entirely in union of the open half-plane H^+ with the extension of $\overline{v_j v_i}$ past v_i (see Figure 4). Similarly, some convex set containing v_j as a vertex must lie entirely in the intersection of the open half-plane H^- with the extension of $\overline{v_i v_j}$ past v_j . However, these two sets are disjoint, so the corresponding convex sets would also be disjoint. This contradicts the condition $|C_i \cap C_j \cap W| = 1$ for all i, j, and is therefore impossible. It follows that every two-vertex convex set is chosen by one of its vertices.

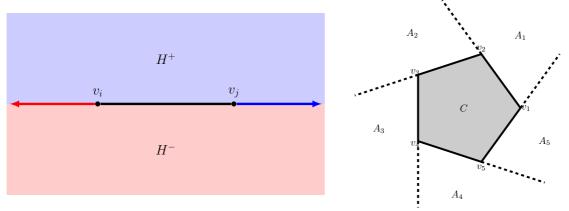


FIGURE 4. Illustration of how edges are chosen

FIGURE 5. Illustration of how twodimensional sets are chosen.

Now it suffices to check the statement for nonedge convex sets. Suppose for the sake of contradiction that a convex set $C = \text{conv}\{v_1, v_2, \dots, v_k\}$ has the property that no v_i , $1 \le i \le k$, chose the wedge corresponding to C. Let v_1, \dots, v_k be ordered in counterclockwise order around the boundary of C. For each $i, 1 \le i \le k$, let R_i denote the ray $v_{i-1}v_i$, and let A_i denote the closed wedge between rays R_i and R_{i+1} with indices taken modulo k. The convex set C, along with A_1, \dots, A_k then form a partition of the plane.

Since all pairs of nonedge convex sets are assumed to be vertex disjoint, the only other sets that could possibly contain the $v_i, 1 \leq i \leq k$, as vertices are edges. If, for any $i, 1 \leq i \leq k$, all rays from v_i (if there are any) point towards the interior of the region A_i , then the vertex v_i would choose the wedge corresponding to C. Furthermore, no ray can point alongside an edge of C, as the intersection of that edge with C would necessarily contain two vertices. Therefore, we may assume that, for every $i, 1 \leq i \leq k$, some ray either points inside the wedge corresponding to C, or points into the union of R_i and the unique open half-plane H_i disjoint from C and whose defining line is $\overleftarrow{v_{i-1}, v_i}$. There are two cases:

Case 1: For every *i*, some ray at v_i points inside *C*. Observe that no two such rays may meet inside *C*. Otherwise, the intersection of either corresponding segment with *C* would necessarily contain two points in *W*. It follows that the edges corresponding to these rays meet outside of *C*, so that every edge must intersect the boundary of *C* internally. Let any segment from v_1 which points inside *C* intersect the boundary of *C* again at a point *Y*.

Since the intersection of this segment and C already contains $v_1 \in W$, it follows that Y is not a vertex of C, so that it lies on some edge. Let $v_i, i \neq 1$ be one of the vertices of the edge of C containing Y. Then any edge with a vertex at v_i must, in order to intersect $\overrightarrow{v_iY}$ outside of C, also point outside of C. This contradicts the assumption that every vertex has some ray pointing inwards, so this case is resolved.

Case 2: For some *i*, there is a ray from v_i which points into H_i .

In this case, no ray from v_{i-1} can point inside C, for then this ray and the above ray from v_i would point into opposite sides of the line $v_{i-1}v_i$. It follows that some ray from v_{i-1} points into H_{i-1} . Repeating this argument k-2 more times, there is some ray r_i for each $i, 1 \le i \le k$ which points into H_i . Let θ_i denote the clockwise angle measured between rays r_i and $\overrightarrow{v_iv_{i-1}}$, and let γ_i denote the measure of $\angle v_{i+1}v_iv_{i-1}$.

For each *i*, the rays r_i and r_{i+1} must intersect, since the corresponding segments intersect. The condition that r_i, r_{i+1} intersect is exactly the condition that the sum of the clockwise angle measures from $\overline{v_i v_{i+1}}$ to r_i and from r_{i+1} to $\overline{v_{i+1} v_i}$ is less than π ; that is, $(2\pi - \gamma_i - \theta_i) + \theta_{i+1} < \pi$, or $\theta_{i+1} - \theta_i < \gamma_i - \pi$ for each $1 \le i \le k$. However, summing these *k* inequalities cyclically gives:

$$0 = \sum_{i=1}^{k} (\theta_{i+1} - \theta_i) < \sum_{i=1}^{k} (\gamma_i - \pi) = -2\pi$$

This is a contradiction, so this case is also impossible.

Since a contradiction was derived in all cases, it follows that some vertex from every convex set does in fact choose that convex set. Since each vertex chooses at most one convex set, this mapping forms a natural surjection from a subset of vertices onto $\{C_1, \ldots, C_m\}$. It follows that $m \leq n$ as required. \Box

The following theorem shows that Conjecture 4.1 holds whenever the transversal set W contains only the vertices and no additional points as in Example 4.2. This is a purely combinatorial statement independent of any geometry of the sets C_i and ambient space.

Theorem 4.4. Let C_1, \ldots, C_m be sets and suppose there exists a transversal of their pairwise intersections W, that is $|C_i \cap C_j \cap W| = 1$ for all $i \neq j$. Then $m \leq |W|$.

Proof. Create a graph where the vertices represent the sets C_i and there is an edge between the vertices if the two corresponding sets intersect. Since every pair of sets must intersect, this graph will be the complete graph on m vertices, K_m . Any point of the transversal set W induces a complete subgraph of sets it intersects. Therefore W induces a decomposition of the complete graph into proper complete subgraphs. The complete graph K_m cannot be decomposed into less than m proper complete subgraphs; see de Brujin and Erdős [9]. Thus $m \leq |W|$.

While this is a purely combinatorial statement, Conjecture 4.1 has geometric content and the analogous statement fails in \mathbb{R}^3 :

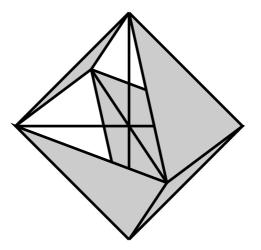


FIGURE 6. Counterexample to Conjecture 4.1 in \mathbb{R}^3 on six vertices with seven convex sets.

5. Higher-dimensional thrackles

A *d*-dimensional simplicial complex is *pure* if every face is contained in a *d*-dimensional face. A pure simplicial complex K of dimension d is called *d*-thrackle if there is a continuous map $f: K \longrightarrow \mathbb{R}^{d+1}$ such that

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- (i) the restriction of f to any facet is an embedding,
- (ii) any two facets intersect in a (d-1)-ball,
- (iii) intersections between faces are *stable*, that is, there is an $\varepsilon > 0$ such that any homotopy that moves points by at most ε cannot remove the intersection.

The (d-1)-faces of a *d*-thrackle are called *ridges*. If the map *f* is linear on each facet then we call *K* linear *d*-thrackle. The classical case of thrackle graphs corresponds to 1-thrackles. Here we prove higher-dimensional extensions of the linear thrackle conjecture:

Theorem 5.1. A linear (d-1)-thrackle with m facets and n ridges satisfies $dm \leq 2n$.

Proof. For any pure (d-1)-dimensional simplicial complex with m facets and n ridges such that any ridge is contained in at most two facets we have that $dm \leq 2n$ by multiple counting. Suppose there is a (d-1)-thrackle K with m facets and n ridges such that dm > 2n. Further suppose that K is a minimal counterexample, that is, any (d-1)-thrackle with at most m-1 facets satisfies the inequality of the theorem.

The simplicial complex K contains a ridge τ that is contained in at least three facets. Let $\sigma_1, \sigma_2, \sigma_3$ be three facets incident to τ . Fix any affine map $f: K \longrightarrow \mathbb{R}^d$ that realizes K as a linear (d-1)-thrackle. Since f embeds each facet, the (d-1)-simplices $f(\sigma_i)$ span affine hyperplanes H_i . These hyperplanes intersect in the (d-2)-plane spanned by $f(\tau)$, and at most two of the hyperplanes can coincide. Thus at least one of the hyperplanes H_j leaves the $f(\sigma_i), i \neq j$, on different sides of it, meaning that $f(\sigma_i) \setminus f(\tau)$, $i \neq j$, are contained in different open halfspaces determined by H_j . We claim that σ_j is only adjacent to other facets through τ and not through any other ridge. This is because any facet σ that shares a ridge with σ_j has its image $f(\sigma)$ entirely contained in one closed halfspace determined by H_j . But unless σ contains τ the (d-1)-simplex $f(\sigma)$ cannot intersect both $f(\sigma_i), i \neq j$, in (d-2)-balls.

Removing σ_j yields a (d-1)-thrackle with m-1 facets and n-d+1 ridges. Now $d(m-1) = dm-d > 2n-d \ge 2(n-d+1)$ and thus we obtained a counterexample with fewer facets than K, in contradiction to the minimality of K.

Any embedding of the boundary of the *d*-simplex into \mathbb{R}^d is a (d-1)-thrackle with d+1 facets and $\binom{d+1}{2}$ ridges. Thus the bound in Theorem 5.1 is tight in any dimension. The proof shows that the only examples of (d-1)-thrackles, $d \geq 3$, with equality dm = 2n are pseudomanifolds in the sense that each ridge is contained in precisely two facets.

If in the definition of d-thrackle we only require that any two facets intersect in a contractible set instead of a (d-1)-ball, Theorem 5.1 fails to hold in this more general setting: consider a square pyramid with base 1, 2, 3, 4 in cyclic order and apex 5. Let the set of facets consist of all triangles of the pyramid in addition to the triangle 1, 2, 3 and its three cyclic copies as well as the triangles 1, 3, 5 and 2, 4, 5. Every pair of facets intersects in a ball of dimension at most two. There are ten facets and ten ridges, which violates the inequality of Theorem 5.1.

Moreover, for a (d-1)-thrackle with m facets, the bound of $m \leq |V|$ will not hold in \mathbb{R}^d as can be seen by the counterexample in the figure below. In this figure, all edges will be extended into triangles to the blue vertex directly above the star, and the three marked edges will be extended to triangles with the red vertex above and to the side of the star.

We conjecture the continuous analog of Theorem 5.1.

Conjecture 5.2. Let K be a (d-1)-thrackle with m facets and n ridges. Then $dm \leq 2n$.

The planar case d = 2 of Conjecture 5.2 is Conway's thrackle conjecture. Under mild assumptions on the map f we can show that Conway's thrackle conjecture in fact implies Conjecture 5.2: suppose $f: K \longrightarrow \mathbb{R}^{d+1}$ realizes K as a d-thrackle in such a way that for every vertex v of K we can find a d-sphere S_v around f(v) that intersect all facets incident to v in (d-1)-balls and for every pair of distinct facets σ and τ incident to v the intersection $f(\sigma) \cap f(\tau) \cap S_v$ is a (d-2)-ball and stable within S_v . That

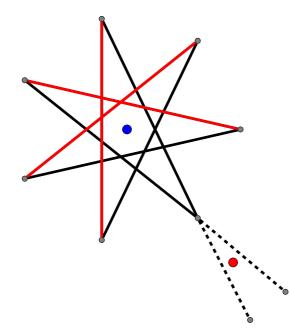


FIGURE 7. Coning all edges to the blue vertex and red edges to the red vertex yields a linear 2-thrackle in \mathbb{R}^3 with ten facets and nine vertices.

is, S_v is a sphere that is in general position with respect to the image of f restricted to the star of v. Then stereographic projection realizes the link of v as a (d-1)-thrackle.

These mild assumptions on f are met, for example, when f embeds each facet σ of K into \mathbb{R}^{d+1} as a d-manifold with boundary (i.e. f can be smoothly extended past $\partial \sigma$), and stability is strengthened to the condition of these manifolds intersecting transversally. If a vertex v is contained in a face σ , then f(v) is on the boundary of $f(\sigma)$, so for sufficiently small ε , the open ball B_{ε} of radius ε centered at f(v) has $B_{\varepsilon} \cap f(\sigma)$ homeomorphic to a closed half plane in \mathbb{R}^d , or equivalently an open d-ball with an open (d-1)-ball pasted on the boundary. If S_{ε} is the sphere of radius ε centered at f(v), we have $S_{\varepsilon} \cap f(\sigma) = (\overline{B_{\varepsilon}} \cap f(\sigma)) - (B_{\varepsilon} \cap f(\sigma))$. The right-hand side is homeomorphic (via the above homeomorphism) to a closed d-ball minus its interior and minus an open (d-1)-ball on its boundary, which is homeomorphic to precisely a closed (d-1)-ball. Note that for faces σ, τ that contain the vertex v, f(v) is also on the boundary of the (d-1)-manifold with boundary $f(\sigma) \cap f(\tau)$, so we can use the same argument to find ε small so that S_{ε} also intersects each $f(\sigma) \cap f(\tau)$ in a (d-2)-ball. Furthermore, the condition that $f(\sigma) \cap S_{\varepsilon}$ and $f(\tau) \cap S_{\varepsilon}$ intersect transversally on the sphere is equivalent to the condition that S_{ε} and $f(\sigma) \cap f(\tau)$ intersect transversally, which is possible for some small perturbation of the sphere since manifold transversality is known to be a generic property; see Chapter 3 of Hirsch [12] for an introduction to transversality.

These assumptions on f allow us to inductively transfer inequalities relating edges and vertices of thrackles in \mathbb{R}^2 to inequalities relating facets and ridges of d-thrackles in \mathbb{R}^{d+1} . Supposing we have the inequality $dm \leq 2cn$ between the number of facets m and the number of ridges n of a (d-1)-thrackle for some constant c, we get the inequality $(d+1)m \leq 2cn$ for d-thrackles in \mathbb{R}^{d+1} realized as above by multiple counting: let m be the number of facets of K and n the number of ridges. Denote by $f_k(v)$ the number of k-faces in the link of v, that is $f_{d-1}(v)$ is the number of facets incident to v. We have the inequality $df_{d-1}(v) \leq 2cf_{d-2}(v)$ for every vertex link. Summing this inequality over all vertex links yields $(d+1)dm \leq 2cdn$.

Thus, when f satisfies the above assumptions, the bound for plane thrackles given by Fulek and Pach [11] shows that any (d-1)-thrackle with m facets and n ridges satisfies the inequality $dm \leq 2.856n$. A proof of the thrackle conjecture immediately implies our Conjecture 5.2 for such f as noted above. High-dimensional versions of this conjecture might be simpler to attack since there are more serious restrictions on (d-1)-thrackles for $d \ge 3$: for example, every vertex link has to be a (d-2)-thrackle.

References

- 1. Roland Bacher, Günter M. Ziegler, and others, An Erdős-Szekeres-type question, June 2011, See http://mathoverflow.net/questions/67762/.
- Imre Bárány, Pavle V. M. Blagojević, and Günter M. Ziegler, Tverberg's Theorem at 50: Extensions and Counterexamples, Notices Amer. Math. Soc. 63 (2016), no. 7, 732–739.
- Imre Bárány and David G. Larman, A colored version of Tverberg's theorem, J. London Math. Soc. 2 (1992), no. 2, 314–320.
- 4. Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler, *Barycenters of Polytope Skeleta and Counterexamples to the Topological Tverberg Conjecture, via Constraints, arXiv preprint arXiv:1510.07984 (2015).*
- Pavle V. M. Blagojević, Benjamin Matschke, and Günter M. Ziegler, Optimal bounds for the colored Tverberg problem, J. Eur. Math. Soc. 17 (2015), no. 4, 739–754.
- 6. Mark de Longueville, Notes on the topological Tverberg theorem, Discrete Math. 241 (2001), no. 1, 207–233.
- Jean-Paul Doignon and G. Valette, Radon partitions and a new notion of independence in affine and projective spaces, Mathematika 24 (1977), no. 01, 86–96.
- 8. Paul Erdős, On sets of distances of n points, Amer. Math. Monthly 53 (1946), no. 5, 248–250.
- Paul Erdős and Nicolaas G. de Bruijn, On a combinational [sic] problem, Indagationes Mathematicae 10 (1948), 421– 423.
- 10. Florian Frick, Counterexamples to the topological Tverberg conjecture, Oberwolfach Reports 12 (2015), no. 1, 318–321.
- Radoslav Fulek and János Pach, A computational approach to Conway's thrackle conjecture, Comput. Geom. 44 (2011), 345–355.
- 12. Morris W. Hirsch, Differential topology, vol. 33, Springer Science & Business Media, 2012.
- László Lovász, János Pach, and Mario Szegedy, On Conway's thrackle conjecture, Discrete Comput. Geom. 18 (1997), no. 4, 369–376.
- Isaac Mabillard and Uli Wagner, Eliminating Higher-Multiplicity Intersections, I. A Whitney Trick for Tverberg-Type Problems, arXiv preprint arXiv:1508.02349 (2015).
- 15. Micha A. Perles and Moriah Sigron, A generalization of Tverberg's theorem, arXiv preprint arXiv:0710.4668 (2007).
- 16. Micha A. Perles and Moriah Sigron, Strong general position, arXiv preprint arXiv:1409.2899 (2014).
- 17. John R. Reay, Several generalizations of Tverberg's theorem, Israel J. Math. 34 (1979), no. 3, 238–244.
- John R. Reay, Twelve general position points always form three intersecting tetrahedra, Discrete Math. 28 (1979), no. 2, 193–199.
- 19. Helge Tverberg, A generalization of Radon's theorem, J. London Math. Soc. 41 (1966), no. 1, 123–128.
- 20. Jacobus H. van Lint and Richard M. Wilson, A course in combinatorics, Cambridge University Press, 2001.

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