

ARITHMETIC SIEGEL–WEIL FORMULA ON THE PRODUCT OF MODULAR CURVES

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ABSTRACT. We prove the full generating series arithmetic Siegel–Weil formula on the product of two modular curves, with arbitrarily singular coefficients. This is the case $\mathrm{GSpin}(2, 2)$ in Kudla’s program.

CONTENTS

1. Introduction	1
Part 1. Analytic side	7
2. Local densities of quadratic lattices	7
3. Weil representations, Eisenstein series and Whittaker functions	10
4. Normalization of Whittaker functions and Eisenstein series	14
5. Singular Fourier coefficients	19
6. Regularized Siegel–Weil formula	22
Part 2. Geometric side	25
7. Kudla–Millson theory on the product of upper half-planes	25
8. Special cycles on the integral model	32
9. Green currents and arithmetic cycles	39
10. Faltings heights of corank 1 cycles	48
11. Local special cycles	52
12. Heights and Rapoport–Zink uniformization	65
Part 3. Arithmetic degrees of special cycles	73
13. Proof of non-Archimedean local identities	73
14. Proof of the arithmetic Siegel–Weil formula	80
Appendix A. Finiteness and boundedness	89
Appendix B. Calculations of local densities and the case $p = 2$	90
Appendix C. The absolute convergence of the genus 3 Green currents	96
References	109

1. INTRODUCTION

1.1. Background. The theory of modular correspondences is one of the earliest settings in which intersection theory was related to modular forms. Let $\mathcal{E}ll \rightarrow \mathrm{Spec} \mathbb{Z}$ denote the moduli stack of elliptic curves. The coarse moduli scheme $Y_0(1)$ of $\mathcal{E}ll_{\mathbb{Q}}$ is isomorphic to $\mathbb{A}_{\mathbb{Q}}^1$ by the j -invariant map, with compactification $X_0(1) \simeq \mathbb{P}_{\mathbb{Q}}^1$. For positive integers m , the modular correspondence $T(m) \subset Y_0(1) \times Y_0(1)$ parametrizes pairs of elliptic curves admitting a degree m isogeny. The closure $\overline{T(m)}$ in $X_0(1) \times X_0(1)$ has divisor class

$$[\overline{T(m)}] = \mathcal{O}(\sigma_1(m), \sigma_1(m)), \quad \sigma_1(m) = \sum_{d|m} d.$$

The numbers $\sigma_1(m)$ are also the positive Fourier coefficients of the nonholomorphic weight 2 Eisenstein series

$$G_2(\tau) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{m \geq 1} \sigma_1(m) q^m, \quad \tau = x + iy, \quad q = e^{2\pi i \tau}.$$

Thus, the divisor classes of modular correspondences are already encoded by an Eisenstein series.

Hurwitz further studied the intersection of two modular correspondences $T(m_1)$ and $T(m_2)$; see [GK93, Proposition 2.4]. In the case of proper intersection, the intersection is supported on the locus of elliptic curves with complex multiplication, and its degree is expressed in terms of Hurwitz class numbers. This gives a remarkable relation between the proper intersections of modular correspondences and the nonsingular Fourier coefficients of a genus 2 Siegel Eisenstein series.

Gross and Keating [GK93] studied the arithmetic analogue by considering the proper intersections of three modular correspondences on the Deligne–Mumford stack $\mathcal{M} := \mathcal{E}ll \times_{\mathrm{Spec} \mathbb{Z}} \mathcal{E}ll$. The new contributions occur at primes where the correspondences meet in the supersingular locus. By studying deformations of endomorphisms of supersingular formal groups, they computed the corresponding local intersection multiplicities and related them to derivatives of nonsingular Fourier coefficients of a genus 3 Siegel Eisenstein series. The Kronecker–Hurwitz and Gross–Keating formulas treat only the case of proper intersection, and therefore account only for the nonsingular Fourier coefficients. For non-proper intersections, the resulting cycles have excess dimension and do not directly define the expected geometric or arithmetic degrees; the arithmetic-geometric analogues of singular Fourier coefficients were thus left undefined and unknown.

Kudla’s program provides a common framework for these phenomena. Special cycles on Shimura varieties associated with orthogonal groups are organized into generating series indexed by symmetric matrices T , and are expected to be related to Siegel Eisenstein series on symplectic groups. In the geometric setting, degrees and intersection numbers of special cycles are expected to equal values of Eisenstein series; in the arithmetic setting, arithmetic degrees and height pairings are expected to equal their central derivatives [Kud97, Kud04]. The product of two modular curves corresponds to $\mathrm{GSpin}(2, 2)$ for the standard split quadratic space. In this language, the formulas of Kronecker and Hurwitz provide the comparison with nonsingular coefficients of the genus 1 and genus 2 geometric Siegel–Weil formulas, while the Gross–Keating formula provides the comparison with nonsingular coefficients of the genus 3 arithmetic Siegel–Weil formula. The unresolved non-proper intersections on $\mathcal{M}$ correspond precisely to the singular Fourier coefficients.

For quaternionic Shimura curves, arithmetic Siegel–Weil formulas were established by Kudla, Rapoport, and Yang [KRY06]. For modular curves $X_0(N)$, related formulas were proved by Du and Yang and by Sankaran, Shi, and Yang at square-free N [DY19, SSY23], and by one of the authors for arbitrary N [Zhu25, Zhu24].

In higher dimensions, the nonsingular coefficients are by now better understood. Kudla and Rapoport formulated precise non-Archimedean identities relating intersections of integral models of special cycles to derivatives of local representation densities [KR11, KR14]. At finite places, Li and Zhang related local arithmetic intersections to derivatives of local representation densities [LZ22a, LZ22b]. At the Archimedean place, García and Sankaran constructed Green forms in arbitrary codimension and related the resulting heights to derivatives of local Whittaker functions [GS19] for compact Shimura varieties; see also Bruinier–Yang [BY20] and the corresponding unitary results of Liu [Liu11].

The singular coefficients remain subtler. For singular matrices T , the associated special cycle has codimension $\text{rank}(T)$ in the generic fiber if nonempty, which is smaller than the codimension required for a top-degree arithmetic class. One then supplies the missing codimension using powers of a metrized tautological bundle, together with suitably normalized Archimedean terms. For compact Shimura varieties, the Archimedean aspects are handled in [GS19], including an “Archimedean part” of the arithmetic Siegel–Weil formula for arbitrarily singular T . In [Che24, §4.4], it is explained how to combine the construction of [GS19, §5.4] with derived modifications in positive characteristic to define arithmetic cycle classes for possibly singular T , though some technical issues remain with non-compact moduli spaces and with possible modifications of Green currents. For singular terms in arithmetic Siegel–Weil, progress includes work by one of the authors on the corank 1 formula for unitary Shimura varieties of arbitrarily large dimension [Che24], and the work of Bruinier and Howard on arithmetic volumes relevant for the most degenerate Fourier coefficients [BH26].

In this article, we establish a complete central-derivative arithmetic Siegel–Weil formula for $\mathcal{M}$, for the entire generating series and all arbitrarily singular Fourier coefficients. One motivation for this work was to illustrate a full-generating-series arithmetic Siegel–Weil formula in higher dimension: our moduli stack $\mathcal{M}$ has generic fiber which is a surface, in contrast with the previously mentioned curve cases in [KRY06] etc.. To our knowledge, ours is the first such example with generic fiber of dimension > 1 .

Just as the classical Siegel mass formula involves a weighted count (“stacky count”), we expect that arithmetic Siegel–Weil formulas are best formulated in stacky language. We thus work with the stack $\mathcal{M}$ rather than its coarse moduli space $\mathbb{A}^1 \times \mathbb{A}^1$.

1.2. Main results. Our arithmetic Siegel–Weil formula is, roughly, an equality between arithmetic degrees of top degree cycle classes on the arithmetic 3-fold $\mathcal{M}$ and the central derivative of a genus 3 Siegel Eisenstein series. We begin by describing the arithmetic degrees.

Let $T \in \text{Sym}_3(\mathbb{Q})$. There is a special cycle $\mathcal{Z}(T)$ over $\mathcal{M}$ defined using modular correspondences. Over the open modular surface, $\mathcal{Z}(T)$ parametrizes two elliptic curves E_1 and E_2 , together with a triple of homomorphisms

$$\mathbf{x} = (x_1, x_2, x_3) \in \text{Hom}(E_1, E_2)^3$$

whose moment matrix is T . More precisely, the quadratic form on $\text{Hom}(E_1, E_2)$ is given by

$$Q(x) = \deg(x),$$

and the moment matrix $T(\mathbf{x}) \in \text{Sym}_3(\mathbb{Q})$ is characterized by

$$Q(a_1x_1 + a_2x_2 + a_3x_3) = {}^t_a T(\mathbf{x}) a, \quad a = (a_1, a_2, a_3) \in \mathbb{Z}^3.$$

The cycle $\mathcal{Z}(T)$ is defined by the condition $T(\mathbf{x}) = T$. In particular, the usual modular correspondence $T(m)$ is roughly the special cycle attached to the 1×1 matrix (m) . In our case, $\mathcal{Z}(T)$ is flat over $\text{Spec } \mathbb{Z}$ whenever $\text{rk}(T) < 3$, and is 0-dimensional whenever $\text{rk}(T) = 3$ (Lemma 8.2.2).

We now describe an arithmetic degree which is a sum of two terms: the height of $\mathcal{Z}(T)$ plus a Green current integral. More precisely, there is a metrized tautological line bundle $\hat{\mathcal{E}}$ on $\mathcal{M}$ given by tensoring the pullbacks of the metrized Hodge bundle from each copy of $\mathcal{E}l$ (Section 8.1). Take the first Chern form $\Omega := c_1(\hat{\mathcal{E}}^\vee)$ of the dual metrized bundle. Then we have $\dim \mathcal{Z}(T) = 3 - \text{rk}(T)$ if $\mathcal{Z}(T)$ is nonempty (Lemma 8.2.2). For any $\mathbf{y} \in \text{Sym}_3(\mathbb{R})_{>0}$, we define the *arithmetic degree*

$$(1.2.1) \quad \widehat{\deg}[\hat{\mathcal{Z}}(T, \mathbf{y})] := \widehat{\deg} \left((\hat{\mathcal{E}}^\vee)^{3-\text{rk}(T)}|_{\mathcal{Z}(T)} \right) + \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathbf{g}(T, \mathbf{y})$$

where the first term is the height of $\mathcal{Z}(T)$ with respect to $\hat{\mathcal{E}}^\vee$, and where the second term is an integral involving a current $\mathbf{g}(T, \mathbf{y})$ whose construction on $\mathcal{M}(\mathbb{C})$ is a small modification of the one considered in [GS19]. More precisely, see Section 9.7. In Appendix C, we explain how $\mathbf{g}(T, \mathbf{y})$ extends to a current on the Deligne–Rapoport compactification $\overline{\mathcal{M}}(\mathbb{C})$, so that the integral in (1.2.1) is well-defined.

If T is nonsingular, we have $\dim \mathcal{Z}(T) = 3 - \text{rk}(T) = 0$ if $\mathcal{Z}(T)$ is nonempty, and the height term in (1.2.1) collapses to $\log p$ times the usual algebro-geometric degree of the 0-cycle $\mathcal{Z}(T)$, where p is the unique prime over which $\mathcal{Z}(T)$ is supported. This is the case of proper intersection studied by Gross and Keating. The case of singular T , or $\dim \mathcal{Z}(T) > 0$ corresponds exactly to the “excess intersection” cases not covered by Gross and Keating. The appearance of $\hat{\mathcal{E}}^\vee$ in (1.2.1) corrects for this excess intersection, and can be used to define a top-codimension arithmetic cycle class $[\hat{\mathcal{Z}}(T, \mathbf{y})] \in \widehat{\text{Ch}}^3(\mathcal{M})$ for arbitrary $(T, \mathbf{y})$, as explained in (9.6.9).

We next describe the analytic side of the formula. Let $E(\mathbf{z}, s)$ be the classical Siegel Eisenstein series of genus 3 and weight 2, given by

$$(1.2.2) \quad E(\mathbf{z}, s) := \sum_{\gamma \in P(\mathbb{Z}) \backslash \text{Sp}_6(\mathbb{Z})} \frac{\det(\mathbf{y})^{s/2}}{|\det(c\mathbf{z} + d)|^s} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $P \subseteq \text{Sp}_6$ the Siegel parabolic, for $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{H}_3$ and $s \in \mathbb{C}$ with $\text{Re}(s) \gg 0$, with meromorphic continuation to $s \in \mathbb{C}$. As explained in (4.4.5), the Eisenstein series $E(\mathbf{z}, s)$ satisfies a functional equation centered at $s = 0$. There is a Fourier expansion

$$(1.2.3) \quad E(\mathbf{z}, s) = \sum_{T \in \text{Sym}_3(\mathbb{Q})} C_T(\mathbf{y}, s) q^T \quad q^T = e^{2\pi i \text{tr}(T\mathbf{z})}.$$

Our main theorem, in generating series form, is as follows.

Theorem 1.2.1 (Theorem 9.7.1). *We have*

$$(1.2.4) \quad \sum_{T \in \text{Sym}_3(\mathbb{Q})} \widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] \cdot e^{2\pi i \text{tr}(Tz)} = -\frac{\text{vol}(\mathcal{M}(\mathbb{C}))}{2} \frac{d}{ds} \Big|_{s=0} E(z, s).$$

More precisely, Theorem 1.2.1 asserts an equality between $\widehat{\deg}[\widehat{\mathcal{Z}}(T, y)]$ and the central derivative of the T -th Fourier coefficient $C_T(y, s)$, for any $T \in \text{Sym}_3(\mathbb{Q})$ and $y \in \text{Sym}_3(\mathbb{R})_{>0}$, and up to the factor $-\text{vol}(\mathcal{M}(\mathbb{C}))/2$. Here $\text{vol}(\mathcal{M}(\mathbb{C})) = 1/288$ denotes the orbifold volume of $\mathcal{M}(\mathbb{C})$ with respect to the volume form Ω^2 .

In Theorem 1.2.1, the terms with positive definite T correspond exactly to the situation studied by Gross and Keating. The terms of ranks 2, 1, and 0 are the singular coefficients arising from non-proper intersections. They are expressed, respectively, through the heights of certain CM cycles (rank 2), the heights of modular correspondences (rank 1), and the arithmetic volume of $\mathcal{M}$ with respect to the dual metrized tautological bundle (rank 0). Thus, a single genus 3 Eisenstein series simultaneously encodes the proper arithmetic intersections of modular correspondences and all of their excess-intersection contributions, along with Green current integrals. We emphasize that the correct arithmetic Siegel–Weil formula must not forget the Green currents: for example, the constant term of $-\frac{1}{2} \text{vol}(\mathcal{M}(\mathbb{C})) \frac{d}{ds} \Big|_{s=0} E(z, s)$ is the arithmetic volume of $\mathcal{M}$ *plus* an additional Green current integral.

1.3. Strategy of the proof. As already explained, the arithmetic degrees (1.2.1) have two contributions: heights of special cycles and integrals of the currents $\mathbf{g}(T, y)$. We discuss their evaluation separately.

1.3.1. Heights of special cycles. Let T be positive semidefinite and put $r = \text{rank}(T)$. For $r = 0$ and $r = 1$, the required calculations reduce to the arithmetic volume of $\mathcal{M}$ and the heights of the Hecke correspondences $T(m)$, respectively. These quantities were computed explicitly by Burgos Gil, Kramer, and Kühn [BGKK07]. The computation is based on explicit *global sections* of line bundles, e.g. the Ramanujan discriminant function, and on Rohrlich’s modular Jensen’s formula [Roh84].

In contrast, the method for $r = 3$ and $r = 2$ is *local*, without explicit sections of line bundles. We first recall the work of Gross–Keating for the case $r = 3$. When $r = 3$, the special cycle $\mathcal{Z}(T)$ is an Artinian Deligne–Mumford stack supported on the supersingular locus. Its height is a sum of stacky lengths of local rings, multiplied by $\log p$. The required local lengths are given by the deformation-theoretic Gross–Keating formulas [GK93]. On the analytic side, there is a factorization of the Fourier coefficients C_T into local Whittaker functions. More precisely, after normalizing $C_T^3(y, s) := (s+1)(s+2)\Lambda(s+2)\Lambda(2s+2)C_T(y, s)$ to produce the symmetric functional equation $C_T^3(y, -s) = -C_T^3(y, s)$ (Section 4.4) we have a factorization

$$(1.3.1) \quad -(32\pi^2)^{-1} C_T^3(y, s) = W_{T, \infty}^*(y, s) \prod_p W_{T, p}^*(s)$$

into normalized local Whittaker functions, as in (4.3.5). Gross–Keating’s result is a comparison of the stacky length in characteristic p with $\frac{d}{ds} \Big|_{s=0} W_{T, p}^*(s)$. We have $\text{vol}(\mathcal{M}(\mathbb{C}))/2 = (32\pi^2)^{-1}(s+1)(s+2)\Lambda(s+2)\Lambda(2s+2) \Big|_{s=0}$, consistent with the constant in Theorem 1.2.1.

The remaining case is $r = 2$, which is the main new contribution of the heights part of our work. Unlike Gross–Keating’s case, the cycle $\mathcal{Z}(T)$ is flat over $\text{Spec } \mathbb{Z}$ and hence its height does not admit an obvious canonical local decomposition. We now describe our local decomposition procedure. The generic fiber of $\mathcal{Z}(T)$ consists of CM points parametrizing pairs of elliptic curves with complex multiplication by orders in an explicit imaginary quadratic field depending on T (Lemma 8.2.2). The height of $\mathcal{Z}(T)$ is then given by weighted averages of the Faltings heights of these elliptic curves, with weights determined by the cycle multiplicities and automorphism groups.

The Faltings height of an elliptic curve with complex multiplication by the full ring of integers $\mathcal{O}_K$ is known explicitly by the formula of Lerch and Chowla–Selberg as recalled in (10.1.2) (also [Del85, p. 29]). The height in the case of CM by a non-maximal order can be found in the work of Nakajima and Taguchi [NT91].

Rather than evaluate these heights individually, we compute the difference between the relevant average height and the maximal-order CM height. This reduces the rank 2 height calculation to a “relative height correction”, with the maximal-order formula providing the reference value. It turns out that this “average relative height correction” for $\mathcal{Z}(T)$ decomposes locally in a canonical way as a sum (see (10.2.4))

$$(1.3.2) \quad \sum_p \frac{\widehat{\deg}_{\mathcal{E}^\vee}(\mathcal{Z}(T))_p}{\deg_{\mathcal{E}^\vee}(\mathcal{Z}(T)_{\mathbb{Q}})}$$

over all the finite primes whenever the denominator is nonzero, where $\deg_{\mathcal{E}^\vee}(\mathcal{Z}(T)_{\mathbb{Q}})$ denotes the degree of the 0-cycle $\mathcal{Z}(T)_{\mathbb{Q}}$ in the generic fiber (Section 10.2). We then make a global-to-local reduction step and explain how the “average relative height correction” can be calculated via Rapoport–Zink uniformization. More precisely, we show

$$(1.3.3) \quad \frac{\widehat{\deg}_{\mathcal{E}^\vee}(\mathcal{Z}(T))_p}{\deg_{\mathcal{E}^\vee}(\mathcal{Z}(T)_{\mathbb{Q}})} = -\frac{\delta_{\text{Fal},p}(T)}{\deg_p(T)} \cdot \log p$$

in (12.3.8), whenever the denominators are nonzero, where the quantities $\delta_{\text{Fal},p}(T)$ and $\deg_p(T)$ are formulated purely in terms of local special cycles on Rapoport–Zink spaces (11.6.3). Thus, the average height calculation for $r = 2$ reduces to a key local comparison of “local height corrections” with derivatives of local Whittaker functions appearing in the Eisenstein series Fourier coefficient. More precisely, let $T^\flat \in \text{Sym}_2(\mathbb{Q})$ be a 2×2 nondegenerate matrix associated to the rank 2 matrix T as in (9.6.1), for example if $T = \text{diag}(0, T^\flat)$. We then have

$$(1.3.4) \quad \delta_{\text{Fal},p}(T) \cdot \log p = -\frac{d}{ds} \Big|_{s=-1/2} W_{T^\flat,p}^\star(s) \quad \deg_p(T) = W_{T^\flat,p}^\star(-1/2)$$

where $W_{T^\flat,p}^\star(s)$ is a normalized local Whittaker function of genus 2. The derivative and special value are taken at the non-central point $s = -1/2$, and the central point is at $s = 0$. This local comparison is proved in Section 13. There, Theorem 13.0.1 states the result in terms of local densities, whose relation with local Whittaker functions is explained in Remark 3.5.1.

Lastly, extracting the height calculation for $\mathcal{Z}(T)$ from the average height calculation amounts to extracting the numerator of the left-hand side of (1.3.3) from the displayed ratio. For this purpose, the left-hand side denominator can be calculated e.g. by complex uniformization and compared with the special value of a genus 2 Eisenstein series Fourier coefficient; this is Lemma 8.5.2.

1.3.2. *Integrals of currents.* For $T \in \text{Sym}_3(\mathbb{Q})$ and $y \in \text{Sym}_3(\mathbb{R})_{>0}$, García and Sankaran [GS19] construct a smooth $(2, 2)$ -form $\nu^\circ(T, y)$ on $\mathcal{M}(\mathbb{C})$. Its Mellin transform defines a family of currents

$$(1.3.5) \quad \mathfrak{g}^\circ(T, y, \rho) = \int_1^\infty \nu^\circ(T, ty) \frac{dt}{t^{1+\rho}},$$

initially for $\text{Re}(\rho)$ sufficiently large; see (9.2.3).

In Appendix C, we show that this family admits a meromorphic continuation to $\text{Re}(\rho) > -1/2$ with values in currents on the compactification $\overline{\mathcal{M}}(\mathbb{C})$, extending the corresponding family on $\mathcal{M}(\mathbb{C})$. Taking the constant term at $\rho = 0$ gives an extension of $\mathfrak{g}^\circ(T, y)$ to the compactification. Its integral, understood as the pairing of this extension with the constant function 1, is therefore well-defined. Moreover, the convergence estimates in the appendix justify interchanging the two integrations in their common domain of absolute convergence. By meromorphic continuation, we obtain

$$(1.3.6) \quad \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}^\circ(T, y) = \text{CT}_{\rho=0} \int_{\mathcal{M}(\mathbb{C})} \left(\int_1^\infty \nu^\circ(T, ty) \frac{dt}{t^{1+\rho}} \right) = \text{CT}_{\rho=0} \int_1^\infty \left(\int_{\mathcal{M}(\mathbb{C})} \nu^\circ(T, ty) \right) \frac{dt}{t^{1+\rho}},$$

where $\text{CT}_{\rho=0}$ denotes the constant term at $\rho = 0$. The correction terms in (9.6.7) contribute geometric degrees of special cycles, which are computed explicitly in §8.5. Thus the evaluation of the Archimedean contribution in (1.2.1) reduces to computing (1.3.6).

Following the strategy of [GS19], we express $\nu^\circ(T, ty)$ in terms of the T -th Fourier coefficient of an explicit theta series. More precisely, (9.4.2) gives

$$(1.3.7) \quad \nu^\circ(T, ty)_z = C_{\theta, T} \left(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{\text{M}_2(\hat{\mathbb{Z}})^3} \right) \Omega_z^2$$

for $z \in \mathcal{M}(\mathbb{C})$, where $C_{\theta, T}$ is a T -th Fourier coefficient of a theta series for the Schwartz function $\tilde{\nu}_3 \otimes \mathbf{1}_{\text{M}_2(\hat{\mathbb{Z}})^3}$, where $\tilde{\nu}_3$ arises from a Schwartz form in Kudla–Millson theory. The integral of $\nu^\circ(T, ty)$ over $\mathcal{M}(\mathbb{C})$ can then be identified, up to an explicit normalization factor, with a theta integral over $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$. Our setting lies outside Weil’s convergence range [Wei65], so the ordinary Siegel–Weil formula does not apply directly. Instead, we use the regularized Siegel–Weil formula [KR94]; see also [HK91]. Together with the absolute convergence of the relevant Fourier-coefficient integrals, this yields the identity predicted by the ordinary Siegel–Weil formula and expresses (1.3.6) in terms of Fourier coefficients of Eisenstein series. The relation with the derivative of the classical Siegel Eisenstein series (formed using a different Schwartz function) then appears via a suitable differential operator, as in [GS19, (5.139)] and [GS19, Lemma 3.5].

1.4. Structure of the paper. In Part 1, we set up the precise normalizations of Eisenstein series and local Whittaker functions, explain explicit formulas relating singular Fourier coefficients of $E(z, s)$ to Fourier coefficients of lower genus Eisenstein series, and record the relevant Siegel–Weil formula for coefficient-by-coefficient convergent theta integrals.

Sections 7 through 9 explain the construction of arithmetic cycle classes. Sections 10 through 12 explain how the global height term for T of corank 1 can be reduced to a collection of local quantities calculated on Rapoport–Zink spaces. This is an analogue of the procedure from [Che24] with some simplifications, e.g. we work with the ratio (12.3.8) which is eventually compared with a log derivative in (14.3.5), so that the groupoid cardinality $\#[\cdots]$ from (12.3.7) does not need to be computed explicitly, unlike in loc. cit.. Once the problem is reduced to a local calculation

on Rapoport–Zink spaces, the comparison with local Whittaker functions (via local densities) is accomplished in Section 13.

The main theorem is proved in Section 14, by assembling our main results with the existing pieces in the literature and aligning the constants.

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An initial version of Appendix C was produced by GPT-6 Astra. The current version has been rewritten from scratch.

Part 1. Analytic side

2. LOCAL DENSITIES OF QUADRATIC LATTICES

Let p be any prime. Let F be a p -adic local field with ring of integers $\mathcal{O}_F$, uniformizer $\varpi \in \mathcal{O}_F$ and residue field $\mathcal{O}_F/(\varpi) \simeq \mathbb{F}_q$ of size q . Let $v_F : F \rightarrow \mathbb{Z} \cup \{\infty\}$ be the valuation with $v_F(\varpi) = 1$.

2.1. Notations on quadratic lattices. A quadratic lattice (M, q) is a finite free $\mathcal{O}_F$ -module equipped with a quadratic form $q : M \rightarrow F$. Let $M_F := M \otimes_{\mathcal{O}_F} F$ be the F -vector space associated to M . The quadratic form q induces a symmetric bilinear form

$$(2.1.1) \quad M_F \times M_F \xrightarrow{(\cdot, \cdot)} F$$

given by $(x, y) = q(x + y) - q(x) - q(y)$. Let $M^\vee = \{x \in M \otimes_{\mathcal{O}_F} F : (x, M) \subset \mathcal{O}_F\}$. We say a quadratic lattice M is *integral* if $q(x) \in \mathcal{O}_F$ for all $x \in M$, and is *self-dual* if it is integral and $M = M^\vee$. Given a tuple $\mathbf{x}$ of elements in M , our convention is that the *normalized Gram matrix* or *moment matrix* is the matrix $\frac{1}{2}(\mathbf{x}, \mathbf{x})$.

Suppose the symmetric bilinear form $(\cdot, \cdot)$ on M_F is nondegenerate. With $n = \text{rank}_{\mathcal{O}_F} M$, let $\{x_i\}_{i=1}^n$ be any $\mathcal{O}_F$ basis of M , with $t_{ij} = \frac{1}{2}(x_i, x_j)$. The *discriminants* of M and M_F are

$$\text{disc}(M) = (-1)^{n(n-1)/2} \det((t_{ij})) \in F^\times / (\mathcal{O}_F^\times)^2 \quad \text{disc}(M_F) = (-1)^{n(n-1)/2} \det((t_{ij})) \in F^\times / (F^\times)^2.$$

Given a nondegenerate matrix $T \in \text{Sym}_n(F)$, we set $\text{disc}(T) := (-1)^{n(n-1)/2} \det(T)$.

Let $\chi_F : F^\times / (F^\times)^2 \rightarrow \{\pm 1, 0\}$ be the function

$$\chi_F(a) := \begin{cases} +1 & \text{if } F[X]/(X^2 - a) \text{ is split over } F \\ -1 & \text{if } F[X]/(X^2 - a) \text{ is inert over } F \\ 0 & \text{if } F[X]/(X^2 - a) \text{ is ramified over } F. \end{cases}$$

We define $\chi_F(M) = \chi_F(\text{disc}(M))$. For nondegenerate $T \in \text{Sym}_n(F)$, we set $\chi_F(T) := \chi_F(\text{disc}(T))$. When $F = \mathbb{Q}_p$, we set $\chi_p := \chi_F$.

Let $\text{Hilb}(\cdot, \cdot)_F$ be the Hilbert symbol on the local field F . If moreover $\{x_i\}_{i=1}^n$ is any orthogonal basis of the quadratic space M_F , i.e. $t_{ij} = 0$ if $i \neq j$ and $t_{ii} \neq 0$, the *Hasse invariant* of the quadratic

space M_F is

$$(2.1.2) \quad \epsilon(M_F) = \prod_{1 \leq i < j \leq n} \text{Hilb}(t_{ii}, t_{jj})_F \in \{\pm 1\}.$$

The invariants $\text{disc}(M_F)$ and $\epsilon(M_F)$ determine M_F up to isomorphism.

The lattice M has an associated sequence $\text{GK}(M) = (a_1, \dots, a_n) \in \mathbb{Z}^n$ of *Gross–Keating invariants*, with $a_1 \leq \dots \leq a_n$. These invariants $\text{GK}(M)$ are useful for uniform statements in all residue characteristics, particularly when including $p = 2$. The invariants $\text{GK}(M)$ were introduced in [GK93, §4] for $F = \mathbb{Q}_p$ and $n = 3$, and the analogous general case is defined in [IK18, Definition 0.1]. See also Appendix B for the definition, there with $F = \mathbb{Q}_p$. From the definition, one finds that M is integral if and only if $a_i \geq 0$ for all i . It follows from this that $a_1 = \min_{x \in M} v_F(q(x))$ for any M , not necessarily integral. We set $\nu(M) := a_1 + \dots + a_n$ for the sum of the Gross–Keating invariants. For the reader’s convenience, we recall [IK18, Theorem 0.1] which reads

$$(2.1.3) \quad \nu(M) = \begin{cases} v_F(4^{\lfloor n/2 \rfloor} \text{disc}(M)) & \text{if } n \text{ is odd} \\ v_F(4^{\lfloor n/2 \rfloor} \text{disc}(M)) & \text{if } n \text{ is even and } \chi_F(M) \neq 0 \\ v_F(4^{\lfloor n/2 \rfloor} \text{disc}(M)) - v_F(\Delta_M) + 1 & \text{if } n \text{ is even and } \chi_F(M) = 0, \end{cases}$$

where $\Delta_M \subseteq \mathcal{O}_F$ denotes the discriminant ideal of the extension $F[X]/(X^2 - \text{disc}(M))$ over F . In particular, if n is even, we find that $\chi_F(M) = 0$ if and only if $\nu(M)$ is odd. The previous sentence also holds for n odd when $p \neq 2$, but fails for n odd and $p = 2$.

2.2. Local densities. Set

$$(2.2.1) \quad \text{Sym}_n(\mathcal{O}_F)^* := \{b \in \text{Sym}_n(F) : \text{tr}(bc) \in \mathcal{O}_F \text{ for all } c \in \text{Sym}_n(\mathcal{O}_F)\}.$$

An element $T \in \text{Sym}_n(F)$ lies in $\text{Sym}_n(\mathcal{O}_F)^*$ if and only if $T_{i,i} \in \mathcal{O}_F$ and $T_{i,j} \in \frac{1}{2}\mathcal{O}_F$ for all $i \neq j$.

Given nonsingular matrices $S \in \text{Sym}_m(F)$ and $T \in \text{Sym}_n(F)$, we recall the *local representation density* (or just *local density*)

$$(2.2.2) \quad \text{Den}(S, T) := \lim_{k \rightarrow \infty} \frac{\text{vol}(\{x \in M_{m,n}(\mathcal{O}_F) : {}^t x S x - T \in \varpi^k \text{Sym}_n(\mathcal{O}_F)^*\})}{q^{-k \cdot n(n+1)/2}},$$

where $M_{m,n}(\mathcal{O}_F)$ is given the Haar measure of total volume 1. The limit argument stabilizes for $k \gg 0$. This depends only on the isomorphism classes of the quadratic lattices defined by S and T , so we also write $\text{Den}(M, L)$ if M and L are quadratic lattices admitting bases with normalized Gram matrices S and T , respectively. If $n > m$, we have $\text{Den}(S, T) = 0$. If $S \in \text{Sym}_n(\mathcal{O}_F)^*$ but $T \notin \text{Sym}_n(\mathcal{O}_F)^*$, we have $\text{Den}(S, T) = 0$.

Remark 2.2.1. Let L, M be two integral non-degenerate quadratic $\mathcal{O}_F$ -lattices. Let $\text{Rep}_{M,L}$ be the scheme of integral representations, an $\mathcal{O}_F$ -scheme such that for any $\mathcal{O}_F$ -algebra R ,

$$\text{Rep}_{M,L}(R) = \text{QHom}(L \otimes_{\mathcal{O}_F} R, M \otimes_{\mathcal{O}_F} R),$$

where QHom denotes the set of module homomorphisms which preserve the quadratic forms. The local density of integral representations is

$$\text{Den}(M, L) = \lim_{d \rightarrow \infty} \frac{\#\text{Rep}_{M,L}(\mathcal{O}_F/\varpi^d)}{q^{d \cdot \dim(\text{Rep}_{M,L})_F}}.$$

Equivalently, $\text{Den}(M, L)$ is the volume of $\text{Rep}_{M,L}(\mathcal{O}_F)$ with respect to the fibral measure associated to the map $M_{m,n}(\mathcal{O}_F) \rightarrow \text{Sym}_n(\mathcal{O}_F)^*$ given by $x \mapsto {}^t x S x - T$ in the previous notation, if the source and target are taken to have total volume 1. The limit in the construction of $\text{Den}(M, L)$ corresponds to a shrinking tubular neighborhood of $\text{Rep}_{M,L}(\mathcal{O}_F)$.

Remark 2.2.2. In Remark 2.2.1, if L, M have ranks n, m respectively, then $(\text{Rep}_{M,L})_F \neq \emptyset$ if and only if $n \leq m$, in which case $\dim(\text{Rep}_{M,L})_F = mn - \frac{n(n+1)}{2}$. If $n > m$, then $\text{Rep}_{M,L}(\mathcal{O}_F/\varpi^d) = \emptyset$ for $d \gg 0$.

2.3. Normalized local density polynomial. Let H be the standard split rank 2 quadratic $\mathcal{O}_F$ -lattice, with quadratic form given by $q(x, y) = xy$. For a nonnegative integer r , define H^r to be the orthogonal sum of r copies of H .

Let L be a quadratic $\mathcal{O}_F$ -lattice of rank $n \geq 0$. We set

$$(2.3.1) \quad \text{Nor}(X, L) := \begin{cases} (1 - q^{-(n+1)/2} X) \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - q^{2i-(n+1)} X^2) & \text{if } n \text{ is odd} \\ (1 - q^{-1/2} \chi_F(L) X)^{-1} (1 - q^{-(n+1)/2} X) \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - q^{2i-(n+1)} X^2) & \text{if } n \text{ is even,} \end{cases}$$

which is taken from [Ike17, §4], up to a shift.

Consider an even integer $m \geq 0$, and set $s_0 = (m - (n + 1))/2$. With m suppressed from notation, the discussion in [Ike17, §4] (see also Remark 3.5.1) implies that there is a (unique) Laurent polynomial $\text{Den}^*(X, L) \in \mathbb{Z}[X, X^{-1/2}]$ such that

$$(2.3.2) \quad \text{Den}^*(X, L)|_{X=q^{-k}} = \begin{cases} \left(X^{-\nu(L)/2} \frac{\text{Den}(H^{m/2+k-2s_0}, L)}{\text{Nor}(q^{s_0} X, L)} \right) \Big|_{X=q^{-k}} & \text{if } n \text{ is odd} \\ \left(X^{-\lfloor \nu(L)/2 \rfloor} \frac{\text{Den}(H^{m/2+k-2s_0}, L)}{\text{Nor}(q^{s_0} X, L)} \right) \Big|_{X=q^{-k}} & \text{if } n \text{ is even} \end{cases}$$

for all integers k with $m/2 + k - 2s_0 \geq 0$ and $\text{Nor}(q^{s_0} X, L)|_{X=q^{-k}} \neq 0$. We moreover have $X^{\nu(L)/2} \text{Den}^*(X, L) \in \mathbb{Z}[X]$ if n is odd and $X^{\lfloor \nu(L)/2 \rfloor} \text{Den}^*(X, L) \in \mathbb{Z}[X]$ if n is even, and these elements of $\mathbb{Z}[X]$ have constant term 1. By [Ike17, Theorem 4.1] combined with the formula for $\nu(L)$ recalled in (2.1.3), we have the functional equation

$$(2.3.3) \quad \begin{cases} \text{Den}^*\left(\frac{1}{q^{2s_0} X}, L\right) = \eta_L \text{Den}^*(X, L) & \text{if } n \text{ is odd} \\ \text{Den}^*\left(\frac{1}{q^{2s_0} X}, L\right) = \text{Den}^*(X, L) & \text{if } n \text{ is even} \end{cases}$$

where

$$(2.3.4) \quad \eta_L = \begin{cases} +1 & \text{if } L \text{ is split} \\ -1 & \text{if } L \text{ is nonsplit} \end{cases}$$

via the explanation after [IK18, Definition 0.4]. We then set

$$(2.3.5) \quad \text{Den}^*(L) := \text{Den}^*(1, L), \quad \partial \text{Den}^*(L) := \frac{d}{dX} \Big|_{X=1} \text{Den}^*(X, L).$$

For n odd, we are abusing notation as in [Che24, Remark 15.5.2] to handle expressions involving $X^{-1/2}$, e.g. $\frac{d}{dX} X^{-1/2} := -\frac{1}{2} X^{-3/2}$ as one expects.

If $\mathrm{GK}(L) = (0, \dots, 0)$, then we have $\mathrm{Den}^*(X, L) = 1$, a constant function: we know $\mathrm{Den}^*(X, L) \in \mathbb{Z}[X]$ must be a constant because of the functional equation (2.3.3), and $\mathrm{Den}^*(X, L)$ has constant term 1.

Example 2.3.1. For this paper, we are interested in the case $F = \mathbb{Q}_p$ with $m = 4$ and $0 \leq n \leq m-1$. In these cases, we have

$$(2.3.6) \quad \mathrm{Den}^*(p^{-k}, L) = \begin{cases} p^{k(\nu(L)/2)} \frac{\mathrm{Den}(H^{k+2}, L)}{(1-p^{-k-2})(1-p^{-2k-2})} & \text{if } \mathrm{rank}_{\mathbb{Z}_p}(L) = 3 \\ p^{k[\nu(L)/2]} \frac{\mathrm{Den}(H^{k+1}, L)}{(1-p^{-k-1})(1-p^{-2k})} \cdot (1 - \chi_p(L)p^{-k}) & \text{if } \mathrm{rank}_{\mathbb{Z}_p}(L) = 2 \\ p^{k(\nu(L)/2)} \frac{\mathrm{Den}(H^k, L)}{1-p^{-k}} & \text{if } \mathrm{rank}_{\mathbb{Z}_p}(L) = 1 \\ 1 & \text{if } \mathrm{rank}_{\mathbb{Z}_p}(L) = 0 \end{cases}$$

for all integers $k \geq 0$ if $n = 3$, for $k \geq 1$ if $n = 2$ or $n = 1$, and for all k if $n = 0$.

3. WEIL REPRESENTATIONS, EISENSTEIN SERIES AND WHITTAKER FUNCTIONS

3.1. Notations on Siegel groups. For a positive integer r , let W_r be the standard symplectic space over $\mathbb{Q}$ of dimension $2r$, equipped with a basis with Gram matrix

$$(3.1.1) \quad w_r := \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}.$$

Let $P \subseteq \mathrm{Sp}_{2r} := \mathrm{Sp}(W_r)$ be the Siegel parabolic, with Levi $M \subseteq P$ and unipotent radical $N \subseteq P$ given explicitly as

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{Sp}(W_r) \right\}$$

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} : a \in \mathrm{GL}_r \right\} \quad N = \left\{ n(b) = \begin{pmatrix} 1_r & b \\ 0 & 1_r \end{pmatrix} : b \in \mathrm{Sym}_r \right\}.$$

We sometimes write P_r, M_r, N_r to emphasize r -dependence. The Hermitian symmetric domain associated to the group $\mathrm{Sp}(W_r)$ is the genus r upper half-plane

$$\mathbb{H}_r = \{z = x + iy : x, y \in \mathrm{Sym}_r(\mathbb{R}), y > 0\}.$$

There is the usual transitive action $\mathrm{Sp}_{2r}(\mathbb{R}) \curvearrowright \mathbb{H}_r$ given by

$$(3.1.2) \quad z \mapsto (az + b)(cz + d)^{-1} \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{R}) \quad \text{and} \quad z \in \mathbb{H}_r,$$

and the stabilizer of $i1_r \in \mathbb{H}_r$ is the standard maximal compact

$$(3.1.3) \quad K_{r,\infty} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : k = a + ib \in U(r) \right\} \simeq U(r).$$

Given an element $z = x + iy \in \mathbb{H}_r$, we write $g_{z,\infty} = n(x)m(a) \in P(\mathbb{R})$ for any element with the property $z = g_{z,\infty} \cdot i1_r$, i.e. $a \cdot {}^t a = y$, such that moreover $\det(a) > 0$. We then write $g_z \in P(\mathbb{A})$ for an element with

$$(3.1.4) \quad g_{z,v} = 1_{2r} \text{ for all } v < \infty \text{ and } g_{z,\infty} \text{ as above.}$$

3.2. Degenerate principal series. Fix the standard additive character $\psi : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{C}^\times$ such that $\psi_\infty(x) = e^{2\pi i x}$. We define the degenerate principal series to be the unnormalized smooth induction

$$(3.2.1) \quad I_r(s) := \text{Ind}_{P_r(\mathbb{A})}^{\text{Sp}(\mathbf{W}_r)(\mathbb{A})} \left(|\det(\cdot)|_{\mathbb{A}}^{s+(r+1)/2} \right), \quad s \in \mathbb{C},$$

where $\det(\cdot) : P_r(\mathbb{A}) \rightarrow \mathbb{A}^\times$ is abuse of notation for $m(a) \mapsto \det(a)$. This is the restricted tensor product of the following local representations of $\text{Sp}(\mathbf{W}_r)(\mathbb{Q}_v)$:

$$(3.2.2) \quad I_{r,v}(s) := \text{Ind}_{P_r(\mathbb{Q}_v)}^{\text{Sp}(\mathbf{W}_r)(\mathbb{Q}_v)} \left(|\det(\cdot)|_v^{s+(r+1)/2} \right), \quad s \in \mathbb{C}.$$

We will use r (resp. r_v) to denote the right regular representation of $\text{Sp}(\mathbf{W}_r)(\mathbb{A})$ (resp. $\text{Sp}(\mathbf{W}_r)(\mathbb{Q}_v)$) on $I_r(s)$ (resp. $I_{r,v}(s)$).

3.3. Representations of $K_{r,\infty}$. The irreducible representations of the group $K_{r,\infty} \simeq U(r)$ (3.1.3) are denoted by π_λ with

$$\lambda = (l_1, \dots, l_r) \in \mathbb{Z}^r, \quad l_1 \geq \dots \geq l_r.$$

The $K_{r,\infty}$ -types appearing in $I_{r,\infty}(s)$ were determined by Kudla and Rallis [KR90, (1.3)] (see also [KR94, Proposition 2.8]) to be precisely those irreducible representations π_λ of $K_{r,\infty}$ with highest weight $\lambda = (l_1, \dots, l_r)$ such that

$$(3.3.1) \quad l_i \in 2\mathbb{Z}, \quad 1 \leq i \leq r.$$

Moreover, these $K_{r,\infty}$ -types appear with multiplicity one in $I_{r,\infty}(s)$. If $\Phi_\infty^\lambda(\cdot, s) \in I_{r,\infty}(s)$ is a non-zero highest weight vector of weight λ , then $\Phi_\infty^\lambda(e, s) \neq 0$ ([KR90, Proposition 1.1]), hence from now on we normalize all such highest weight vectors so that $\Phi_\infty^\lambda(e, s) = 1$. Note that the restriction of Φ_∞^λ to $K_{r,\infty}$ is independent of s . For scalar weights $\lambda = l(1, \dots, 1)$ with $l \in 2\mathbb{Z}$, we have

$$\Phi_\infty^l(k, s) := \Phi_\infty^{l(1, \dots, 1)}(k, s) = \det(k)^l, \quad k \in K_{r,\infty}.$$

3.4. Notations on Eisenstein series. For a standard section $\Phi(-, s) \in I_r(s)$ (i.e., its restriction to the standard maximal compact subgroup of $\text{Sp}(\mathbf{W}_r)(\mathbb{A})$ is independent of s), define the associated Siegel Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \text{Sp}(\mathbf{W}_r)(\mathbb{Q})} \Phi(\gamma g, s),$$

which converges for $\text{Re}(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$.

We have a Fourier expansion

$$(3.4.1) \quad E(g, s, \Phi) = \sum_{T \in \text{Sym}_r(\mathbb{Q})} E_T(g, s, \Phi),$$

where

$$E_T(g, s, \Phi) = \int_{\text{Sym}_r(\mathbb{Q}) \backslash \text{Sym}_r(\mathbb{A})} E(n(b)g, s, \Phi) \psi(-\text{tr}(Tb)) \, dn(b),$$

for $\text{Re}(s) \gg 0$, where the Haar measure $dn(b)$ is normalized to be self-dual with respect to $(b, c) \mapsto \psi(\text{tr}(bc))$.

When T is nonsingular, for factorizable $\Phi = \otimes_v \Phi_v$, we have a factorization of the Fourier coefficient into a product

$$(3.4.2) \quad E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v)$$

where the local (generalized) Whittaker function is defined by

$$(3.4.3) \quad W_{T,v}(g_v, s, \Phi_v) = \int_{N_r(\mathbb{Q}_v)} \Phi_v(w_r^{-1}n(b)g, s) \psi_v(-\text{tr}(Tb)) \, \text{dn}(b)$$

for $\text{Re}(s) \gg 0$, where the Haar measure is self-dual for $(b, c) \mapsto \psi_v(\text{tr}(bc))$. For an arbitrary matrix $T \in \text{Sym}_r(\mathbb{Q})$, we also have the global Whittaker function:

$$(3.4.4) \quad W_T(g, s, \Phi) = \int_{N_r(\mathbb{A})} \Phi(w_r^{-1}n(b)g, s) \psi(-\text{tr}(Tb)) \, \text{dn}(b)$$

for $\text{Re}(s) \gg 0$, again for the self-dual Haar measure. When the section Φ is factorizable, the above global Whittaker function is the product of all the local Whittaker functions. It is exactly the T -th Fourier coefficient of $E(g, s, \Phi)$ when T is nonsingular.

In this article, we focus on standard sections $\Phi \in I_r(s)$ of the following form:

$$(3.4.5) \quad \Phi_r^\lambda(-, s) = \Phi_{r,\infty}^\lambda(-, s) \otimes \bigotimes_{v < \infty} \Phi_{r,v}(-, s),$$

where the weight vector $\lambda = (l_1, \dots, l_r)$ satisfies (3.3.1) and the function $\Phi_{r,v}(-, s) \in I_v(s)$ is the unique spherical function such that $\Phi_{r,v}(e, s) = 1$ for all $s \in \mathbb{C}$. If $\lambda = l(1, \dots, 1)$ for some $l \in 2\mathbb{Z}$, we use $\Phi_r^l(-, s)$ to denote this section.

For $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{H}_r$, and for $m = 2l$, we define the classical weight l Eisenstein series to be

$$(3.4.6) \quad E(\mathbf{z}, s, \Phi_r^l) = (\det \mathbf{y})^{-m/4} E(g_{\mathbf{z}}, s, \Phi_r^l)$$

for any $g_{\mathbf{z}}$ as in Section 3.1. This is independent of the choice of $g_{\mathbf{z}}$ because different choices differ by an element in $\text{SO}_r \subset U(r)$, whose determinant is 1. A computation yields the more classical form

$$(3.4.7) \quad E(\mathbf{z}, s, \Phi_r^l) = \sum_{\gamma \in P(\mathbb{Z}) \backslash \text{Sp}_{2r}(\mathbb{Z})} \frac{\det(\mathbf{y})^{(s-s_0)/2}}{|\det(c\mathbf{z} + d)|^{m/2} |\det(c\mathbf{z} + d)|^{s-s_0}} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $\text{Re}(s) \gg 0$, where $s_0 := (m - (r + 1))/2$, and where $P(\mathbb{Z})$ and $\text{Sp}_{2r}(\mathbb{Z})$ are defined as in Section 3.1 but with $\mathbb{Z}$ instead of $\mathbb{Q}$. For

$$(3.4.8) \quad C_T(\mathbf{y}, s, \Phi) := (\det \mathbf{y})^{-m/4} E_T(g_{i\mathbf{y}}, s, \Phi) \cdot e^{2\pi i \text{tr}(T\mathbf{y})},$$

we have the following Fourier expansion of the classical Eisenstein series (3.4.6):

$$(3.4.9) \quad E(\mathbf{z}, s, \Phi) = \sum_{T \in \text{Sym}_r(\mathbb{Q})} C_T(\mathbf{y}, s, \Phi) \cdot q^T, \quad q^T := e^{2\pi i \text{tr}(T\mathbf{z})}.$$

3.5. Siegel–Weil section. Let $\mathbb{V}$ be a rank m quadratic space over $\mathbb{A}$, giving a collection of m -dimensional quadratic spaces $\mathbb{V}_v$ over $\mathbb{Q}_v$ for all places v of $\mathbb{Q}$, with bilinear pairing $(-, -)$ and quadratic form $x \mapsto \frac{1}{2}(x, x)$. We moreover assume that $\mathbb{V} \otimes_{\mathbb{A}} \mathbb{A}^{\mathbf{S}} \cong V \otimes_{\mathbb{Q}} \mathbb{A}^{\mathbf{S}}$ for some finite set of places $\mathbf{S}$ and some quadratic space V over $\mathbb{Q}$, where $\mathbb{A}^{\mathbf{S}}$ denotes the away-from- $\mathbf{S}$ adèle ring. In this paper, we are primarily interested in the case where

$$(3.5.1) \quad V = \text{M}_2(\mathbb{Q}) \quad \mathbb{V}_v = V_v = \text{M}_2(\mathbb{Q}_v) \text{ if } v < \infty \quad \mathbb{V}_{\infty} \text{ is positive definite}$$

where V is equipped with the quadratic form $x \mapsto \det(x)$. In this case, we can immediately verify that $\prod_v \epsilon(\mathbb{V}_v) = -1$, where the Archimedean Hasse invariant is also defined by the formula (2.1.2).

We allow general $\mathbb{V}$ for the moment, but simplify slightly by requiring m to be even. For each place v of $\mathbb{Q}$, consider the quadratic character $\chi_{\mathbb{V}_v}: \mathbb{Q}_v^\times \rightarrow \{\pm 1\}$, defined by the Hilbert symbol $\chi_{\mathbb{V}_v}(a_v) := (a_v, \text{disc}(\mathbb{V}_v))_{\mathbb{Q}_v}$. and define $\chi_{\mathbb{V}}: \mathbb{A}^\times \rightarrow \{\pm 1\}$ by $\chi_{\mathbb{V}}(a) = \prod_v \chi_{\mathbb{V}_v}(a_v)$. Assume that $\chi_{\mathbb{V}}|_{\mathbb{Q}^\times}$ is trivial. In the situation of (3.5.1), our $\chi_{\mathbb{V}}$ is the trivial character.

Let $r \geq 0$ be a nonnegative integer, and let $\mathcal{S}(\mathbb{V}^r)$ be the space of Schwartz functions on $\mathbb{V}^r$. The fixed choice of ψ gives a Weil representation ω_r of $\text{Sp}(\mathbf{W}_r)(\mathbb{A}) \times \text{O}(\mathbb{V})$ on $\mathcal{S}(\mathbb{V}^r)$, which may be described explicitly by the Schrödinger model

$$(3.5.2) \quad (\omega(m(a))\varphi)(\mathbf{x}) = \chi_{\mathbb{V}}(a)|a|_{\mathbb{A}}^{m/2}\varphi(\mathbf{x} \cdot a) \quad m(a) \in M(\mathbb{A})$$

$$(3.5.3) \quad (\omega(n(b))\varphi)(\mathbf{x}) = \psi(\tfrac{1}{2} \text{tr}(b(\mathbf{x}, \mathbf{x})))\varphi(\mathbf{x}) \quad n(b) \in N(\mathbb{A})$$

$$(3.5.4) \quad (\omega(w_r)\varphi)(\mathbf{x}) = \gamma_\psi(\mathbb{V})^r \hat{\varphi}(\mathbf{x})$$

$$(3.5.5) \quad (\omega(h)\varphi)(\mathbf{x}) = \varphi(h^{-1} \cdot \mathbf{x}) \quad h \in \text{O}(V(\mathbb{A}))$$

as in [Kud94, §5] and [Kud97, §I.1]; here $\gamma_\psi(\mathbb{V}) = \prod_v \gamma_{\psi_v}(\mathbb{V}_v)$ is a product of local Weil indices, all but finitely many of which are equal to 1, where $\gamma_{\psi_v}(\mathbb{V}_v)$ means the Weil index associated to the “non-degenerate character of the second degree” $x \mapsto \psi_v(\frac{1}{2}(x, x))$ on $\mathbb{V}_v$. In the situation of (3.5.1), we have $\gamma_{\psi_v}(\mathbb{V}_v) = 1$ if $v < \infty$ by e.g. [Che24, Lemma 14.1.1], and we have $\gamma_{\psi_\infty}(\mathbb{V}_\infty) = -1$ by [RR93, Proposition A.10(2)]. Above, we abuse notation $\chi_{\mathbb{V}}(a) = \chi_{\mathbb{V}}(\det a)$ and $|a|_{\mathbb{A}} = |\det a|_{\mathbb{A}}$, as usual.

Set $s_0 := (m - (r + 1))/2$. For $\varphi \in \mathcal{S}(\mathbb{V}^r)$, define a function

$$\Phi_\varphi(g, s) := (\omega_r(g)\varphi)(0) \cdot |\det(a(g))|_{\mathbb{A}}^{s-s_0}, \quad g \in \text{Sp}(\mathbf{W}_r)(\mathbb{A})$$

with $g = m(a(g))n(b)k$ for k in the standard maximal compact subgroup. Then $\Phi_\varphi(g, s) \in I_r(s, \chi_{\mathbb{V}})$ where $I_r(s, \chi_{\mathbb{V}})$ is defined as in (3.2.1) but with $\chi_{\mathbb{V}}| - |_{\mathbb{A}}^{s+(r+1)/2}$ instead of $| - |_{\mathbb{A}}^{s+(r+1)/2}$ from loc. cit.. This Φ_φ is called the standard Siegel–Weil section associated to φ . For $\varphi \in \mathcal{S}(\mathbb{V}^r)$, we write $E(g, s, \varphi) := E(g, s, \Phi_\varphi)$.

Given $\varphi_v \in \mathcal{S}(\mathbb{V}_v^r)$, we similarly have Φ_{φ_v} , defined using the local Weil representation, i.e. replacing $\mathbb{A}$ with $\mathbb{Q}_v$ above. If $v = \infty$ and $\mathbb{V}_\infty$ is positive definite with $m \equiv 0 \pmod{4}$, the formula [GS19, (2.66)] implies that the Gaussian $\varphi_\infty \in \mathcal{S}(\mathbb{V}_\infty^r)$ given by

$$(3.5.6) \quad \varphi_\infty(\mathbf{x}) = e^{-\pi \text{tr}(\mathbf{x}, \mathbf{x})}$$

is such that $\Phi_{\varphi_\infty} = \Phi_\infty^{m/2}$ with the latter as defined in Section 3.3. If $v < \infty$, if $\mathbb{V}_v$ is isomorphic to the standard split quadratic space of rank m , and if $L \subseteq \mathbb{V}_v$ is a lattice which is isomorphic to the standard split quadratic lattice of rank m , i.e. admitting a basis with the quadratic form

$$(3.5.7) \quad \frac{1}{2} \begin{pmatrix} 0 & 1_{m/2} \\ 1_{m/2} & 0 \end{pmatrix},$$

then the function $\varphi_v = \mathbf{1}_{L^r}$ is such that Φ_{φ_v} is the unique spherical standard section with $\Phi_{\varphi_v}(1, s) = 1$. This applies, for example, to the situation of (3.5.1).

Remark 3.5.1. For a non-Archimedean place v , consider a standard section $\Phi_{\varphi_v}(-, s) \in I_{r,v}(s)$ associated to a Schwartz function $\varphi_v = \mathbf{1}_{M^r}$ where $M \subset \mathbb{V}_v$ is a $\mathbb{Z}_v$ -lattice of rank m . Let L be a quadratic $\mathbb{Z}_v$ -lattice of rank r with normalized Gram matrix T , for some basis. We have the

following relation between local Whittaker function and the local density for all integers $k \geq 0$ by a generalization of the calculation in [Kud97, Proposition A.4, Proposition A.6],¹

$$(3.5.8) \quad W_{T,v}(e, k + s_0, \Phi_{\varphi_v}) = \gamma_{\psi_v}(\mathbb{V}_v)^{-r} |2|_v^{r(r-1)/4 + mr/2} |\text{disc}(M)|_v^{r/2} \cdot \text{Den}(M \oplus H^k, L).$$

The same formula holds if $\mathbb{Q}_p$ is replaced with an arbitrary p -adic field. The same formula also holds for possibly odd m , if one replaces the symplectic group with its metaplectic double cover.

3.6. Pullback map and intertwining operator. In this section we fix some notation which will be used repeatedly in the next section. Given nonnegative integers $r_1 \leq r_2$, there is a standard inclusion $W_{r_1} \rightarrow W_{r_2}$ inducing an embedding $\eta_{r_1}^{r_2} : \text{Sp}(W_{r_1})(\mathbb{A}) \rightarrow \text{Sp}(W_{r_2})(\mathbb{A})$ as in [GS19, (5.39)]. Define the following pullback map for every place v [GS19, (5.45)]

$$(3.6.1) \quad (\eta_{r_1}^{r_2})^* : I_{r_2,v}(s) \rightarrow I_{r_1,v}\left(s + \frac{r_2 - r_1}{2}\right).$$

Let $\mathcal{M}_r(s) = \otimes_v \mathcal{M}_{r,v}(s)$ be the standard intertwining operator, where

$$(3.6.2) \quad \mathcal{M}_{r,v}(s) : I_{r,v}(s) \longrightarrow I_{r,v}(-s)$$

$$\Phi \longmapsto (\mathcal{M}_{r,v}(s)\Phi)(g) := \int_{N_r(\mathbb{Q}_v)} \Phi(w_r^{-1}n(b)g, s) \, dn(b)$$

is defined for $s \in \mathbb{C}$ with $\text{Re}(s) \gg 0$. Both $\mathcal{M}_{r,v}(s)$ and $\mathcal{M}_r(s)$ admit meromorphic continuation to $s \in \mathbb{C}$.

4. NORMALIZATION OF WHITTAKER FUNCTIONS AND EISENSTEIN SERIES

4.1. Normalized Archimedean local Whittaker functions. Fix a nonnegative $m \in 4\mathbb{Z}$, set $s_0 = (m - (r+1))/2$, and consider the standard section $\Phi_{\infty}^{m/2}(g, s)$ as in Section 3.3. Fix $T \in \text{Sym}_r(\mathbb{R})$ nonsingular. Given $y \in \text{Sym}_r(\mathbb{R})_{>0}$ and any $a \in \text{GL}_r(\mathbb{R})$ with $a \cdot {}^t a = y$ and $\det a > 0$, a change of variables gives

$$(4.1.1) \quad W_{T,\infty}(m(a), s, \Phi_{\infty}^{m/2}) = (\det y)^{-s/2 + (r+1)/4} W_{t_a T a, \infty}(1, s, \Phi_{\infty}^{m/2}).$$

We set

$$(4.1.2) \quad W_{T,\infty}(y, s, \Phi_{\infty}^{m/2}) := (\det y)^{-m/4} W_{T,\infty}(m(a), s, \Phi_{\infty}^{m/2}),$$

and this does not depend on the choice of a .

For any $b \in \text{Sym}_r(\mathbb{R})$, a computation gives

$$(4.1.3) \quad \Phi_{\infty}^{m/2}(w_r^{-1}n(b), s) = \det(i1 + b)^{-(s-s_0+m)/2} \cdot \det(-i1 + b)^{-(s-s_0)/2}$$

¹We find that a few constants should be modified in loc. cit. in the case of residue characteristic $p = 2$. In the proof of [Kud97, Proposition A.4], the self-dual Haar volume for $\text{Sym}_n(\mathcal{O})$ should contribute an additional factor of $|2|_v^{n(n-1)/4}$. In the proof of [Kud97, Proposition A.6], we find that the measure dx should be self-dual for the pairing defined by $2S_r$, for compatibility with [Kud94, §5]. This contributes an additional factor of $|2|_v^{mn/2}$. Our normalization of the local density differs from loc. cit., i.e. we do not have the $|2|_v^{n(n+1+2r)}$ factor. We also mention that $\gamma_F(\det V, \psi)$ and $\gamma_F(\psi)$ in [Kud97] are what would have been denoted $\gamma_F(\det V, \frac{1}{2}\psi)$ and $\gamma_F(\frac{1}{2}\psi)$ in [Kud94], and that Kudla's $\gamma(V)$ in [Kud97, Proposition A.4] is our $\gamma_{\psi_v}(\mathbb{V}_v)^{-r}$.

with log branch convention as in [Shi82, (1.11)] (see also [Che24, Remark 13.2.1]), so we have

$$(4.1.4) \quad W_{T,\infty}(1, s, \Phi_\infty^{m/2}) = \int_{\text{Sym}_r(\mathbb{R})} \det(i1 + b)^{-(s-s_0+m)/2} \cdot \det(-i1 + b)^{-(s-s_0)/2} \psi_\infty(-\text{tr}(Tb)) \, db$$

for the Haar measure db which is self-dual for the pairing $(b, c) = \psi_\infty(\text{tr}(bc))$, i.e. $2^{r(r-1)/4}$ times the Euclidean measure.

We have

$$(4.1.5) \quad W_{T,\infty}(1, s, \Phi_\infty^{m/2}) = 2^{r(r-1)/4} \xi(1, T; \alpha, \beta), \quad \alpha = (s - s_0 + m)/2 \quad \beta = (s - s_0)/2$$

with $\xi(-, -, -, -)$ as in [Shi82, (1.25)] and $s_0 = (m - (r + 1))/2$. Recall the Archimedean local L -factors

$$(4.1.6) \quad L_{\mathbb{R}}(s, \mathbf{1}) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \quad L_{\mathbb{R}}(s, \text{sgn}) = \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right).$$

Let (r_1, r_2) be the signature of T . We set

$$(4.1.7) \quad W_{T,\infty}^\star(\mathbf{y}, s) := \frac{(2\pi)^{r(r-1)/4}}{(-2\pi i)^{rm/2}} \pi^{-r(s-s_0)/2} \left(\prod_{j=0}^{r-1} \Gamma((s - s_0 + m - j)/2) \right) |\det T|^{-(s+s_0)/2} e^{-2\pi i \text{tr}(iT\mathbf{y})} W_{T,\infty}(\mathbf{y}, s, \Phi_\infty^{m/2})$$

if r is odd, and we set

$$(4.1.8) \quad W_{T,\infty}^\star(\mathbf{y}, s) := \frac{(2\pi)^{r(r-1)/4}}{(-2\pi i)^{rm/2}} \pi^{-r(s-s_0)/2} \left(\prod_{j=0}^{r-1} \Gamma((s - s_0 + m - j)/2) \right) |\det T|^{-(s+s_0)/2} e^{-2\pi i \text{tr}(iT\mathbf{y})} W_{T,\infty}(\mathbf{y}, s, \Phi_\infty^{m/2}) \\ \cdot \frac{L_{\mathbb{R}}(s + 1/2, \text{sgn}^{r(r-1)/2})}{L_{\mathbb{R}}(s + 1/2, \text{sgn}^{r(r-1)/2+r_2})}$$

if r is even, where the normalizing factor should be compared with [GS19, (3.3.14)], and also Shimura [Shi82], at least without the additional Archimedean L -factors. For any $a \in \text{GL}_r(\mathbb{R})$ with $a \cdot {}^t a = \mathbf{y}$ and $\det a > 0$, we have

$$(4.1.9) \quad W_{T,\infty}^\star(\mathbf{y}, s) = W_{t_a T a, \infty}^\star(s) := W_{t_a T a, \infty}^\star(1_r, s).$$

We have

$$(4.1.10) \quad W_{T,\infty}^\star(\mathbf{y}, s_0) = \begin{cases} 1 & \text{if } T \text{ is positive definite} \\ 0 & \text{if } r+1 \leq m \text{ and } T \text{ is not positive definite} \end{cases}$$

and have the functional equation

$$(4.1.11) \quad W_{T,\infty}^\star(\mathbf{y}, s) = \begin{cases} (-1)^{r_1 r_2 / 2} W_{T,\infty}^\star(\mathbf{y}, -s) & \text{if } r \text{ is odd} \\ W_{T,\infty}^\star(\mathbf{y}, -s) & \text{if } r \text{ is even.} \end{cases}$$

as can be shown using [Shi82, (3.15), (3.7)] in the positive definite case and [Shi82, (4.12.K), (4.34.K)] in the other cases.

Lemma 4.1.1. *Let $T \in \text{Sym}_r(\mathbb{R})$ be a nondegenerate matrix.*

(a). If T is positive definite, for all $\mathbf{y} \in \text{Sym}_r(\mathbb{R})_{>0}$

$$(4.1.12) \quad \left. \frac{d}{ds} \right|_{s=s_0} W_{T,\infty}^*(t\mathbf{y}, s) = O(t^{-1}) \quad \text{as } t \rightarrow +\infty.$$

For all fixed s' , there is a positive constant C such that

$$(4.1.13) \quad \frac{d}{dt} \left[\left. \frac{d}{ds} \right|_{s=s'} W_{T,\infty}^*(t\mathbf{y}, s) \right] = O(t^{-1-C}) \quad \text{as } t \rightarrow +\infty.$$

(b). If T is not positive definite, for all fixed s' and $\mathbf{y} \in \text{Sym}_r(\mathbb{R})_{>0}$, there are positive constants C' and C'' such that

$$\left. \frac{d}{ds} \right|_{s=s'} W_{T,\infty}^*(t\mathbf{y}, s) = O(e^{-C't}) \quad \text{and} \quad \frac{d}{dt} \left[\left. \frac{d}{ds} \right|_{s=s'} W_{T,\infty}^*(t\mathbf{y}, s) \right] = O(e^{-C''t}) \quad \text{as } t \rightarrow +\infty.$$

Proof. We first prove (4.1.12). Set $\mathcal{W}_{T,\infty}(\mathbf{y}, s) = W_{T,\infty}(\mathbf{y}, s, \Phi^{m/2})e^{2\pi\text{tr}(T\mathbf{y})}$. By the definition of $W_{T,\infty}^*(\mathbf{y}, s)$, we have

$$(4.1.14) \quad \frac{\left(W_{T,\infty}^* \right)'(\mathbf{y}, s)}{W_{T,\infty}^*(\mathbf{y}, s)} = -\frac{1}{2} \log \det(\pi T) + \frac{\Gamma'_r((s-s_0+m)/2)}{2\Gamma_r((s-s_0+m)/2)} + \frac{\mathcal{W}'_{T,\infty}(\mathbf{y}, s)}{\mathcal{W}_{T,\infty}(\mathbf{y}, s)}$$

where $\Gamma_r(s) := \pi^{r(r-1)/4} \prod_{j=0}^{r-1} \Gamma(s-j/2)$. It is proved in [GS19, Proposition 3.2 (iii), (iv)] that for $t > 0$,

$$(4.1.15) \quad \frac{\mathcal{W}'_{T,\infty}(t\mathbf{y}, s)}{\mathcal{W}_{T,\infty}(t\mathbf{y}, s)} = \frac{1}{2} \log \det(\pi T) - \frac{\Gamma'_r((s-s_0+m)/2)}{2\Gamma_r((s-s_0+m)/2)} + O(t^{-1}) \quad \text{as } t \rightarrow +\infty.$$

Then (4.1.12) follows by combining (4.1.14) and (4.1.15).

Equation (4.1.13) and part (b) follow, respectively, from Proposition 3.2 (ii) and Proposition 3.3 of loc. cit.. $\square$

4.2. Normalized non-Archimedean local Whittaker functions. Suppose $\Phi_{r,p}(-, s) \in I_{r,p}(s)$ is the normalized spherical section, i.e. the unique K_p -invariant function with $\Phi_{r,p}(1, s) = 1$. Fix a nonnegative $m \in 2\mathbb{Z}$ and let $s_0 = (m - (r+1))/2$ as usual. Given any nonsingular $T \in \text{Sym}_r(\mathbb{Q}_p)$, we define the *normalized local Whittaker function* $W_{T,p}^*(s)_m \in \mathbb{C}[p^s, p^{-s/2}]$ to be the function given by

$$(4.2.1) \quad W_{T,p}^*(s-s_0)_m = \text{Den}^*(p^{-s}, L)$$

for L any quadratic $\mathbb{Z}_p$ -lattice admitting a basis with normalized Gram matrix T , and $\text{Den}^*(X, L)$ as defined in Section 2.3. More explicitly, we have

$$(4.2.2) \quad W_{T,p}^*(s)_m = |2|_p^{-r(r-1)/4} |4|_p^{\lfloor r/2 \rfloor} |\text{disc}(T)|_p^{-(s+s_0)/2} \zeta_p(s + (r+1)/2) \cdot \left(\prod_{i=1}^{\lfloor r/2 \rfloor} \zeta_p(2s + r + 1 - 2i) \right) W_{T,p}(1, s, \Phi_{r,p})$$

if r is odd and

$$(4.2.3) \quad W_{T,p}^\star(s)_m = |2|_p^{-r(r-1)/4} |4|_p^{\lfloor r/2 \rfloor} \text{disc}(T) \Delta_T^{-1} |p|_p^{-(s+s_0)/2} L_p(s+1/2, \eta_T)^{-1} \zeta_p(s+(r+1)/2) \\ \cdot \left(\prod_{i=1}^{\lfloor r/2 \rfloor} \zeta_p(2s+r+1-2i) \right) W_{T,p}(1, s, \Phi_{r,p})$$

if r is even, where $\Delta_T \subseteq \mathbb{Z}_p$ is the discriminant ideal for the extension $\mathbb{Q}_p[X]/(X^2 - \text{disc}(T))$ of $\mathbb{Q}_p$ where $\eta_T: \mathbb{Q}_p^\times \rightarrow \{\pm 1\}$ is the quadratic character associated to this extension of $\mathbb{Q}_p$, and where $\zeta_p(s) := (1 - p^{-s})^{-1}$, and $L_p(s, \eta_T) = (1 - \chi_p(T)p^{-s})^{-1}$ are the usual local L -factors.

The formulas in (2.3.3) translate to the functional equations

$$(4.2.4) \quad W_{T,p}^\star(-s)_m = \begin{cases} \pm W_{T,p}^\star(s)_m & \text{if } r \text{ is odd} \\ W_{T,p}^\star(s)_m & \text{if } r \text{ is even} \end{cases}$$

where the sign “ $\pm$ ” is “ $+$ ” if T defines a split quadratic space, and is “ $-$ ” otherwise. We typically lighten notation and write $W_{T,p}^\star(s)$ instead of $W_{T,p}^\star(s)_m$, when m is understood.

If $T \notin \text{Sym}_r(\mathbb{Z}_p)^*$, we have $W_{T,p}^\star(s) = 0$ identically, as $\text{Den}^\star(X, L) = 0$ in the above notation as because $\text{Den}(H^k, L) = 0$ for all $k \geq 0$ via the explanation in Section 2.2.

4.3. Normalized global Whittaker functions. Consider a nonsingular matrix $T \in \text{Sym}_r(\mathbb{Q})$, for some $r \geq 0$. Fix an integer $m \equiv 0 \pmod{4}$. A computation using (4.1.11) and (4.2.4) gives the functional equation

$$(4.3.1) \quad W_{T,\infty}^\star(y, s) \prod_p W_{T,p}^\star(s) = \begin{cases} (-1)^{(r^2-1)/8} W_{T,\infty}^\star(y, -s) \prod_p W_{T,p}^\star(-s) & \text{if } r \text{ is odd} \\ W_{T,\infty}^\star(y, -s) \prod_p W_{T,p}^\star(-s) & \text{if } r \text{ is even.} \end{cases}$$

Consider any degree 2 étale algebra $E/\mathbb{Q}$ with associated quadratic character $\eta: \mathbb{A}^\times \rightarrow \{\pm 1\}$. Let $\Delta \subseteq \mathbb{Z}$ be the discriminant ideal. For a place v of $\mathbb{Q}$, we write $L_v(s, \eta)$ for the associated local L -factor, and we let $\Lambda(s, \eta) = \prod_v L_v(s, \eta)$ be the completed L -function, where the product runs over all places v . Recall that there is a functional equation

$$(4.3.2) \quad \Lambda(s, \eta) = \epsilon(s, \eta) \Lambda(1-s, \eta)$$

where $\epsilon(s, \eta) = |\Delta|^{-s+1/2}$. If we set

$$(4.3.3) \quad \Lambda^*(s, \eta) = |\Delta|^{s/2} \Lambda(s, \eta)$$

then we have

$$(4.3.4) \quad \Lambda^*(s, \eta) = \Lambda^*(1-s, \eta).$$

If η is the trivial character, we write $\Lambda(s) = \Lambda(s, \eta)$ for short.

In the special case of $m = 4$, we compute

$$(4.3.5) \quad \frac{W_{T,\infty}^\star(y, s) \prod_p W_{T,p}^\star(s)}{C_T(y, s, \Phi_r^{m/2})} = \begin{cases} -\frac{1}{32\pi^2} (s+1)(s+2) \Lambda(s+2) \Lambda(2s+2) & \text{if } r = 3 \\ \frac{1}{16\pi^2} (s+1/2)(s+3/2) \frac{\Lambda(2s+1) \Lambda(s+3/2)}{\Lambda^*(s+1/2, \eta_T)} & \text{if } r = 2 \\ -\frac{1}{8\pi} (s+1) \Lambda(s+1) & \text{if } r = 1 \end{cases}$$

where η_T denotes the quadratic character associated to $\mathbb{Q}[X]/(X^2 - \text{disc}T)$.

4.4. Normalized Eisenstein series. The following lemma follows from [KR88, Lemma 4.1, Lemma 4.6] (see also [Ste21, Theorem 3.10] and [KR94, (4.20)]).

Lemma 4.4.1. *For all positive integers r , we have*

$$(4.4.1) \quad (\mathcal{M}_{r,\infty} \Phi_{r,\infty}^2)(g_\infty, s) = (-1)^r \cdot \frac{2^{(1-s)r+r(r-1)/4} \pi^{r(r+1)/2} \Gamma_r(s)}{\Gamma_r(s/2 + (r+5)/4) \Gamma_r(s/2 + (r-3)/4)} \cdot \Phi_{r,\infty}^2(g_\infty, -s), \quad g_\infty \in \text{Sp}(W_r)(\mathbb{R}).$$

For a non-Archimedean place p of $\mathbb{Q}$, we have

$$(4.4.2) \quad (\mathcal{M}_{r,p} \Phi_{r,p})(g_p, s) = |2|_p^{r(r-1)/4} \Phi_{r,p}(g_p, -s) \cdot \begin{cases} \frac{\zeta_p(s-1)\zeta_p(2s-1)}{\zeta_p(s+2)\zeta_p(2s+2)}, & \text{if } r = 3; \\ \frac{\zeta_p(s-1/2)\zeta_p(2s)}{\zeta_p(s+3/2)\zeta_p(2s+1)}, & \text{if } r = 2; \\ \frac{\zeta_p(s)}{\zeta_p(s+1)}, & \text{if } r = 1, \end{cases}$$

where $\zeta_p(s) = (1 - p^{-s})^{-1}$. Therefore we have

$$(4.4.3) \quad (\mathcal{M}_r \Phi_r^2)(g, s) = \Phi_r^2(g, -s) \cdot \begin{cases} -\frac{(1-s)(2-s)}{(1+s)(2+s)} \cdot \frac{\Lambda(s-1)\Lambda(2s-1)}{\Lambda(s+2)\Lambda(2s+2)}, & \text{if } r = 3; \\ \frac{(1/2-s)(3/2-s)}{(1/2+s)(3/2+s)} \cdot \frac{\Lambda(s-1/2)\Lambda(2s)}{\Lambda(s+3/2)\Lambda(2s+1)}, & \text{if } r = 2; \\ -\frac{s-1}{s+1} \cdot \frac{\Lambda(s)}{\Lambda(s+1)}, & \text{if } r = 1. \end{cases}$$

Proof. We remark that in (4.4.1), there is an extra factor of $2^{r(r-1)/4}$ compared to [KR88, Lemma 4.6]. This factor arises because of the difference between the measure $dn(b)$ that we are using and the standard Euclidean measure dx in loc. cit.. The extra factor $|2|_p^{r(r-1)/4}$ in (4.4.2) also arises in this way (see also Remark 3.5.1).

The formula (4.4.3) follows from combining (4.4.1), (4.4.2), [GS19, (5.94)] and the duplication formula

$$\Gamma(s)\Gamma(s+1/2) = 2^{1-2s} \pi^{1/2} \Gamma(2s), \quad s \in \mathbb{C}.$$

□

Define the normalized Eisenstein series to be

$$(4.4.4) \quad E^r(g, s) := E(g, s, \Phi_r^2) \cdot \begin{cases} (1+s)(2+s)\Lambda(2+s)\Lambda(2+2s), & \text{if } r = 3; \\ (1/2+s)(3/2+s)\Lambda(s+3/2)\Lambda(2s+1), & \text{if } r = 2; \\ (1+s)\Lambda(s+1), & \text{if } r = 1. \end{cases}$$

By (4.4.3) and the functional equation $E(g, s, \Phi) = E(g, -s, \mathcal{M}_r(s)\Phi)$ for any section $\Phi \in I_r(s)$, we have

$$(4.4.5) \quad E^r(g, -s) = \pm E^r(g, s)$$

where the sign “ $\pm$ ” is “ $+$ ” if $r = 1, 2$ and is “ $-$ ” if $r = 3$. For $z = x + iy \in \mathbb{H}_r$, define $E^r(z, s) = (\det y)^{-1} E^r(g_z, s)$. We have the following Fourier expansion by (3.4.9):

$$(4.4.6) \quad E^r(z, s) = \sum_{T \in \text{Sym}_r(\mathbb{Q})} C_T^r(y, s) \cdot q^T$$

where $C_T^r(y, s)$ is given by normalizing $C_T(y, s, \Phi_r^2)$ by the same factor from (4.4.4).

Example 4.4.2. For $r = 1$ and $s = 1$, the Eisenstein series $E^1(z, 1)$ is the classical non-holomorphic Eisenstein series of weight 2 whose explicit Fourier expansion is

$$(4.4.7) \quad E^1(z, 1) = \frac{\pi}{3} - \frac{1}{\text{Im}(z)} - 8\pi \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

5. SINGULAR FOURIER COEFFICIENTS

In this section we focus on the Fourier coefficients in the decomposition (3.4.1) of the genus r Eisenstein series $E(g, s, \Phi)$ when the matrix T is singular, especially for $m = 4$ and $r = 3$.

5.1. Decomposition of the singular Fourier coefficients. Let $T \in \text{Sym}_r(\mathbb{Q})$ be a matrix

$$(5.1.1) \quad T = \begin{pmatrix} \mathbf{0}_{r-t} & \\ & T^\flat \end{pmatrix}$$

where $0 \leq t \leq r$ and $T^\flat \in \text{Sym}_t(\mathbb{Q})$ is nonsingular if $t \neq 0$, and $t = \text{rk}(T)$. By [GS19, Lemma 5.4], there is a decomposition

$$(5.1.2) \quad E_T(g, s, \Phi) = \sum_{j=t}^r \sum_{a \in P_{r-t, j-t}(\mathbb{Q}) \backslash \text{GL}_{r-t}(\mathbb{Q})} B_T^j(m \begin{pmatrix} a & \\ & 1_t \end{pmatrix} g, s, \Phi),$$

where $P_{r-t, j-t}(\mathbb{Q})$ is a parabolic subgroup of GL_{r-t} whose explicit expression can be found in [GS19, (5.40)]. The functions B_T^j are defined by the following integral

$$(5.1.3) \quad B_T^j(g, s, \Phi) = \int_{N_j(\mathbb{A})} \Phi \left(\eta_j^r \left(w_j^{-1} n(b) \right) g, s \right) \psi \left(-\text{tr} \left(T \begin{pmatrix} \mathbf{0}_{r-j} & \\ & b \end{pmatrix} \right) \right) dn(b).$$

It turns out that the most important terms in (5.1.2) are the terms corresponding to $j = t$ and $j = r$. For $j = r$, we have the following identity by combining [GS19, Lemma 5.4, Remark 5.6, (5.102), (5.103)],

$$(5.1.4) \quad \begin{aligned} B_T^r(g, s, \Phi) &= W_{T^\flat} \left(e, s - \frac{r-t}{2}, [U_{t,r}(s) \circ r(g)] \Phi \right) \\ &= W_{T^\flat} \left(e, \frac{r-t}{2} - s, [(\eta_t^r)^* \circ \mathcal{M}_r(s) \circ r(g)] \Phi \right) \\ &= W_{T^\flat} \left(e, \frac{r-t}{2} - s, [(\eta_t^r)^* \circ r(g) \circ \mathcal{M}_r(s)] \Phi \right), \end{aligned}$$

where the definition of the operator $U_{j,r}(s)$ can be found in [GS19, (5.46), Lemma 5.5] (We omit the details because the explicit definition will not be needed.). For $j = t$,

$$(5.1.5) \quad B_T^t(g, s, \Phi) = W_{T^\flat} \left(e, s + \frac{r-t}{2}, [(\eta_t^r)^* \circ r(g)] \Phi \right).$$

Moreover, if $t = 0$, i.e., $T = \mathbf{0}_r$,

$$(5.1.6) \quad B_{\mathbf{0}_r}^0(g, s, \Phi) = \Phi(g, s), \quad B_{\mathbf{0}_r}^r(g, s, \Phi) = (\mathcal{M}_r(s) \Phi)(g, -s).$$

Lemma 5.1.1. Let $y = \text{diag}\{y_1, y_2\} \in \text{Sym}_r(\mathbb{R})$ be a positive definite symmetric matrix where $y_1 \in \text{Sym}_{r-t}(\mathbb{R})$ and $y_2 \in \text{Sym}_t(\mathbb{R})$. Let $h \in \text{Sp}(W_t)(\mathbb{A})$ be an element. For all sections $\Phi \in I_r(s)$, we

have

$$([\eta_t^r]^* \circ r(g_{iy}))\Phi \left(h, s - \frac{r-t}{2} \right) = (\det y_1)^{s/2+(t+1)/4} \cdot ([r(g_{iy_2}) \circ (\eta_t^r)^*]\Phi) \left(h, s - \frac{r-t}{2} s \right).$$

Proof. By the definition of degenerate principal series, we have

$$\begin{aligned} ([\eta_t^r]^* \circ r(g_{iy}))\Phi(h, s) &= \Phi \left(\eta_t^r(h)g_{iy}, s - \frac{r-t}{2} \right) \\ &= \det y_1^{s/2+(t+1)/4} \Phi \left(\eta_t^r(hg_{iy_2}), s - \frac{r-t}{2} \right). \end{aligned}$$

The last expression is $\det y_1^{s/2+(t+1)/4} \cdot ([r(g_{iy_2}) \circ (\eta_t^r)^*]\Phi)(h, s)$. $\square$

5.2. The case of corank 1. Let $m = 4$, $r = 3$ and $t = 2$. We obtain the following lemma by combining the explicit formula of the intertwining operator in Lemma 4.4.1 and Lemma 5.1.1.

Lemma 5.2.1. *Let $T^\flat \in \text{Sym}_2(\mathbb{Q})$ be a matrix. Let $y = \text{diag}\{y_1, y_2\} \in \text{Sym}_3(\mathbb{R})$ be a positive definite symmetric matrix where $y_1 \in \text{Sym}_1(\mathbb{R})$ and $y_2 \in \text{Sym}_2(\mathbb{R})$. We have*

$$\begin{aligned} (5.2.1) \quad W_{T^\flat} \left(e, \frac{1}{2} - s, [(\eta_2^3)^* \circ r(g_{iy}) \circ \mathcal{M}_3(s)]\Phi_3^2 \right) \\ = -y_1^{-s/2+1} \cdot \frac{(1-s)(2-s)}{(1+s)(2+s)} \cdot \frac{\Lambda(s-1)\Lambda(2s-1)}{\Lambda(s+2)\Lambda(2s+2)} \cdot W_{T^\flat} \left(g_{iy_2}, \frac{1}{2} - s, \Phi_2^2 \right). \end{aligned}$$

Let $T \in \text{Sym}_3(\mathbb{Q})$ be of the form in (5.1.1) and consider a positive definite 3×3 matrix of the form $y = \text{diag}\{y_1, y_2\} \in \text{Sym}_3(\mathbb{R})$ where $y_1 \in \text{Sym}_1(\mathbb{R})$ and $y_2 \in \text{Sym}_2(\mathbb{R})$. Combining Lemma 5.1.1 and 5.2.1 gives

$$\begin{aligned} (5.2.2) \quad E_T^3(g_{iy}, s) &= (1+s)(2+s)\Lambda(2+s)\Lambda(2+2s) \cdot y_1^{s/2+1} \cdot W_{T^\flat} \left(g_{iy_2}, \frac{1}{2} + s, \Phi_2^2 \right) \\ &\quad - (1-s)(2-s)\Lambda(2-s)\Lambda(2-2s) \cdot y_1^{-s/2+1} \cdot W_{T^\flat} \left(g_{iy_2}, \frac{1}{2} - s, \Phi_2^2 \right) \\ &= y_1^{s/2+1} E_{T^\flat}^2 \left(g_{iy_2}, \frac{1}{2} + s \right) - y_1^{-s/2+1} E_{T^\flat}^2 \left(g_{iy_2}, \frac{1}{2} - s \right). \end{aligned}$$

Therefore

$$(5.2.3) \quad C_T^3(y, s) = y_1^{s/2} C_{T^\flat}^2 \left(y_2, \frac{1}{2} + s \right) - y_1^{-s/2} C_{T^\flat}^2 \left(y_2, \frac{1}{2} - s \right).$$

5.3. The case of corank 2. Let $m = 4$, $r = 3$ and $t = 1$.

Lemma 5.3.1. *Let $T = \text{diag}\{\mathbf{0}_2, T^\flat\}$ where $T^\flat \in \text{Sym}_1(\mathbb{Q}) \setminus \{(0)\}$, let $y = \text{diag}\{y_1, y^\flat\} \in \text{Sym}_3(\mathbb{R})$ be a positive definite symmetric matrix such that $y_1 \in \text{Sym}_2(\mathbb{R})$ and $y^\flat \in \text{Sym}_1(\mathbb{R})$. We have*

$$\begin{aligned} (5.3.1) \quad C_T^3(y, s) &= (\det y_1)^{s/2} (1+s)\Lambda(2+2s) C_{T^\flat}^1(y^\flat, 1+s) \\ &\quad - (\det y_1)^{-s/2} (1-s)\Lambda(2-2s) C_{T^\flat}^1(y^\flat, 1-s) \\ &\quad + f_T(y, s), \end{aligned}$$

where $f_T(y, -s) = -f_T(y, s)$ and there exists a positive number C such that

$$(5.3.2) \quad \lim_{t \rightarrow +\infty} \frac{\partial f_T}{\partial s}(ty, 0) = 0, \quad \lim_{t \rightarrow +\infty} \frac{\partial^2 f_T}{\partial t \partial s}(ty, 0) = O(t^{-1-C}).$$

Moreover, we have

$$(5.3.3) \quad \text{CT}_{\rho=0} \int_1^{+\infty} \frac{d}{dt} \left((C_T^3)'(ty, 0) \right) \frac{dt}{t^\rho} - \frac{\pi}{3} \cdot \text{CT}_{\rho=0} \int_1^{+\infty} \frac{d}{dt} \left((C_{T^b}^1)'(ty^b, 1) \right) \frac{dt}{t^\rho} = -\frac{\partial f_T}{\partial s}(y, 0).$$

Proof. By a similar strategy as used in Lemma 5.2.1, we obtain

$$(5.3.4) \quad \begin{aligned} E_T^3(g_{iy}, s) &= (\det y_1)^{s/2+1} (1+s) \Lambda(2+2s) E_{T^b}^1(g_{iy^b}, 1+s) \\ &\quad - (\det y_1)^{-s/2+1} (1-s) \Lambda(2-2s) E_{T^b}^1(g_{iy^b}, 1-s) \\ &\quad + \mathcal{F}_T(y, s), \end{aligned}$$

where $\mathcal{F}_T(y, s)$ corresponds to the (normalized) “middle” terms in (5.1.2), i.e. $j = 2$. Equation (5.3.1) follows upon setting $f_T(y, s) = \det y^{-1} \mathcal{F}_T(y, s) q^{-T}$, and the functional equation for $C_T^3(y, s)$ implies that $f_T(y, s)$ is an odd function of s . Formula (5.3.2) follows from [GS19, Proposition 5.8].

Finally we prove (5.3.3). By (5.3.1),

$$(5.3.5) \quad (C_T^3)'(y, 0) = \left(\frac{\pi}{6} \log \det y_1 + \frac{\pi}{3} + 4\Lambda'(2) \right) C_{T^b}^1(y^b, 1) + \frac{\pi}{3} (C_{T^b}^1)'(y^b, 1) + \frac{\partial f_T}{\partial s}(y, 0).$$

Since $T^b \neq (0)$, the coefficient $C_{T^b}^1(y^b, 1)$ is independent of the matrix y^b by (4.4.7), hence for $t > 0$,

$$\begin{aligned} & (C_T^3)'(ty, 0) - \frac{\pi}{3} (C_{T^b}^1)'(ty^b, 1) \\ &= \left(\frac{\pi}{3} \log t + \frac{\pi}{6} \log \det y_1 + \frac{\pi}{3} + 4\Lambda'(2) \right) C_{T^b}^1(y^b, 1) + \frac{\partial f_T}{\partial s}(ty, 0) \\ &= \left(\frac{\pi}{6} \log \det y_1 + \frac{\pi}{3} + 4\Lambda'(2) \right) C_{T^b}^1(y^b, 1) + \frac{\pi}{3} C_{T^b}^1(y^b, 1) \log t + \frac{\partial f_T}{\partial s}(ty, 0). \end{aligned}$$

Then (5.3.3) follows from the estimate (5.3.2) and [GS19, Lemma 5.11]. $\square$

5.4. The case of corank 3. Let $m = 4$, $r = 3$, and $t = 0$, so $T = \mathbf{0}_3$.

Lemma 5.4.1. *Let $y \in \text{Sym}_3(\mathbb{R})$ be a positive definite symmetric matrix, then*

$$(5.4.1) \quad \begin{aligned} C_{\mathbf{0}_3}^3(y, s) &= (1+s)(2+s)\Lambda(2+s)\Lambda(2+2s) \det y^{s/2} \\ &\quad - (1-s)(2-s)\Lambda(2-s)\Lambda(2-2s) \det y^{-s/2} \\ &\quad + f(y, s), \end{aligned}$$

where $f(y, -s) = -f(y, s)$ and there exists a positive number C such that

$$(5.4.2) \quad \lim_{t \rightarrow +\infty} \frac{\partial f}{\partial s}(ty, 0) = 0, \quad \lim_{t \rightarrow +\infty} \frac{\partial^2 f}{\partial t \partial s}(ty, 0) = O(t^{-1-C}).$$

Moreover, we have

$$(5.4.3) \quad \text{CT}_{\rho=0} \int_1^{+\infty} \frac{d}{dt} \left((C_{\mathbf{0}_3}^3)'(ty, 0) \right) \frac{dt}{t^\rho} = -\frac{\partial f}{\partial s}(y, 0).$$

Proof. A calculation gives that $\Phi_3^2(g_{iy}, s) = \det y^{s/2+1}$. Therefore, by (5.1.6) and Lemma 4.4.1,

$$(5.4.4) \quad \begin{aligned} E_{\mathbf{0}_3}^3(g_{iy}, s) &= (1+s)(2+s)\Lambda(2+s)\Lambda(2+2s) \det y^{s/2+1} \\ &\quad - (1-s)(2-s)\Lambda(2-s)\Lambda(2-2s) \det y^{-s/2+1} \\ &\quad + \mathcal{F}(y, s), \end{aligned}$$

where $\mathcal{F}(\mathbf{y}, s)$ is a normalized sum of all the “middle” terms in (5.1.2), i.e. $j = 1$ and $j = 2$. Equation (5.4.1) now follows upon setting $f(\mathbf{y}, s) = \det \mathbf{y}^{-1} \mathcal{F}(\mathbf{y}, s)$, and the functional equation for $f(\mathbf{y}, s)$ follows from the functional equation for $C_{\mathbf{0}_3}^3(\mathbf{y}, s)$. Formula (5.4.2) follows from [GS19, Proposition 5.8].

Finally we prove (5.4.3). By (5.4.1),

$$(5.4.5) \quad (C_{\mathbf{0}_3}^3)'(\mathbf{y}, 0) = 2\pi\Lambda'(2) + \frac{\pi^2}{6} + \frac{\pi^2}{18} \log \det \mathbf{y} + \frac{\partial f}{\partial s}(\mathbf{y}, 0).$$

Hence for $t > 0$,

$$(C_{\mathbf{0}_3}^3)'(t\mathbf{y}, 0) = 2\pi\Lambda'(2) + \frac{\pi^2}{6} + \frac{\pi^2}{18} \log \det \mathbf{y} + \frac{\pi^2}{6} \log t + \frac{\partial f}{\partial s}(t\mathbf{y}, 0).$$

Then (5.4.3) follows from the estimate (5.4.2) and [GS19, Lemma 5.11]. $\square$

6. REGULARIZED SIEGEL–WEIL FORMULA

6.1. Some Schwartz functions at ∞ . Consider the quadratic space $V_\infty = M_2(\mathbb{R})$ with quadratic form $x \mapsto \det(x)$ and symmetric bilinear pairing $(-, -)$ with $\frac{1}{2}(x, x) = \det(x)$. We specialize some formulas from [GS19] to these explicit coordinates.

Let z_0 be the oriented negative 2-plane in $M_2(\mathbb{R})$ with oriented basis

$$(6.1.1) \quad \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

In the coordinates of Section 7.1, we have $z_0 = (i, i)$. The Siegel majorant (as in [GS19, (2.43)]) associated to z_0 is

$$\|x\|^2 = a^2 + b^2 + c^2 + d^2 \quad \text{for } x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) = V_\infty.$$

For $1 \leq r \leq 3$ and $\mathbf{x} = (x_1, \dots, x_r) \in V_\infty^r$, define the following real-valued functions:

$$(6.1.2) \quad \nu(\mathbf{x}) := e^{-\pi \sum_{i=1}^r \|x_i\|^2}, \quad R(\mathbf{x}) := \sum_{i=1}^r \left(\frac{\|x_i\|^2}{2} - \det(x_i) \right) \quad T(\mathbf{x}) = \frac{1}{2}(\mathbf{x}, \mathbf{x})$$

where $T(\mathbf{x})$ is an $r \times r$ symmetric matrix. When $r = 1$, the function ν is the Siegel Gaussian from [GS19, (2.44)], and we have $R(x) = R(x, z_0)$ where the latter is $R(x, z_0) = h_{z_0}(s_x)$ in the notation of [GS19, §2.3].

We also set $\phi(\mathbf{x}) := R(\mathbf{x}) + 2 \sum_{i=1}^r \det(x_i)$, and for $1 \leq i \leq r$, define $\xi_{\text{KM}, i}^r \in \mathcal{S}(V_\infty^r)$ by:

Let $\mathbf{x} = (x_1, \dots, x_r)$ and $\hat{\mathbf{x}}_i = (\dots, \hat{x}_i, \dots)$,

- For $r = 1, 2$,

$$(6.1.3) \quad \xi_{\text{KM}, i}^r(\mathbf{x}) = \nu(\mathbf{x}) (1 - 2\pi\phi(\hat{\mathbf{x}}_i))$$

- For $r = 3$ and $j < k$, define $f(x_j, x_k) = (a_j + d_j)(c_k - b_k) - (a_k + b_k)(c_j - b_j)$,

$$(6.1.4) \quad \xi_{\text{KM}, i}^3(\mathbf{x}) = \nu(\mathbf{x}) (1 - 2\pi\phi(\hat{\mathbf{x}}_i) + 2\pi^2 f(\hat{\mathbf{x}}_i)^2)$$

We set $\xi_{\text{KM}}^r := \sum_{i=1}^r \xi_{\text{KM},i}^r(\mathbf{x}) \in \mathcal{S}(V_\infty^r)$. These functions appear naturally in the Kudla–Millson forms on the Hermitian symmetric domain associated to the quadratic space V_∞ (the original construction can be found in [KM86, KM87], see also [GS19] for another construction). We refer the reader to §7.6, especially Lemma 7.6.1, for a more precise statement.

Lemma 6.1.1. *For $1 \leq r \leq 3$, the Siegel–Weil section associated to the Schwartz function $\xi_{\text{KM},r}^r$ is a highest weight vector of weight λ_r where*

$$(6.1.5) \quad \lambda_r = \begin{cases} 0, & \text{if } r = 1; \\ (2, 0), & \text{if } r = 2; \\ (2, 2, 0), & \text{if } r = 3. \end{cases}$$

More precisely,

$$(6.1.6) \quad \Phi^{\lambda_r}(g, s) = (\omega_{r,\infty}(g) \xi_{\text{KM},r}^r)(0) \cdot |\det(a(g))|_\infty^{s-(3-r)/2}.$$

Proof. By Lemma 7.6.1 and Remark 7.6.2, the Schwartz function $\xi_{\text{KM},r}^r$ is exactly $(-1)^{r-1} \tilde{\nu}_i$ of [GS19, (3.87)]. Formula (6.1.6) follows from [GS19, Lemma 3.4]. $\square$

6.2. An auxiliary Eisenstein series. The group $H := \text{GSpin}(V)$ is isomorphic to $\text{GL}_2 \times_{\mathbb{G}_m} \text{GL}_2$. We consider the action of $(h_1, h_2) \in \text{GL}_2 \times_{\mathbb{G}_m} \text{GL}_2$ on V by the following formula:

$$(6.2.1) \quad (h_1, h_2) \curvearrowright x \in V \mapsto \det(h_1)^{-1} \cdot h_1 x^t h_2.$$

This action induces the following exact sequence:

$$(6.2.2) \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow H \longrightarrow \text{SO}(V) \longrightarrow 1.$$

This action also induces an action of $H(\mathbb{Q}_v)$ (resp. $O(V)(\mathbb{Q}_v)$) on $\mathcal{S}(V_v^r)$ for all positive integers r and all places v of $\mathbb{Q}$ given by

$$h = (h_1, h_2) \curvearrowright \varphi \in \mathcal{S}(V_v^r) \mapsto (x \in V_v^r \mapsto \varphi(h^{-1}x)).$$

We also use $\omega_{r,v}$ to denote this action.

Let $V = V_2 + V_2'$ be a Witt decomposition of V where V_2 is a totally isotropic subspace of dimension 2, whose dual is V_2' . Let $Q_2 \subset O(V)$ be the parabolic subgroup which stabilizes the subspace V_2' . Let $Q_2 = M_2 \cdot U_2$ where U_2 is the unipotent radical of Q_2 and M_2 is the Levi component which preserves the Witt decomposition [KR94, §5.4]. In the current situation, we have $M_2 \simeq \text{GL}_2$ ([KRS92, §3.2]). Let $K \subset O(V)(\mathbb{A})$ be a maximal compact subgroup so that we have an Iwasawa decomposition

$$O(V)(\mathbb{A}) = Q_2(\mathbb{A}) \cdot K.$$

For $h \in O(V)(\mathbb{A})$, we write $h = u \cdot m(a) \cdot k$, let $|a(h)| = |\det(a)|_{\mathbb{A}}$. Define the Eisenstein series

$$(6.2.3) \quad E(h, s) = \sum_{\gamma \in Q_2(\mathbb{Q}) \backslash O(V)(\mathbb{Q})} |a(\gamma h)|^{s+1/2}.$$

It is proved in [KR94, Proposition 5.4.1] that this series is absolutely convergent for $\text{Re}(s) > \frac{1}{2}$ and has a meromorphic analytic continuation. The Eisenstein series has a simple pole at $s = \frac{1}{2}$ with a

non-zero constant residue:

$$(6.2.4) \quad \kappa = \operatorname{Res}_{s=1/2} E(h, s).$$

6.3. A (regularized) Siegel–Weil formula. Kudla–Rallis [KR94, Corollary 5.1.2] defines an operator D in the center of the universal enveloping algebra for $\operatorname{Sp}(W_{3,\infty})$ such that the regularized theta series $\theta(g, h, \omega_{3,\infty}(D)\varphi)$ decreases rapidly on $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$ for all $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$. Therefore the integral

$$\int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta(g, h, \omega_{3,\infty}(D)\varphi) E(h, s) dh$$

is absolutely convergent for all s for which $E(h, s)$ is holomorphic, where we give $O(V)(\mathbb{A})$ the Haar measure with $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$ of total volume 1.

By Howe’s result [How89], there exists an element D' in the center of the universal enveloping algebra for $O(V)(\mathbb{R})$ such that

$$\omega_{3,\infty}(D) = \omega_{3,\infty}(D'),$$

where $\omega_{3,\infty}$ on the right means the action of the group $O(V)(\mathbb{R})$ on $\mathcal{S}(V_\infty^3)$, and we have [KR94, Lemma 5.5.3]

$$E(h, s) * D' = P(s)E(h, s)$$

where $P(s) = (s - 5/2)(s + 5/2)(s - 1/2)(s + 1/2)$ by [KR94, Lemma 5.5.3]. Define

$$(6.3.1) \quad \mathcal{E}(g, s, \varphi) := \frac{1}{P(s)} \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta(g, h, \omega_{3,\infty}(D)\varphi) E(h, s) dh.$$

Proposition 6.3.1. *Let $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$ be a Schwartz function. Let κ be the constant in (6.2.4). The associated Eisenstein series $E(g, s, \varphi)$ is holomorphic at $s = 0$. The regularized theta integral $\mathcal{E}(g, s, \varphi)$ has a pole at $s = \frac{1}{2}$ of order at most 1 with residue $B_{-1}(g, \varphi)$. Moreover,*

$$(6.3.2) \quad E(g, 0, \varphi) = 2\kappa^{-1} \cdot B_{-1}(g, \varphi) = \frac{1}{6} \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta(g, h, \omega_{3,\infty}(D)\varphi) dh.$$

Proof. The statement can be found in [HK91, Theorem 4.1]. □

6.4. Absolutely convergent theta integrals.

Lemma 6.4.1. *Let $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$ be a Schwartz function. Let $T \in \operatorname{Sym}_3(\mathbb{Q})$ be a matrix. If the function $\theta_T(g, h, \varphi)$ is absolutely integrable over $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$, then*

$$(6.4.1) \quad \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T(g, h, \varphi) dh = \frac{1}{12} \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T(g, h, \omega_{3,\infty}(D)\varphi) dh = \frac{1}{2} \cdot E_T(g, 0, \varphi).$$

Proof. By the absolute convergence of the theta integral, we have

$$\begin{aligned} \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T(g, h, \omega_{3,\infty}(D_3)\varphi) dh &= \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T(g, h, \varphi) \cdot (D'_3)^* \{1\} dh \\ &= P\left(\frac{1}{2}\right) \cdot \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T(g, h, \varphi) dh = 12 \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T(g, h, \varphi) dh \end{aligned}$$

and the relation with $E_T(g, 0, \varphi)$ follows upon applying Proposition 6.3.1 to the left-hand side. $\square$

Part 2. Geometric side

7. KUDLA–MILLSON THEORY ON THE PRODUCT OF UPPER HALF-PLANES

In a series of papers [KM86, KM87, KM90], Kudla and Millson developed a systematic construction of closed Schwartz forms representing the Poincaré duals of special cycles on symmetric domains of orthogonal and unitary type. García and Sankaran [GS19] constructed Green currents for special cycles on these domains (see also Kudla’s work [Kud03, Proposition 4.10] in the divisor case). The product $\mathbb{H}_1^2$ is a symmetric domain of orthogonal type. In this section, we make these constructions explicit in this case, following the appendix to Berndt and Kühn’s work [BK12a].

7.1. Orthogonal description of $\mathbb{H}_1^2$. There is a symmetric bilinear pairing on $V_\infty = M_2(\mathbb{R})$ given by the following formula:

$$(7.1.1) \quad \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) = a_1 d_2 + a_2 d_1 - b_1 c_2 - b_2 c_1.$$

Recall that the group $H(\mathbb{R}) \simeq GL_2(\mathbb{R}) \times_{\mathbb{R}^\times} GL_2(\mathbb{R})$ has an action on V_∞ by the formula (6.2.1):

$$(h_1, h_2) \cdot x := (\det h_1)^{-1} h_1 x^t h_2.$$

Let $\mathbb{D}^\pm$ be the space parameterizing oriented negative 2-planes $X \subset V_\infty = M_2(\mathbb{R})$, which carries an induced action by $H(\mathbb{R})$. The space $\mathbb{D}^\pm$ has two connected components $\mathbb{D}^+$ and $\mathbb{D}^-$. We will fix an isomorphism $\iota : \mathbb{H}_1^2 \xrightarrow{\sim} \mathbb{D}^+$ in the following way:

$$(7.1.2) \quad z = (z_1, z_2) \in \mathbb{H}_1^2 \longmapsto \text{Span}_{\mathbb{R}}\{\text{Re } Z, \text{Im } Z\},$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and

$$(7.1.3) \quad Z = \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix}, \quad \text{Re } Z = \begin{pmatrix} x_1 x_2 - y_1 y_2 & x_1 \\ x_2 & 1 \end{pmatrix}, \quad \text{Im } Z = \begin{pmatrix} x_1 y_2 + x_2 y_1 & y_1 \\ y_2 & 0 \end{pmatrix}.$$

The right-hand side of (7.1.2) should be viewed as the span of an oriented basis. The isomorphism (7.1.2) also respects the actions of the group $H(\mathbb{R})^\circ \simeq GL_2^+(\mathbb{R}) \times_{\mathbb{R}_+^\times} GL_2^+(\mathbb{R})$ on both sides [BK12a, (A.4.3)]. We remark that [BK12a, §A.8] also gives another isomorphism between $\mathbb{H}_1^2$ and $\mathbb{D}^+$ under a different action of H on V . From now on, we denote $\mathbb{D}^+$ simply by $\mathbb{D}$.

7.2. Alternative coordinates for $\mathbb{D}^\pm$. A slightly different description will also be useful. This description is closer (but not identical) to [BK12a, §A.8]. This alternative is natural for complex uniformization of special cycles. Outside of Section 7.2, we will default to the conventions of Section 7.1 unless specified otherwise.

Let $W_1 \cong \mathbb{Q}^{\oplus 2}$ and $W_2 \cong \mathbb{Q}^{\oplus 2}$ be 2-dimensional vector spaces over $\mathbb{Q}$ with chosen bases. View W_1 and W_2 as consisting of column vectors and consider $M_2 = V := \text{Hom}(W_1, W_2)$. Equip W_1 and W_2 with symplectic pairings $\langle -, - \rangle_1$ and $\langle -, - \rangle_2$ respectively, given by the Gram matrix

$$(7.2.1) \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for the given bases. The symplectic pairings induce an involution $\text{Hom}(W_1, W_2) \rightarrow \text{Hom}(W_2, W_1)$ denoted $M \mapsto M^\dagger$, i.e. $\langle x, My \rangle_2 = \langle M^\dagger x, y \rangle_1$ for all $x \in W_2$ and $y \in W_1$. There is an induced symmetric bilinear pairing $(M, M') := \text{tr}(M^\dagger M')$ on M_2 , and this recovers the formula in (7.1.1).

Recall that we use $\mathbb{H}_1$ to denote the complex upper plane. Let $\mathbb{H}_1^-$ denote the complex lower half-planes, and write $\mathbb{H}_1^\pm$ for disjoint union of $\mathbb{H}_1$ and $\mathbb{H}_1^-$. For $W := \mathbb{Q}^{\oplus 2}$, we view $\mathbb{H}_1^\pm$ as parameterizing Hodge structures h on W with Hodge types $(-1, 0)$ and $(0, -1)$ in a standard way, for the following conventions. We let $z \in \mathbb{H}_1^\pm$ correspond to the filtration $F^0 W_{\mathbb{C}} \subseteq W_{\mathbb{C}}$ where $F^0 W_{\mathbb{C}}$ is spanned by the column vector $(z, 1)$. There is a corresponding complex structure $\mathbb{C} \curvearrowright W_{\mathbb{R}}$ where $a \in \mathbb{C}^\times$ acts by $h(a^{-1})$. This is arranged so that if E_z is the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}z)$ and $W \rightarrow H_1(E_z, \mathbb{Q})$ sends the standard basis to the ordered pair $(1, -z)$, then $W_{\mathbb{R}} \rightarrow \text{Lie } E_z$ is $\mathbb{C}$ -linear.

We consider two copies of the above situation associated to W_1 and W_2 from before, and write e.g., $z_j \in \mathbb{H}_{1,j}^\pm$ for $j = 1, 2$. The spaces $\mathbb{H}_{1,j}^\pm$ are also equipped with *tautological bundles* ω_j , where ω_j is the complex line subbundle of $\mathbb{H}_{1,j}^\pm \times W_{j,\mathbb{C}}$ whose fiber over a point $F^0 W_{j,\mathbb{C}} \in \mathbb{H}_{1,j}^\pm$ is the complex line $F^0 W_{j,\mathbb{C}} \subseteq W_{j,\mathbb{C}}$. Recall that $W_{j,\mathbb{C}}$ has a Hermitian pairing, which we normalize by the formula $x, y \mapsto \langle \bar{x}, iy \rangle_j$. We thus obtain a positive definite metric on ω_j by restriction from $W_{j,\mathbb{C}}$, after changing the pairing by a sign if necessary (the sign will be locally constant on $\mathbb{H}_{1,j}^\pm$). We write $\hat{\omega}_{j,\text{nat}}$ for the resulting metrized line bundles.² In coordinates, the metric on $\hat{\omega}_{j,\text{nat}}$ has square norm $\|-\|_{\text{nat}}^2$ given by

$$(7.2.2) \quad \left\| \begin{pmatrix} z_j \\ 1 \end{pmatrix} \right\|_{\text{nat}}^2 = 2|\text{Im}(z_j)| \quad \text{for} \quad \begin{pmatrix} z_j \\ 1 \end{pmatrix} \in F^0 W_{j,\mathbb{C}}.$$

Next, extend $(-, -)$ to a $\mathbb{C}$ -bilinear pairing on $V_{\mathbb{C}}$. This gives a Hermitian pairing $Y, Z \mapsto (\bar{Y}, Z)$ on $V_{\mathbb{C}}$.

The set $\mathbb{D}^\pm$ of oriented negative 2-planes $X \subseteq V_{\mathbb{R}}$ is in bijection with the set of complex lines $F^1 V_{\mathbb{C}} \subseteq V_{\mathbb{C}}$ which are negative definite for the Hermitian pairing and with $F^1 V_{\mathbb{C}} \perp \overline{F^1 V_{\mathbb{C}}}$ for the Hermitian pairing. This bijection is canonical up to the following convention: if $F^1 V_{\mathbb{C}} = \text{span}_{\mathbb{C}} Z$ for some $Z \in V_{\mathbb{C}}$, we take the associated oriented 2-plane $X \subseteq V_{\mathbb{R}}$ with oriented basis $\{\text{Re}(Z), \text{Im}(Z)\}$. This convention is such that $X \rightarrow X_{\mathbb{C}}/F^1 V_{\mathbb{C}}$ is orientation preserving.

The complex manifold structure on $\mathbb{D}^\pm$ is given by the parameterization of these filtrations $F^1 V_{\mathbb{C}} \subseteq V_{\mathbb{C}}$. We obtain a *tautological bundle* $\mathcal{E}$ on $\mathbb{D}^\pm$, which is the complex line subbundle of $\mathbb{D}^\pm \times V_{\mathbb{C}}$ whose fiber over a point $F^1 V_{\mathbb{C}} \in \mathbb{D}^\pm$ is the complex line $F^1 V_{\mathbb{C}} \subseteq V_{\mathbb{C}}$. We give $\mathcal{E}$ a Hermitian metric by restricting the Hermitian pairing from $V_{\mathbb{C}}$ to $F^1 V_{\mathbb{C}}$ and taking the negative. We write $\hat{\mathcal{E}}_{\text{nat}} = (\mathcal{E}, \|-\|_{\text{nat}})$ for the resulting metrized line bundle.

Giving $F^1 V_{\mathbb{C}} \subseteq V_{\mathbb{C}}$ as in the previous paragraph is the same as giving a Hodge structure on $V_{\mathbb{R}}$ with Hodge types among $(1, -1), (0, 0), (-1, 1)$, with the additional requirements that the Hodge decomposition is orthogonal for the Hermitian pairing, that the Hermitian pairing is negative definite on $F^1 V_{\mathbb{C}}$, and that we have $\dim_{\mathbb{C}} F^1 V_{\mathbb{C}} = 1$. Using the definition $V = \text{Hom}(W_1, W_2)$, we find that such Hodge structures are exactly those Hodge structures induced by Hodge structures h_j on W_j ,

²The subscript $\|-\|_{\text{nat}}$ (“natural”) is to indicate a particular choice of normalization. We will eventually also consider $\|-\|_{\text{Pet}}$ (“Petersson”) and $\|-\|$, with the latter appearing in our main theorem.

subject to the condition that the Hermitian pairings

$$(7.2.3) \quad \langle x, iy \rangle_j + i \langle x, y \rangle_j$$

on $W_{j,\mathbb{R}}$ are either both positive definite or both negative definite for $j = 1, 2$. Here i acts on $W_{j,\mathbb{R}}$ via the complex structure associated to the Hodge structure h_j , as described above. For the reader's convenience, we recall that the above Hermitian pairing on $W_{j,\mathbb{R}}$ agrees with the Hermitian pairing on $\overline{F^0 W_{j,\mathbb{C}}}$ restricted from $W_{j,\mathbb{C}}$, under the canonical $\mathbb{C}$ -linear isomorphisms $\overline{F^0 W_{j,\mathbb{C}}} \rightarrow W_{j,\mathbb{C}}/F^0 W_{j,\mathbb{C}}$ and $W_{j,\mathbb{R}} \rightarrow W_{j,\mathbb{C}}/F^0 W_{j,\mathbb{C}}$.

Unwinding the above identification gives an isomorphism of complex manifolds

$$(7.2.4) \quad (\mathbb{H}_{1,1} \times \mathbb{H}_{1,2}) \sqcup (\mathbb{H}_{1,1}^- \times \mathbb{H}_{1,2}^-) \xrightarrow{i_{\mathbb{D}^\pm}} \mathbb{D}^\pm$$

$$(z_1, z_2) \longmapsto \text{span}_{\mathbb{C}} \begin{pmatrix} z_2 & -z_1 z_2 \\ 1 & -z_1 \end{pmatrix} \quad \begin{pmatrix} z_2 & -z_1 z_2 \\ 1 & -z_1 \end{pmatrix} = \begin{pmatrix} z_2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -z_1 \end{pmatrix}.$$

In other words, the pair $(F^0 W_{1,\mathbb{C}}, F^0 W_{2,\mathbb{C}})$ is sent to $F^1 V_{\mathbb{C}} := \text{Hom}_{\mathbb{C}}(W_{1,\mathbb{C}}/F^0 W_{1,\mathbb{C}}, F^0 W_{2,\mathbb{C}})$, and we can recover the pair $(F^0 W_{1,\mathbb{C}}, F^0 W_{2,\mathbb{C}})$ from $F^1 V_{\mathbb{C}}$ as the kernel and image of any nonzero element of $F^1 V_{\mathbb{C}}$. With this convention, the connected component $\mathbb{D}^+ \subseteq \mathbb{D}^\pm$ from Section 7.1 corresponds to $\mathbb{H}_{1,1} \times \mathbb{H}_{1,2}$.

Recall that each $\mathbb{H}_{1,j}^\pm$ carries a tautological bundle ω_j , as described above. There is a second line bundle over each $\mathbb{H}_{1,j}^\pm$ given by the quotients $W_{j,\mathbb{C}}/F^0 W_{j,\mathbb{C}}$ as $F^0 W_{j,\mathbb{C}}$ varies. The fixed symplectic pairing on $W_{j,\mathbb{C}}$ identifies this second line bundle with the dual $\omega_j^\vee$, and this identification is canonical up to a sign convention.³ Up to this sign, we have a canonical identification of tautological bundles $i_{\mathbb{D}^\pm}^* \mathcal{E} \cong p_1^* \omega_1 \otimes p_2^* \omega_2$ for pullback along $i_{\mathbb{D}^\pm}$ from (7.2.4), where $p_1: \mathbb{H}_{1,1}^\pm \times \mathbb{H}_{1,2}^\pm \rightarrow \mathbb{H}_{1,1}^\pm$ and $p_2: \mathbb{H}_{1,1}^\pm \times \mathbb{H}_{1,2}^\pm \rightarrow \mathbb{H}_{1,2}^\pm$ are the projections. In the above coordinates, this isomorphism (for one of the two choices of sign) is

$$(7.2.5) \quad p_1^* \omega_1 \otimes p_2^* \omega_2 \xrightarrow{\sim} i_{\mathbb{D}^\pm}^* \mathcal{E}$$

$$\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} z_2 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} z_2 & -z_1 z_2 \\ 1 & -z_1 \end{pmatrix}.$$

The other choice of sign gives the negative of this isomorphism. By construction, the above isomorphism must preserve the Hermitian metrics. This can also be verified by direct calculation, i.e. both sides of (7.2.5) have square norm $4|\text{Im}(z_1)\text{Im}(z_2)|$. We thus have a canonical (up to sign) isomorphism of metrized tautological bundles

$$(7.2.6) \quad i_{\mathbb{D}^\pm}^* \hat{\mathcal{E}}_{\text{nat}} \cong p_1^* \hat{\omega}_{1,\text{nat}} \otimes p_2^* \hat{\omega}_{2,\text{nat}}.$$

We next describe actions of $H = \text{GL}_2 \times_{\mathbb{G}_m} \text{GL}_2$ in this setup. View the first and second factors of GL_2 as acting on W_1 and W_2 , respectively. This induces actions $\text{GL}_2(\mathbb{R}) \subset \mathbb{H}_{1,j}^\pm$ for $j = 1, 2$ by pushforward of Hodge structures, and this agrees with the usual action of $\text{GL}_2(\mathbb{R})$ by Möbius

³ Here, we simply mean the convention on whether the alternating pairing on W_j induces $W_j \rightarrow W_j^\vee$ given by $x \mapsto \langle x, - \rangle_j$ or $x \mapsto \langle -, x \rangle_j$.

transformations. For each tautological bundle ω_j , we also define an action $\mathrm{GL}_2(\mathbb{R}) \curvearrowright \omega_j$ to be $|\det g|^{-1/2}$ times the naive one induced by $\mathrm{GL}_2(\mathbb{R}) \curvearrowright W_{j,\mathbb{C}}$. This is arranged so that the action preserves Hermitian metrics: for $g \in \mathrm{GL}_2(\mathbb{R})$ and s any local section of ω_j , we have $\|g^*(s)\|_{\mathrm{nat}} = g^* \|s\|_{\mathrm{nat}}$.

There is also an action of H on V given by $(h_1, h_2) \circ x = h_2 x h_1^{-1}$. Note that this is different from the action in Section 7.1; the two actions intertwine via the involution $(h_1, h_2) \mapsto (h_1, J h_2 J^{-1})$ of $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ for J the matrix from (7.2.1). This induces an action $H \curvearrowright \mathbb{D}^\pm$, and an action $H \curvearrowright \mathcal{E}$ on the tautological bundle which preserves the Hermitian metrics: for $h = (h_1, h_2) \in H(\mathbb{R})$ and s any local section of $\mathcal{E}$, we have $\|h^*(s)\|_{\mathrm{nat}} = h^* \|s\|_{\mathrm{nat}}$.

With these conventions, the description of $\mathbb{D}^\pm$ in (7.2.4) and the identification of tautological bundles in (7.2.5) are both $H(\mathbb{R})$ -equivariant. In these alternative coordinates, the first Chern form $c_1(\hat{\mathcal{E}}^\vee)$ is also given by the formula (7.4.5) (which was calculated using the coordinates from Section 7.1), i.e.,

$$(7.2.7) \quad c_1(\hat{\mathcal{E}}^\vee) = \Omega.$$

We refer to (7.4.5) for an explicit expression of Ω .

7.3. Special cycles on $\mathbb{D}^\pm$. We next recall local special cycles on $\mathbb{D}^\pm$, which should trace back to the work of Hirzebruch and Zagier [HZ76]. For $r \geq 0$ and $\mathbf{x} = (x_1, \dots, x_r) \in V_\infty^r$, we have the Kudla local special cycle

$$(7.3.1) \quad \mathbb{D}_{\mathbf{x}}^\pm := \{Z \in \mathbb{D}^\pm : (Z, x_i) = 0 \text{ for } 1 \leq i \leq r\} \subseteq \mathbb{D}^\pm$$

where $Z \in \mathbb{D}^\pm$ denotes an oriented negative 2-plane in V_∞ . The subset $\mathbb{D}_{\mathbf{x}}^\pm \subseteq \mathbb{D}^\pm$ is a complex submanifold. We set $\mathbb{D}_{\mathbf{x}} = \mathbb{D} \cap \mathbb{D}_{\mathbf{x}}^\pm$.

We have $\mathbb{D}_{\mathbf{x}}^\pm \neq \emptyset$ if and only if $\mathrm{span}_{\mathbb{R}}\{x_1, \dots, x_r\}$ is positive definite or the zero space. We have $\mathbb{D}_{\mathbf{x}}^\pm = \bigcap_{i=1}^r \mathbb{D}_{x_i}^\pm$, as follows immediately from the definition. These assertions also hold with $\mathbb{D}^\pm$ replaced by $\mathbb{D}$.

Any element $x \in V_{\mathbb{R}}$ gives a $\mathbb{C}$ -linear functional $(x, -)$ on $V_{\mathbb{C}}$, and hence a section s_x of the dual tautological bundle $\mathcal{E}^\vee$ on $\mathbb{D}^\pm$. In the notation of Section 7.2, recall that any point $Z \in \mathbb{D}^\pm$ gives Hodge filtrations on $V_{\mathbb{C}}$, $W_{1,\mathbb{C}}$, and $W_{2,\mathbb{C}}$. Given such x and Z , the followings are equivalent.

- (1) The map $x : W_{1,\mathbb{R}} \rightarrow W_{2,\mathbb{R}}$ preserves Hodge structures.
- (2) We have $x \in F^0 V_{\mathbb{C}}$, or equivalently, $x \in F^0 V_{\mathbb{C}} \cap \overline{F^0 V_{\mathbb{C}}}$.
- (3) We have $(x, Z) = 0$ (here Z means the corresponding oriented negative 2-plane).
- (4) The point $Z \in \mathbb{D}^\pm$ lies in the zero locus of s_x .

The local special cycle $\mathbb{D}_{\mathbf{x}}^\pm \subseteq \mathbb{D}^\pm$ is the locus where these equivalent conditions are satisfied. For any $h \in H(\mathbb{R})$, we have $h \cdot \mathbb{D}_{\mathbf{x}}^\pm = \mathbb{D}_{h \cdot \mathbf{x}}^\pm$ for the actions $H(\mathbb{R}) \curvearrowright V_\infty$ and $H(\mathbb{R}) \curvearrowright \mathbb{D}^\pm$ from Section 7.1. The same holds for the alternative action by $H(\mathbb{R})$ used in Section 7.2.

In the coordinates $\mathbb{D} \cong \mathbb{H}_1^2$ from Section 7.1, if $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the set $\mathbb{D}_x \subseteq \mathbb{H}_1^2$ consists of solutions (z_1, z_2) to the equation

$$(7.3.2) \quad dz_1 z_2 - c z_1 - b z_2 + a = 0.$$

7.4. Kudla–Millson form. For integers $r \geq 0$, a *Schwartz form* α on $\mathbb{D}$ of bidegree (p, q) with coefficients in $\mathcal{S}(V_\infty^r)$ will mean an assignment $\mathbf{x} \mapsto \alpha(\mathbf{x})$ with

$$(7.4.1) \quad \alpha(\mathbf{x}) \in A^{p,q}(\mathbb{D}) \quad \text{for all } \mathbf{x} \in V_\infty^r, \quad \text{and} \quad \alpha(-)_z \in \mathcal{S}(V_\infty^r) \otimes \bigwedge^{p,q} T_z^* \mathbb{D} \quad \text{for all } z \in \mathbb{D}$$

where $A^{p,q}(\mathbb{D})$ is the space of smooth (p, q) -forms on $\mathbb{D}$. The Schwartz form α is said to be $H(\mathbb{R})^\circ$ -invariant if $h^* \alpha(h\mathbf{x}) = \alpha(\mathbf{x})$ for all $h \in H(\mathbb{R})^\circ$, and is said to be closed if $\alpha(\mathbf{x})$ is closed for all $\mathbf{x}$. For any fixed $z \in \mathbb{D}$, giving an $H(\mathbb{R})^\circ$ -invariant Schwartz form of degree (p, q) is the same as giving an element of $\bigwedge^{p,q} T_z^* \mathbb{D}$ which is $K_z^\circ := \text{Stab}_{H(\mathbb{R})^\circ}(z)$ -invariant.

Suppose α is an $H(\mathbb{R})^\circ$ -invariant Schwartz form on $\mathbb{D}$ with coefficients in $\mathcal{S}(V_\infty^r)$, not necessarily of pure degree (p, q) (allow sums of forms of different degree). Then $d\alpha$ is also an $H(\mathbb{R})^\circ$ -invariant Schwartz form, which may be described as follows. Let $\mathfrak{h} := \text{Lie } H(\mathbb{R})^\circ$. Given $z \in \mathbb{D}$, set $\mathfrak{k}_z := \text{Lie } K_z^\circ$, and consider the associated Cartan decomposition $\mathfrak{h} = \mathfrak{p}_z \oplus \mathfrak{k}_z$. Pick any basis $\{X_i\}_i$ of $\mathfrak{p}_z$, with associated dual basis $\{X_i^*\}_i$ of $\mathfrak{p}_z^*$. After pulling back to $H(\mathbb{R})^\circ$, a calculation using the invariant formula for the exterior derivative shows that $\alpha \mapsto d\alpha$ is given by

$$(7.4.2) \quad \begin{aligned} [\mathcal{S}(V_\infty^r) \otimes \bigwedge^* T_z^* \mathbb{D}]^{K_z^\circ} &\longrightarrow [\mathcal{S}(V_\infty^r) \otimes \bigwedge^* T_z^* \mathbb{D}]^{K_z^\circ} \\ \sum_j \phi_j \otimes \beta_j &\longmapsto \sum_{i,j} (X_i \phi_j) \otimes (X_i^* \wedge \beta_j). \end{aligned}$$

For this, we use $H(\mathbb{R})^\circ$ -invariance of α and the fact that $[\mathfrak{p}_z, \mathfrak{p}_z] \subseteq \mathfrak{k}_z$.

A special case of a main result from [KM86] and [KM87] gives a certain closed $H(\mathbb{R})^\circ$ -invariant Schwartz form φ_{KM} on $\mathbb{D}$. We refer to [KM86, Theorem 3.1 (ii)] for more details. Following the Appendix of [BK12a], we have an explicit expression for φ_{KM} : For $z = (z_1, z_2) \in \mathbb{H}_1^2 \simeq \mathbb{D}$, and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define

$$(7.4.3) \quad R(x, z) := \frac{|dz_1 z_2 - cz_1 - bz_2 + a|^2}{2y_1 y_2} \quad \|x\|_z^2 := 2R(x, z) + 2\det(x),$$

where $R(x, z)$ corresponds to $h_z(s_x)$ from [GS19, §2.3], and $\|\cdot\|_z$ is the Siegel majorant associated to z , as in [GS19, (2.43)]. Let $h_{z_i} = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_i^{1/2} & 0 \\ 0 & y_i^{-1/2} \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, and define $h_z = (h_{z_1}, h_{z_2}) \in H(\mathbb{R})^\circ$. If $h_z^{-1} \cdot x = (h_{z_1})^{-1} x^t h_{z_2}^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, we have $R(x, z) = R(h_z^{-1} \cdot x)$ (recall our notation from §6.1). The formula [BK12a, (B.4.1)] implies that

$$(7.4.4) \quad \begin{aligned} \varphi_{\text{KM}}(x)_z &= \frac{e^{-\pi(a'^2 + b'^2 + c'^2 + d'^2)}}{4} \times \\ &\quad \left(((a' + d')^2 + (c' - b')^2 - \pi^{-1}) \cdot \left(\frac{dx_1 \wedge dy_1}{y_1^2} + \frac{dx_2 \wedge dy_2}{y_2^2} \right) \right. \\ &\quad \left. + ((a' + d')^2 - (c' - b')^2) \cdot \frac{dx_1 \wedge dy_2 - dy_1 \wedge dx_2}{y_1 y_2} \right. \\ &\quad \left. - 2(a' + d')(c' - b') \cdot \frac{dx_1 \wedge dx_2 + dy_1 \wedge dy_2}{y_1 y_2} \right). \end{aligned}$$

Our sign is the opposite of [BK12a, (B.4.1)]; see the remark on [BK12a, Page 70]. From this explicit expression, we have

$$(7.4.5) \quad \varphi_{\text{KM}}(0) = \Omega := -\frac{1}{4\pi} \cdot \left(\frac{dx_1 \wedge dy_1}{y_1^2} + \frac{dx_2 \wedge dy_2}{y_2^2} \right).$$

Notice that $-\Omega$ is an invariant Kähler form on $\mathbb{D}$.

7.5. Polynomial Fock space. For fixed $z \in \mathbb{D}$, the *polynomial Fock space* $S(V_\infty^r)_z \subseteq \mathcal{S}(V_\infty^r)$ consists of those Schwartz functions of the form

$$(7.5.1) \quad \mathbf{x} \mapsto (\text{polynomial in the coordinates of } \mathbf{x}) \cdot e^{-\pi \|\mathbf{x}\|_z^2}$$

for $\mathbf{x} \in V_\infty^r$, where $\|\cdot\|_z$ is the Siegel majorant associated to z , as in (7.4.3). The action by the universal enveloping algebra for $\text{Sp}(W_{r,\infty})$ preserves $S(V_\infty^r)_z$ (see e.g. [BF10, §3]).

Lemma 7.5.1. *Let α be an $H(\mathbb{R})^\circ$ -invariant Schwartz form on $\mathbb{D}$ with coefficients in the polynomial Fock space. For any fixed $\mathbf{x} = (x_1, \dots, x_r)$, and for all $\rho \in \mathbb{C}$, the integral*

$$(7.5.2) \quad \int_1^{+\infty} \alpha^\circ(\sqrt{t}\mathbf{x}) \frac{dt}{t^{1+\rho}} \quad \alpha^\circ(\mathbf{x}) := e^{2\pi \sum_{i=1}^r \det(x_i)} \alpha(\mathbf{x})$$

converges uniformly absolutely on compact subsets of $\mathbb{D} \setminus \mathbb{D}_\mathbf{x}$ to a smooth form, and the convergence is also uniform for ρ in compact subsets.

Proof. Without loss of generality, we may assume the polynomial-in- $\mathbf{x}$ part of $\alpha(\mathbf{x})_z$, as in (7.5.1), is homogeneous of some degree $k \geq 0$ in the coordinates of $\mathbf{x}$ for some z (equivalently, any z). We then have

$$(7.5.3) \quad \alpha^\circ(\sqrt{t}\mathbf{x}) = t^k e^{-2\pi t R(\mathbf{x}, z)} \cdot (\text{smooth form on } \mathbb{D} \text{ independent of } t).$$

But $R(\mathbf{x}, z)$ is continuous and positive away from $\mathbb{D}_\mathbf{x}$; then note that the integral

$$(7.5.4) \quad \int_1^{+\infty} t^k e^{-2\pi c t} \frac{dt}{t^{1+\rho}}$$

is absolutely convergent for any $c > 0$ and any $\rho \in \mathbb{C}$. The same argument shows that any order z -partial derivative of (7.5.3) gives an integrand which is uniformly absolutely convergent on compact subsets of $\mathbb{D} \setminus \mathbb{D}_\mathbf{x}$, which implies smoothness. $\square$

7.6. Green currents of special cycles. For $\mathbf{x} = (x_1, \dots, x_r) \in V_\infty^r$ where $1 \leq r \leq 3$, define

$$(7.6.1) \quad \varphi_{\text{KM}}(\mathbf{x}) := \varphi_{\text{KM}}(x_1) \wedge \dots \wedge \varphi_{\text{KM}}(x_r).$$

This is an (r, r) -form on $\mathbb{D}$, and is hence identically 0 when $r = 3$. For $1 \leq i \leq r$, define

$$(7.6.2) \quad \nu_{r,i}(\mathbf{x}) = e^{-2\pi(R(x_i, z) + \det(x_i))} \cdot \varphi_{\text{KM}}(x_1, \dots, \widehat{x_i}, \dots, x_r)$$

and $\nu_r(\mathbf{x}) = \sum_{i=1}^r \nu_{r,i}(\mathbf{x})$. Both ν_r and $\nu_{r,i}$ are $(r-1, r-1)$ -smooth forms on $\mathbb{D}$, and ν_r is an $H(\mathbb{R})^\circ$ -invariant Schwartz form [GS19, Proposition 2.6]. At the point $z = (i, i)$, we have

$$(7.6.3) \quad (\nu_{r,i}(\mathbf{x}) \wedge \Omega^{3-r})_{(i,i)} = \tilde{\nu}_{r,i}(\mathbf{x}) \cdot \Omega_{(i,i)}^2, \quad (\nu_r(\mathbf{x}) \wedge \Omega^{3-r})_{(i,i)} = \tilde{\nu}_r(\mathbf{x}) \cdot \Omega_{(i,i)}^2,$$

for some Schwartz functions $\tilde{\nu}_{r,i}, \tilde{\nu}_r \in \mathcal{S}(V_\infty^r)$. With notation as in §6.1, we have explicit expressions for these Schwartz functions.

Lemma 7.6.1. *Consider $\mathbf{x} = (x_1, \dots, x_r) \in V_\infty^r$ where $1 \leq r \leq 3$. We have the following identities:*

$$(7.6.4) \quad \tilde{\nu}_{r,i}(\mathbf{x}) = \xi_{\text{KM},i}^r(\mathbf{x}), \quad 1 \leq i \leq r.$$

Proof. The explicit formula of $\tilde{\nu}_{r,i}$ for $r = 1$ can be found in [BK12a, (B.4)]. For $r = 2, 3$, the explicit formula follows from its defining equation (7.6.2). The final computation gives the expression exactly in the definition of $\xi_{\text{KM},i}^r$ from §6.1. $\square$

Remark 7.6.2. In [GS19, (3.87)], their Ω corresponds to our $-\Omega$, so that our $\tilde{\nu}_{r,i}(\mathbf{x})$ corresponds to $(-1)^{r-1}\tilde{\nu}_i(\mathbf{x})$ of loc. cit..

Following [GS19, §2.6], consider the following integral on $\mathbb{D} \setminus \mathbb{D}_{\mathbf{x}}$:

$$(7.6.5) \quad \mathfrak{g}_r^\circ(\mathbf{x}, \rho) := \int_1^{+\infty} \nu_r^\circ(\sqrt{t}\mathbf{x}) \frac{dt}{t^{1+\rho}}. \quad \nu_r^\circ(\mathbf{x}) := e^{2\pi \sum_{i=1}^r \det(x_i)} \nu_r(\mathbf{x}).$$

Over $\mathbb{D} \setminus \mathbb{D}_{\mathbf{x}}$, this integral converges to a smooth form for all $\rho \in \mathbb{C}$, e.g. by Lemma 7.5.1. For the convergence over $\mathbb{D}$, we record the following lemma from [GS19].

Lemma 7.6.3 ([GS19, Proposition 2.12]). *The integral (7.6.5) converges to a locally integrable⁴ form on $\mathbb{D}$ if $\text{Re}(\rho) > 0$, and extends meromorphically to the right half-plane $\text{Re}(\rho) > -1/2$ as a current. Moreover, if $\mathbb{D}_{\mathbf{x}}$ is either empty or of codimension r in $\mathbb{D}$, the integral converges locally uniformly on $\mathbb{D}$ and in ρ to a locally integrable form on $\mathbb{D}$ for $\text{Re}(\rho) > -1/2$.*

When $r = 1$, the integral (7.6.5) converges at $\rho = 0$ if $x \neq 0$, and $e^{-2\pi \det(x)} \mathfrak{g}_1^\circ(x, 0)$ recovers Kudla's Green function [Kud03, (4.28)], as in [GS19, Example 2.10].

As in [GS19, Definition 2.13], we use the meromorphic continuation of $\mathfrak{g}_r^\circ(\mathbf{x}, \rho)$ to define

$$(7.6.6) \quad \mathfrak{g}_r^\circ(\mathbf{x}) := \text{CT}_{\rho=0} \mathfrak{g}_r^\circ(\mathbf{x}, \rho),$$

where the symbol “CT” means the constant term of the expansion at $\rho = 0$, i.e., for a compactly supported form $\eta \in A_c^{3-r, 3-r}(\mathbb{D})$,

$$(7.6.7) \quad \langle \mathfrak{g}_r^\circ(\mathbf{x}), \eta \rangle = \text{CT}_{\rho=0} \langle \mathfrak{g}_r^\circ(\mathbf{x}, \rho), \eta \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between currents and compactly supported forms. We also use the notation

$$\int_{\mathbb{D}} \mathfrak{g}_r^\circ(\mathbf{x}) \wedge \eta := \langle \mathfrak{g}_r^\circ(\mathbf{x}), \eta \rangle,$$

but the readers should keep in mind that we do not claim $\mathfrak{g}_r^\circ(\mathbf{x})$ is a literal form.

For $r' = \dim_{\mathbb{R}} \text{span}_{\mathbb{R}}\{\mathbf{x}\}$, Garcia and Sankaran [GS19, Proposition 2.12 (vi)] proved that $\mathfrak{g}_r^\circ(\mathbf{x})$ satisfies

$$(7.6.8) \quad \text{dd}^c \mathfrak{g}_r^\circ(\mathbf{x}) + \delta_{\mathbb{D}_{\mathbf{x}}} \wedge \Omega^{r-r'} = \varphi_{\text{KM}}^\circ(\mathbf{x}) \quad \varphi_{\text{KM}}^\circ(\mathbf{x}) := e^{2\pi \sum_{i=1}^r \det(x_i)} \cdot \varphi_{\text{KM}}(\mathbf{x})$$

⁴The smoothness assertion of loc. cit. does not hold, as can be seen in the case of $r = 1$ and $\dim \mathbb{D} = 1$. In general, at points on $\mathbb{D}_{\mathbf{v}}$, the claim of boundedness of $\nu^\circ(\sqrt{t}\mathbf{v})$ as $t \rightarrow \infty$ also does not hold, as can be seen e.g. by inspection of (7.4.4). The same unboundedness applies to $\varphi^\circ(\sqrt{t}\mathbf{v}'_i)$ in the proof of loc. cit..

with the usual convention $dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$.

8. SPECIAL CYCLES ON THE INTEGRAL MODEL

8.1. Moduli problems. Let $\mathcal{E}ll$ be the moduli stack of elliptic curves. It is a smooth separated Deligne–Mumford stack of finite type over $\mathrm{Spec}\mathbb{Z}$. Let $\pi : \mathcal{A} \rightarrow \mathcal{E}ll$ be the universal elliptic curve over $\mathcal{E}ll$, and define the Hodge bundle on $\mathcal{E}ll$ to be $\omega_{\mathcal{A}} := \pi_*\Omega_{\mathcal{A}/\mathcal{E}ll}^1$. This is a line bundle over $\mathcal{E}ll$, and carries a Hermitian metric in the complex fiber. We take the normalizations

$$(8.1.1) \quad (\alpha, \beta)_{\mathrm{nat}} = \frac{i}{2} \int_{A(\mathbb{C})} \beta \wedge \bar{\alpha} \quad (\alpha, \beta)_{\mathrm{Pet}} := \frac{1}{\pi} (\alpha, \beta)_{\mathrm{nat}} \quad (\alpha, \beta) := (4\pi e^\gamma)^{-1/2} (\alpha, \beta)_{\mathrm{Pet}}$$

for $A \in \mathcal{E}ll(\mathbb{C})$ and $\alpha, \beta \in H^0(A, \omega_A)$ (viewed as holomorphic 1-forms on $A(\mathbb{C})$), where γ is the Euler–Mascheroni constant. The resulting metrized line bundles for $\|\cdot\|_{\mathrm{nat}}$, $\|\cdot\|_{\mathrm{Pet}}$, and $\|\cdot\|$ will be denoted, $\hat{\omega}_{\mathcal{A}, \mathrm{nat}}$, $\hat{\omega}_{\mathcal{A}, \mathrm{Pet}}$, and $\hat{\omega}_{\mathcal{A}}$, respectively, with the only difference being the normalization of the metric.

Define $\mathcal{M} := \mathcal{E}ll \times_{\mathbb{Z}} \mathcal{E}ll$. Denote by p_1 (resp. p_2) the first (resp. second) projection morphism from $\mathcal{M}$ to $\mathcal{E}ll$. We define the metrized tautological bundle on $\mathcal{M}$ to be $\hat{\mathcal{E}} := p_1^* \hat{\omega}_{\mathcal{A}} \boxtimes p_2^* \hat{\omega}_{\mathcal{A}}$. We also set $\hat{\omega}_j := p_j^* \hat{\omega}_{\mathcal{A}}$ for $j = 1, 2$. We also write $\hat{\mathcal{E}}_{\mathrm{nat}}$ and $\hat{\mathcal{E}}_{\mathrm{Pet}}$ as helpful, to indicate the same line bundle but with metric defined using $\|\cdot\|_{\mathrm{nat}}$ or $\|\cdot\|_{\mathrm{Pet}}$.

We write $\overline{\mathcal{E}ll}$ for the Deligne–Rapoport compactification of $\mathcal{E}ll$ as in [DR73, §III, Remarque 2.6], i.e. $\overline{\mathcal{E}ll}$ parameterizes families of generalized elliptic curves whose geometric fibers are irreducible. Here $\overline{\mathcal{E}ll} \rightarrow \mathrm{Spec}\mathbb{Z}$ is a smooth proper Deligne–Mumford stack. Define $\overline{\mathcal{M}} = \overline{\mathcal{E}ll} \times_{\mathbb{Z}} \overline{\mathcal{E}ll}$, which has dimension 3, is also a smooth proper Deligne–Mumford over $\mathrm{Spec}\mathbb{Z}$, and contains $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ naturally as a dense open substack. Via the classical j -invariant, the coarse moduli scheme associated to $\overline{\mathcal{M}}$ (resp. $\mathcal{M}$) is $\overline{M} \simeq \mathbb{P}_{\mathbb{Z}}^1 \times_{\mathbb{Z}} \mathbb{P}_{\mathbb{Z}}^1$ (resp. $M \simeq \mathbb{A}_{\mathbb{Z}}^1 \times_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}}^1 \simeq \mathbb{A}_{\mathbb{Z}}^2$), as follows from the corresponding statements for $\mathcal{E}ll$ and $\overline{\mathcal{E}ll}$, given in [DR73, §VI], and e.g. compatibility of coarse moduli maps with flat base change.

8.2. Special cycles on $\mathcal{M}$. In the coming sections, we discuss special cycles and associated Green currents on the Deligne–Mumford stack $\mathcal{M}$.

Definition 8.2.1. For $1 \leq r \leq 3$, let $T = (T_{ij}) \in \mathrm{Sym}_r(\mathbb{Q})$ be a symmetric matrix. Denote by $\mathcal{Z}(T)$ a stack whose fiber category over a scheme S consists of the following objects,

$$((E, E'), \mathbf{x} = (x_1, \dots, x_r)),$$

where (E, E') is an object in $\mathcal{M}(S)$ and $x_i \in \mathrm{Hom}(E, E')$ satisfy $T = \frac{1}{2}(\mathbf{x}, \mathbf{x})$, i.e.

$$T_{ij} = \frac{1}{2} (\deg(x_i + x_j) - \deg(x_i) - \deg(x_j)) = \frac{1}{2} (x_i \circ x_j^\vee + x_j \circ x_i^\vee).$$

There is a natural finite, unramified, and relatively representable (see the works of Howard and Madapusi [HM22, Definition 2.2.11]) morphism $i_r : \mathcal{Z}(T) \rightarrow \mathcal{M}$ given by forgetting the morphisms $\mathbf{x}$ of an object $((E, E'), \mathbf{x})$ of $\mathcal{Z}(T)$. In particular, we know that $\mathcal{Z}(T)$ is a Deligne–Mumford stack. We call $\mathcal{Z}(T) \rightarrow \mathcal{M}$ a “special cycle”. Étale locally on the target, the morphism $\mathcal{Z}(T) \rightarrow \mathcal{M}$ becomes a disjoint union of finitely many closed immersions (equivalent to the finite unramified and relatively representable property).

Since the quadratic space $\text{Hom}(E, E')$ is positive-definite (positivity of the Rosati involution), the stack $\mathcal{Z}(T)$ is non-empty only if T is positive semi-definite. We also see that $\mathcal{Z}(T)$ is empty unless T is the normalized Gram matrix for an integral quadratic form. By the above definition, the special cycle is $\text{GL}_r(\mathbb{Z})$ -invariant, i.e., for an element $\gamma \in \text{GL}_r(\mathbb{Z})$, we have a natural isomorphism:

$$(8.2.1) \quad \begin{aligned} \mathcal{Z}(T) &\xrightarrow{\sim} \mathcal{Z}({}^t\gamma T \gamma) \\ ((E, E'), \mathbf{x}) &\mapsto ((E, E'), \mathbf{x} \cdot \gamma). \end{aligned}$$

If T is of the form $\text{diag}\{\mathbf{0}_{r-\text{rk}(T)}, T^\flat\}$ for a nonsingular matrix $T^\flat$, then we have $\mathcal{Z}(T) = \mathcal{Z}(T^\flat)$ (by positivity of $\text{Hom}(E, E')$ again). For general T , it is always possible to find a matrix $\gamma \in \text{GL}_r(\mathbb{Z})$ such that ${}^t\gamma T \gamma$ is of the form $\text{diag}\{\mathbf{0}_{r-\text{rk}(T)}, T^\flat\}$ for a nonsingular matrix $T^\flat \in \text{Sym}_{\text{rk}(T)}(\mathbb{Q})$. In this situation, we set $\det'(T) := \det(T^\flat)$. This is independent of the choice of γ . We caution the reader that this notation $\det'$ differs from e.g. [GS19, Page 929].

Lemma 8.2.2. *For $1 \leq r \leq 3$, let $T = (T_{ij}) \in \text{Sym}_r(\mathbb{Q})$ be a nonzero matrix.*

- (a). *The morphism $\mathcal{Z}(T) \rightarrow \mathcal{M}$ is a local complete intersection morphism. The generic fiber $\mathcal{Z}(T)_{\mathbb{Q}} \rightarrow \text{Spec } \mathbb{Q}$ is smooth. If $\mathcal{Z}(T)$ is nonempty, then it is equidimensional with $\dim \mathcal{Z}(T) = \dim \mathcal{M} - \text{rk}(T)$.*
- (b). *If $\text{rk}(T) = 1$, then $\mathcal{Z}(T) \rightarrow \mathcal{M}$ is a generalized Cartier divisor in the sense of [HM22, Definition 2.4.1].*
- (c). *If $\text{rk}(T) = 2$, then $\mathcal{Z}(T) \rightarrow \text{Spec } \mathbb{Z}$ is proper quasi-finite and flat with étale generic fiber. For any field k and any object $((E, E'), \mathbf{x})$ in $\mathcal{Z}(T)(k)$, the elliptic curves E and E' both admit an action by $\mathbb{Q}(\sqrt{-\det' T})$. In particular, if k has characteristic $p > 0$, then E and E' are both supersingular (resp. ordinary) if p is nonsplit (resp. split) in $\mathbb{Q}(\sqrt{-\det' T})$.*
- (d). *If $\text{rk}(T) = 3$, if $\mathcal{Z}(T)$ is nonempty, it is supported on the supersingular locus over a unique prime p determined by T in the following way:*

$$(8.2.2) \quad \{p\} = \text{Diff}(T) := \{p : T \text{ cannot be represented in } \text{M}_2(\mathbb{Q}_p)\}.$$

Proof. By the remarks preceding the lemma, it suffices to prove the case when T is nonsingular by choosing some $\gamma \in \text{GL}_r(\mathbb{Z})$ such that ${}^t\gamma T \gamma$ is of the form $\text{diag}\{\mathbf{0}_{r-\text{rk}(T)}, S\}$ for a nonsingular matrix $S \in \text{Sym}_{\text{rk}(T)}(\mathbb{Q})$. Therefore we assume T is nonsingular for the rest of the proof.

(b) We first consider the case $r = 1$. Let $T = (t)$ where $t \in \mathbb{Q}^\times$, the special cycle $\mathcal{Z}(T)$ is nonempty if and only if $t \in \mathbb{Z}_{>0}$, and it parametrizes degree t isogenies between elliptic curves. This is exactly the notion of $[t - \text{Isog}]$ moduli problem introduced by Katz and Mazur [KM85, §6.5]. That the special cycle $\mathcal{Z}(T)$ is a generalized Cartier divisor is essentially proved in [KM85, Proposition 6.8.6]. The flatness over $\mathcal{Z}(T)$ over $\mathbb{Z}$ is proved in Theorem 6.8.1 of [KM85]. The projection map $\text{pr}_1 : \mathcal{Z}(T)_{\mathbb{Q}} \rightarrow \mathcal{E}ll_{\mathbb{Q}}$ is étale, so $\mathcal{Z}(T)_{\mathbb{Q}} \rightarrow \text{Spec } \mathbb{Q}$ is smooth.

For general r , there is a finite unramified morphism from $\mathcal{Z}(T)$ to an open and closed substack of $\mathcal{Z}(T_{11}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{Z}(T_{rr})$. If $\mathcal{Z}(T)$ is nonempty, then T is positive definite, so each T_{ii} is positive. Since $\mathcal{M}$ is regular and each $\mathcal{Z}(T_{ii}) \rightarrow \mathcal{M}$ is a generalized Cartier divisor, the local complete intersection and dimension claims will follow as soon as we show $\dim \mathcal{Z}(T) \leq \dim \mathcal{M} - \text{rk}(T)$ (see also [HM22, Proposition 2.4.4]).

(c) We next consider the case $r = 2$. Suppose $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ is such that ${}^t\gamma T \gamma = \mathrm{diag}(t_1, t_2)$. Given any field-valued point $(E, E', \mathbf{x})$ of $\mathcal{Z}(T)$, consider $(x'_1, x'_2) := \mathbf{x} \cdot \gamma$. If $x := x_2^\vee \circ x'_1$, then $x^\vee = -x$ and $x^\vee \circ x = \det T \cdot (\det \gamma)^2 \equiv \det T \pmod{\mathbb{Q}^{\times 2}}$, so the $\mathbb{Q}$ -subalgebra of $\mathrm{End}^\circ(E')$ generated by x is isomorphic to $\mathbb{Q}(\sqrt{-\det T})$. Since E and E' are isogenous (via e.g. x'_1), we conclude that E must also have CM by the same field. Since every CM elliptic curve may be defined over either $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}}_p$, we conclude that every point of $\mathcal{Z}(T)_{\mathbb{Q}}$ is closed (in the generic fiber) and that every point of $\mathcal{Z}(T)$ in positive characteristic is also closed. This implies that the generic fiber $\mathcal{Z}(T)_{\mathbb{Q}}$ cannot have any positive dimensional components, and that $\mathcal{Z}(T)$ cannot have any positive dimensional components supported in positive characteristic, i.e. $\dim \mathcal{Z}(T) \leq 1$. We now know that $\mathcal{Z}(T) \rightarrow \mathcal{M}$ is a local complete intersection, and since $\mathcal{M}$ is regular, we also know that $\mathcal{Z}(T)$ is Cohen–Macaulay, so $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ is flat by miracle flatness.

Every $\overline{\mathbb{Q}}$ -point of $\mathcal{Z}(T)$ descends to $\mathrm{Spec} \mathcal{O}_L$ for some number field L , because CM elliptic curves have potentially everywhere good reduction, and $\mathbf{x}$ extends by the Néron mapping property. Since $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ is flat and 0-dimensional in the generic fiber, the generic points of the topological space $|\mathcal{Z}(T)|$ are all (images of) $\overline{\mathbb{Q}}$ -points and there are only finitely many of them. This implies that $\mathcal{Z}(T)$ can be covered by finitely many schemes of the form $\mathrm{Spec} \mathcal{O}_L$, which is enough to imply that $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ is proper and quasi-finite (we already know that $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ is separated and finite type).

It remains to show that $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ has étale generic fiber. This can be shown in the style of e.g. [How12, Theorem 2.2.1]: for all but finitely many primes p , the morphism $\mathcal{Z}(T)_{\mathrm{Spf} \mathbb{Z}_p} \rightarrow \mathrm{Spf} \mathbb{Z}_p$ is formally étale, because (for all but finitely many primes p) the associated deformation problem can be identified (Serre–Tate) with the deformation problem for pairs (X, ι) where X is a p -divisible group of height 2 and dimension 1, and ι is an action on X by the *maximal* order in $\mathbb{Q}_p(\sqrt{-\det T})$. One can show that this deformation problem is formally étale using Grothendieck–Messing theory, after further discarding the primes which ramify in $\mathbb{Q}(\sqrt{-\det T})$.

(d) For $r = 3$, let's assume that $\mathcal{Z}(T) \neq \emptyset$. Let k be an algebraically closed field such that $\mathcal{Z}(T)(k) \neq \emptyset$. We will show that $\mathrm{char}(k)$ is positive. Let $((E, E'), (x_1, x_2, x_3))$ be an object in $\mathcal{Z}(T)(k)$. The nonsingularity of T implies that x_1, x_2 and x_3 are three linearly independent vectors in the quadratic space $\mathrm{Hom}(E, E')^\circ := \mathrm{Hom}(E, E') \otimes_{\mathbb{Z}} \mathbb{Q}$, therefore

$$\dim_{\mathbb{Q}} \mathrm{Hom}(E, E')^\circ \geq 3.$$

Since the dimension of the space $\mathrm{Hom}(E, E')^\circ$ can only be 1, 2 or 4, and it is 4 if and only if E and E' are two isogenous supersingular elliptic curves over k , we have $\mathrm{char}(k) = p$ for some finite prime p , and $((E, E'), (x_1, x_2, x_3))$ lies on the supersingular locus of $\mathcal{M}$ over p . Moreover, the quadratic space $\mathrm{Hom}(E, E')^\circ$ is isometric to the unique division quaternion algebra ramified at p and ∞ over $\mathbb{Q}$. Therefore $\mathrm{Diff}(T) = \{p\}$ by [Kud97, Proposition 1.3]. $\square$

8.3. Special cycles in Chow groups. Let $Z^r(\mathcal{M})$ be the group of codimension r cycles on $\mathcal{M}$, i.e. formal $\mathbb{Z}$ -linear combinations of integral closed substacks of codimension r . As defined, $\mathcal{Z}(T)$ is not an actual cycle on $\mathcal{M}$, and it only admits a finite unramified representable morphism i_r to $\mathcal{M}$. Define $\mathcal{Z}_{\mathrm{sub}}(T) \in Z^{\mathrm{rk} T}(\mathcal{M})$ be the cycle given by the stack-theoretic image of i_r .

Example 8.3.1. For $T = (t)$ where $t \in \mathbb{Z}_{>0}$, we set $\mathcal{T}_t := \mathcal{Z}_{\text{sub}}(T)$. Generically, the morphism $i_1 : \mathcal{Z}(T) \rightarrow \mathcal{T}_t$ has degree 2 because over each object (E, E') of $\mathcal{T}_t$, there are two isogenies x and $-x$ with degree t . For later use, we denote by $T_t \subseteq M$ (resp. $\overline{T}_t \subseteq \overline{M}$) the image of $\mathcal{T}_t$ (resp. $\overline{\mathcal{T}}_t$) under the coarse moduli map $\mathcal{M} \rightarrow M$ (resp. $\overline{\mathcal{M}} \rightarrow \overline{M}$). Alternatively, T_t is defined by the product of all modular polynomials of degree t/d^2 where d ranges over all positive integers such that $d^2 | t$.

8.4. Complex uniformization. Let $H := \text{GL}_2 \times_{\mathbb{G}_m} \text{GL}_2$. We write $h = (h_1, h_2)$ for elements of H . By the *standard open compact subgroup* of $H(\mathbb{A}_f)$, we mean $K^\circ := H(\hat{\mathbb{Z}})$. We are mostly interested in the level group K° , but will make occasional use of general open compact $K \subseteq K^\circ$ for the purpose of obtaining a scheme rather than a stack.

Let $\mathbf{S}$ be a (possibly infinite) set of primes, and write e.g. $\mathbb{Z}[1/\mathbf{S}]$ for the localization of $\mathbb{Z}$ given by inverting the multiplicative set generated by elements of $\mathbf{S}$. We say that an open compact subgroup $K \subseteq K^\circ$ is *good away from $\mathbf{S}$* if $K = K_{\mathbf{S}} \times K^{\mathbf{S}}$ for open compact subgroups $K_{\mathbf{S}} \subseteq H(\mathbb{A}_{f,\mathbf{S}})$ and $K^{\mathbf{S}} \subseteq H(\mathbb{A}_f^{\mathbf{S}})$ with $K^{\mathbf{S}} = K^{\circ\mathbf{S}}$. In particular, K° is good away from $\mathbf{S} = \emptyset$.

Let $L := \mathbb{Z}^{\oplus 2}$ be the $\mathbb{Z}$ -lattice of rank 2 with chosen basis, and equip L with the symplectic form with Gram matrix

$$(8.4.1) \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in this basis. Let L' be another copy of L , with the same alternating pairing.

Given an open compact subgroup $K \subseteq K^\circ$ good away from $\mathbf{S}$, we define a stack in groupoids $\mathcal{M}_K \rightarrow \text{Spec } \mathbb{Z}[1/\mathbf{S}]$ whose fiber category $\mathcal{M}_K(S)$ over a connected locally Noetherian scheme S over $\text{Spec } \mathbb{Z}[1/\mathbf{S}]$ is the groupoid of tuples $(E, E', \overline{\eta}, \overline{\eta}')$ with

$$\begin{aligned} E, E' & \quad \text{elliptic curves over } S \\ (\overline{\eta}, \overline{\eta}') & \quad \text{a } \pi_1(S, \overline{s})\text{-stable } K_{\mathbf{S}}\text{-orbit of isomorphisms of } \hat{\mathbb{Z}}_{\mathbf{S}}\text{-modules} \\ & \quad \eta : L_{\hat{\mathbb{Z}}_{\mathbf{S}}} \xrightarrow{\sim} \prod_{\ell \in \mathbf{S}} T_\ell(E) \quad \eta' : L'_{\hat{\mathbb{Z}}_{\mathbf{S}}} \xrightarrow{\sim} \prod_{\ell \in \mathbf{S}} T_\ell(E') \end{aligned}$$

which preserve alternating pairings up to a common scalar

where E and E' are equipped with their principal polarizations, which induce alternating pairings on their Tate modules, and where $\overline{s}$ is any choice of geometric point on S ; the choice of $\overline{s}$ does not matter and is suppressed from notation. The isomorphisms in $\mathcal{M}_K(S)$ are pairs of isomorphisms of elliptic curves which respect the orbits $(\overline{\eta}, \overline{\eta}')$. As usual, the alternating pairings on Tate modules depend on trivializations of roots of unity, but the above definition does not depend on this trivialization as we only require identification of alternating pairings up to common scalar.

There is a smooth separated Deligne–Mumford stack of finite type $\mathcal{M}_K \rightarrow \text{Spec } \mathbb{Z}[1/\mathbf{S}]$ representing the above moduli problem (after restriction to connected locally Noetherian test objects) and the forgetful map $\mathcal{M}_K \rightarrow \mathcal{M}[1/\mathbf{S}]$ is finite étale of degree $[K^\circ : K]$. Indeed, one can show that the given moduli description of $\mathcal{M}_K$ gives an étale local system on $\mathcal{M}$ (such a local system may be described only on connected locally Noetherian test objects). The morphism $\mathcal{M}_K \rightarrow \mathcal{M}$ is determined by its description on connected locally Noetherian test objects. We also have a (somewhat indirect)

description of the groupoid $\mathcal{M}_K(S)$ for an arbitrary scheme S over $\mathrm{Spec} \mathbb{Z}[1/\mathbb{S}]$ (not necessarily connected or locally Noetherian): we can recover $\mathcal{M}_K \rightarrow \mathcal{M}$ from its “sheaf of sections”, i.e. we obtain a “sheaf of K -level structures” Level_K on $\mathcal{M}$ such that $\mathcal{M}_K(S)$ parameterizes pairs $(E, E') \in \mathcal{M}(S)$ along with a section of Level_K over (E, E') , and after restricting to connected locally Noetherian S , sections of Level_K may be functorially identified with data $(\bar{\eta}, \bar{\eta}')$ as above.

The formation of $\mathcal{M}_K$ is compatible with enlargement of $\mathbb{S}$: in the situation above, if $\bar{s}$ is a point in characteristic p (possibly $p = 0$), then at any $\ell \notin \mathbb{S}$ with $\ell \neq p$, we have a unique $\pi_1(S, \bar{s})$ -stable K_ℓ -orbit of isomorphisms $\eta_\ell: L_{\mathbb{Z}_\ell} \rightarrow T_\ell(E)$ and $\eta'_\ell: L'_{\mathbb{Z}_\ell} \rightarrow T_\ell(E')$ which preserve alternating pairings up to the same scalar. When $K = K^\circ$, we have $\mathcal{M} = \mathcal{M}_K$. For sufficiently small K , the stack $\mathcal{M}_K$ is (representable by) a scheme. Whenever $\mathcal{M}_K$ is a scheme, it is also quasi-projective over $\mathrm{Spec} \mathbb{Z}[1/\mathbb{S}]$. It is enough for K to act trivially on L/NL and $L'/N'L'$ for some integers $N, N' \geq 3$.

For any $T \in \mathrm{Sym}_r(\mathbb{Q})$ with $1 \leq r \leq 3$, recall that we have a “special cycle” $\mathcal{Z}(T) \rightarrow \mathcal{M}$ as defined in Section 8.2. We write $\mathcal{Z}(T)_K \rightarrow \mathcal{M}_K$ for the pullback over $\mathcal{M}_K$.

We have complex uniformization of $\mathcal{M}_K$ and $\mathcal{Z}(T)_K \rightarrow \mathcal{M}_K$ in the standard way, as we now describe. In Section 8.4, we take the conventions of Section 7.2, except we slightly switch notation and write W, W' instead of W_1, W_2 , etc. Take the canonical embeddings $L \rightarrow W$ and $L' \rightarrow W'$, using the given bases (i.e. $\mathbb{Z}^{\oplus 2} \hookrightarrow \mathbb{Q}^{\oplus 2}$). Fix r , and consider the characteristic function $\varphi := \mathbf{1}_{\mathrm{Hom}(L, L') \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}^{\otimes r} \in C_c^\infty(V(\mathbb{A}_f)^r)$. Given any r -tuple $\mathbf{x}^\infty \in V(\mathbb{A}_f)^r$, we define the “away-from- ∞ special cycle” (nonstandard terminology) to be the discrete set

$$(8.4.2) \quad \mathcal{Z}(\mathbf{x}^\infty)_K := \{h \in H(\mathbb{A}_f)/K : \varphi(h^{-1}\mathbf{x}^\infty) = 1\}.$$

Consider the setoid $\mathcal{Z}(T)_K(\mathbb{C})_{\mathrm{framed}}$, consisting of data

$$(8.4.3) \quad \begin{aligned} (E, E', \mathbf{x}, \bar{\eta}, \bar{\eta}') &\in \mathcal{Z}(T)_K(\mathbb{C}) \\ \phi: H_1(E, \mathbb{Q}) &\xrightarrow{\sim} W \quad \phi': H_1(E', \mathbb{Q}) \xrightarrow{\sim} W' \end{aligned}$$

where ϕ and ϕ' are isomorphisms of $\mathbb{Q}$ vector spaces which preserve symplectic pairings up to a common scalar, where the alternating pairings on $H_1(E, \mathbb{Q})$ come from the principal polarizations on E and E' . We call (ϕ, ϕ') a “framing”. Morphisms in this category are morphisms in $\mathcal{Z}(T)_K(\mathbb{C})$ which are compatible with (ϕ, ϕ') . The group $H(\mathbb{Q})$ acts on the isomorphism classes of $\mathcal{Z}(T)_K(\mathbb{C})$ by acting on framings via the formula $h \cdot (\phi, \phi') = (h_1\phi, h_2\phi')$ for $h = (h_1, h_2) \in H(\mathbb{Q})$.

Hodge theory provides a canonical bijection

$$(8.4.4) \quad \mathcal{Z}(T)_K(\mathbb{C})_{\mathrm{framed}} \xrightarrow{\sim} \coprod_{\substack{\mathbf{x} \in V(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} \mathbb{D}_{\mathbf{x}}^\pm \times \mathcal{Z}(\mathbf{x}^\infty)_K$$

which is $H(\mathbb{Q})$ -equivariant (after passing to isomorphism classes), described as follows. Given an object as in (8.4.3), we obtain $\mathbf{x} \in V(\mathbb{Q})^r$ and $h = (h_1, h_2) \in \mathcal{Z}(\mathbf{x}^\infty)_K$ defined by the commutative

diagram

(8.4.5)

$$\begin{array}{ccccccc}
 & & & h_1 & & & \\
 & & & \curvearrowright & & & \\
 W_{\mathbb{A}_f} & \xleftarrow{\quad} & W & \xleftarrow{\phi} & H_1(E, \mathbb{Q}) & \xleftarrow{\quad} & H_1(E, \mathbb{Z}) \longrightarrow \prod_{\ell} T_{\ell}(E) \xleftarrow{\eta} L_{\hat{\mathbb{Z}}} \longrightarrow W_{\mathbb{A}_f} \\
 & & \downarrow x & & \downarrow x & & \\
 W'_{\mathbb{A}_f} & \xleftarrow{\quad} & W' & \xleftarrow{\phi'} & H_1(E', \mathbb{Q}) & \xleftarrow{\quad} & H_1(E', \mathbb{Z}) \longrightarrow \prod_{\ell} T_{\ell}(E') \xleftarrow{\eta'} L'_{\hat{\mathbb{Z}}} \longrightarrow W'_{\mathbb{A}_f} \\
 & & & h_2 & & &
 \end{array}$$

A datum as in (8.4.3) also gives a point of $\mathbb{D}_{\mathbf{x}}^{\pm}$, as the Hodge structures on $H_1(E, \mathbb{Q})$ and $H_1(E', \mathbb{Q})$ induce Hodge structures on W and W' respectively (using ϕ and ϕ'). Here we are using the identification $(\mathbb{H}_1^+ \times \mathbb{H}_1^+) \sqcup (\mathbb{H}_1^- \times \mathbb{H}_1'^-) \cong \mathbb{D}^{\pm}$ from Section 7.2 and the description of local special cycles from Section 7.3.

The canonical forgetful map $\mathcal{Z}(T)_K(\mathbb{C})_{\text{framed}} \rightarrow \mathcal{Z}(T)_K(\mathbb{C})$ now induces a canonical isomorphism (equivalence) of orbifolds⁵

$$(8.4.6) \quad \left[H(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in V(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} \mathbb{D}_{\mathbf{x}}^{\pm} \times \mathcal{Z}(\mathbf{x}^{\infty})_K \right) \right] \xrightarrow{\sim} \mathcal{Z}(T)_{K, \mathbb{C}}^{\text{an}}$$

where $\mathcal{Z}(T)_{K, \mathbb{C}}^{\text{an}}$ means analytification of $\mathcal{Z}(T)_{K, \mathbb{C}}$. The case $T = \mathbf{0}_r$ gives complex uniformization

$$(8.4.7) \quad \left[H(\mathbb{Q}) \backslash \left(\mathbb{D}^{\pm} \times H(\mathbb{A}_f)/K \right) \right] \xrightarrow{\sim} \mathcal{M}_{K, \mathbb{C}}^{\text{an}}.$$

Outside of Section 8.4, we will typically be looser and just write $\mathcal{Z}(T)_K(\mathbb{C})$ and $\mathcal{M}_K(\mathbb{C})$ instead of $\mathcal{Z}(T)_{K, \mathbb{C}}^{\text{an}}$ and $\mathcal{M}_{K, \mathbb{C}}^{\text{an}}$, etc..

The complex uniformization also gives an analytic description of the metrized tautological bundle $\hat{\mathcal{E}}$ on $\mathcal{M}$ in the complex fiber. We spell this out, as some care is needed for the metrics to align. To simplify slightly, we assume that K is chosen such that $\mathcal{M}_K$ is a scheme.

Given a point $(E, E', \bar{\eta}, \bar{\eta}') \in \mathcal{M}_K(\mathbb{C})$, the analytification of $\hat{\omega}_{1, \text{nat}}$ has fiber over this point which may be identified with the line $F^0 H_1(E, \mathbb{C}) \subseteq H_1(E, \mathbb{C})$. This identification is canonical up to a sign convention (Footnote 3); pick either of the two conventions. The same holds for $\hat{\omega}_{2, \text{nat}}$ and $F^0 H_1(E', \mathbb{C})$. Given a lift $(E, E', \bar{\eta}, \bar{\eta}', \phi, \phi') \in \mathcal{M}_K(\mathbb{C})_{\text{framed}}$, the $\mathbb{C}$ -lines $F^0 W_{\mathbb{C}} \subseteq W_{\mathbb{C}}$ and $F^0 W'_{\mathbb{C}} \subseteq W'_{\mathbb{C}}$ are (by definition) the images $\phi(F^0 H_1(E, \mathbb{C}))$ and $\phi'(F^0 H_1(E', \mathbb{C}))$. Recall that the maps ϕ and ϕ' preserve alternating pairing only up to scalar. If “ $\det \phi \in \mathbb{Q}^{\times}$ ” and “ $\det \phi' \in \mathbb{Q}^{\times}$ ” are the similitude factors (e.g. the alternating pairing on W pulls back to $\det \phi$ times the alternating pairing on $H_1(E, \mathbb{Q})$), then rescaling gives isomorphisms

$$(8.4.8) \quad F^0 W_{\mathbb{C}} \xleftarrow{|\det \phi|^{-1/2} \phi} F^0(E, \mathbb{C}) \quad F^0 W'_{\mathbb{C}} \xleftarrow{|\det \phi'|^{-1/2} \phi'} F^0(E', \mathbb{C})$$

⁵For us, a (complex) *orbifold* is a stack over complex analytic spaces with the étale topology, which admits a representable étale cover by complex manifolds.

which preserve Hermitian pairings exactly. This pair of isomorphisms is $H(\mathbb{Q})$ -equivariant (recall the rescaled action from Section 7.2). In other words, we have written down an $H(\mathbb{Q})$ -equivariant metric-preserving isomorphism between (1) the analytification of $\hat{\mathcal{E}}_{\text{nat}}$ on $\mathcal{M}_K$, pulled back to $\mathbb{D}^\pm \times H(\mathbb{A}_f)/K$ and (2) the metrized tautological bundle on $\mathbb{D}^\pm$ (from Section 7.2) pulled back to $\mathbb{D}^\pm \times H(\mathbb{A}_f)/K$.

We can form the first Chern form $c_1(\hat{\mathcal{E}}_{\text{nat}}^\vee) = c_1(\hat{\mathcal{E}}_{\text{Pet}}^\vee) = c_1(\hat{\mathcal{E}}^\vee)$ on $\mathcal{M}_{K,\mathbb{C}}^{\text{an}}$. Since the $(1,1)$ -form $c_1(\hat{\mathcal{E}}^\vee)$ on $\mathbb{D}^\pm$ is $H(\mathbb{R})$ -invariant, it descends (uniquely) to a $(1,1)$ -form $\Omega := c_1(\hat{\mathcal{E}}^\vee)$ on the orbifold $\mathcal{M}_{K,\mathbb{C}}^{\text{an}}$ even for possibly stacky level K .

8.5. Degrees of special cycles. For a matrix $T \in \text{Sym}_r(\mathbb{Q})$, the generic fiber $\mathcal{Z}(T)_\mathbb{Q}$ is nonempty only if $\text{rk}(T) \leq 2$. Define the degree of $\mathcal{Z}(T)$ (resp. $\mathcal{Z}(T)_\mathbb{Q}$) when $\text{rk}(T) \leq 2$ with respect to the line bundle $\mathcal{E}$ in the following way:

$$(8.5.1) \quad \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{Z}(T)_\mathbb{Q}) := \int_{\mathcal{Z}(T)(\mathbb{C})} c_1(i_r^* \hat{\mathcal{E}}^\vee)^{2-\text{rk}(T)}.$$

8.5.1. The case of rank 0. Let $T = \mathbf{0}_r$. The special cycle is $\mathcal{Z}(T) = \mathcal{M}$. We have the usual complex uniformizations $\mathcal{M}(\mathbb{C}) \cong [\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}_1]^2$ and $M(\mathbb{C}) \cong (\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}_1)^2$, where $[-]$ means stacky quotient and $(-)$ means non-stacky quotient. Using the quotient integral formula and the trivial action of the size 4 group $\{\pm 1\} \times \{\pm 1\} \subseteq \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$ on $\mathbb{H}_1 \times \mathbb{H}_1$, we compute

$$(8.5.2) \quad \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{Z}(T)_\mathbb{Q}) = \int_{\mathcal{M}(\mathbb{C})} c_1(\hat{\mathcal{E}}^\vee)^2 = \frac{1}{4} \int_{M(\mathbb{C})} \Omega^2 = \frac{1}{4} \int_{(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}_1)^2} \frac{dx_1 dx_2 dy_1 dy_2}{8\pi^2 y_1^2 y_2^2} = \frac{1}{288}.$$

8.5.2. The case of rank 1. Consider $T \in \text{Sym}_r(\mathbb{Q})$ with $\text{rk}(T) = 1$. Let $\alpha \in \text{GL}_r(\mathbb{Z})$ be a matrix such that (9.6.1) holds, i.e., ${}^t \alpha T \alpha = \text{diag}\{\mathbf{0}_{r-1}, T^\flat = (t)\}$. Clearly we have $\mathcal{Z}(T) = \mathcal{Z}(T^\flat)$. Generically, the morphism $i_1 : \mathcal{Z}(T^\flat) \rightarrow \mathcal{Z}_{\text{sub}}(T^\flat) = \mathcal{T}_t$ has degree 2, while the generic automorphism group of $\mathcal{T}_t$ has size 4,

$$(8.5.3) \quad \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{Z}(T)_\mathbb{Q}) = \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{Z}(T^\flat)_\mathbb{Q}) = 2 \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{T}_t)_\mathbb{Q} = \frac{1}{2} \int_{|\mathcal{T}_t(\mathbb{C})|} \Omega|_{|\mathcal{T}_t(\mathbb{C})|}.$$

Lemma 8.5.1. *For $T \in \text{Sym}_r(\mathbb{Q})$ with $\text{rk}(T) = 1$ and all $y^\flat \in \text{Sym}_1(\mathbb{R})_{>0}$,*

$$(8.5.4) \quad \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{Z}(T)_\mathbb{Q}) = (96\pi)^{-1} C_{T^\flat}^1(y^\flat, 1).$$

Proof. If $T^\flat$ is not positive definite and integral, the special cycle $\mathcal{Z}(T) = \mathcal{Z}(T^\flat) = \emptyset$. The Fourier coefficient $C_{T^\flat}^1(y^\flat, 1)$ is 0 by Example 4.4.2. Hence both sides of (8.5.4) are 0. Now we consider the case that $T^\flat$ is positive definite and integral. By [Kud03, Theorem 4.23], there exists a number C such that

$$(8.5.5) \quad \deg_{\mathcal{E}_\mathbb{Q}^\vee}(\mathcal{Z}(T^\flat)_\mathbb{Q}) = C \cdot C_{T^\flat}^1(y^\flat, 1).$$

For $T^\flat = (1)$, the left hand side of (8.5.5) is given by (8.5.3) as

$$\frac{1}{2} \int_{|\mathcal{T}_1(\mathbb{C})|} \Omega|_{|\mathcal{T}_1(\mathbb{C})|} = \frac{1}{2} \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}_1} -\frac{dx \wedge dy}{2\pi y^2} = -\frac{1}{12}.$$

However, the right hand side of (8.5.5) is $C \times -8\pi$. Hence $C = (96\pi)^{-1}$. $\square$

8.5.3. *The case of rank 2.* Consider $T \in \mathrm{Sym}_r(\mathbb{Q})$ with $\mathrm{rk}(T) = 2$. By Lemma 8.2.2, we know that $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ is flat proper quasi-finite with étale generic fiber. We will need to compute the (stacky) degree $\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}}) \in \mathbb{Q}$ of the generic fiber $\mathcal{Z}(T)_{\mathbb{Q}} \rightarrow \mathrm{Spec} \mathbb{Q}$ (in the sense of e.g. [Che24, (A.1.11)]), which should largely go back to Vistoli).

Lemma 8.5.2. *Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a positive definite matrix. For all $y \in \mathrm{Sym}_2(\mathbb{R})_{>0}$,*

$$(8.5.6) \quad \deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}}) = \deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}}) = (16\pi^2)^{-1} C_T^2 \left(y, \frac{1}{2} \right).$$

Proof. Denote by $H(\Delta)$ the Hurwitz class number, i.e. the weighted number of equivalence classes of positive definite binary quadratic forms with discriminant $-\Delta$. For half-integral positive definite matrix $T \in \mathrm{Sym}_2(\mathbb{Q})$, the following formula follows from [GK93, (2.19)] and [Gö07b, Proposition 1.1] (see also [CY20, Theorem 7.3]),

$$\begin{aligned} \deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}}) &= \sum_{\alpha \in |\mathcal{Z}(T)(\mathbb{C})|} \frac{1}{\#\mathrm{Aut}(\alpha)} \\ &= \sum_{(E, E') \in |\mathcal{Z}_{\mathrm{sub}}(T)(\mathbb{C})|} \frac{R_{\mathrm{Hom}(E, E')}(T)}{\#\mathrm{Aut}(E) \cdot \#\mathrm{Aut}(E')} \\ &= \sum_{d | \mathrm{cont}(T)} d \cdot H\left(\frac{4 \det(T)}{d^2}\right), \end{aligned}$$

where $R_{\mathrm{Hom}(E, E')}(T)$ is the representation number of the quadratic form corresponding to T in the quadratic space $\mathrm{Hom}(E, E')$, and for $T = \begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}$, the number $\mathrm{cont}(T) = \gcd(t_1, t_2, t)$.

By [Nag92, Theorem 2.1.1],

$$C_T \left(y, \frac{1}{2}, \Phi_2^2 \right) = 288 \sum_{d | \mathrm{cont}(T)} d \cdot H\left(\frac{4 \det(T)}{d^2}\right).$$

Hence

$$\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}}) = 288^{-1} C_T \left(y, \frac{1}{2}, \Phi_2^2 \right) = (16\pi^2)^{-1} C_T^2 \left(y, \frac{1}{2} \right).$$

If T is positive definite but not half-integral, both sides of (8.5.6) are 0. Therefore (8.5.6) holds for all positive definite matrices $T \in \mathrm{Sym}_2(\mathbb{Q})$. $\square$

9. GREEN CURRENTS AND ARITHMETIC CYCLES

9.1. Hermitian vector bundles. We recall some basics on stacky degrees of Hermitian line bundles, which is summarized and specialized from [Che24, §4.1].

Given a smooth algebraic stack $\mathcal{X}$ over $\mathrm{Spec} \mathbb{C}$, a Hermitian vector bundle $\hat{\mathcal{L}}$ on $\mathcal{X}$ is the following functorial assignment: for every morphism $S \rightarrow \mathcal{X}$ with S a scheme smooth over $\mathrm{Spec} \mathbb{C}$, the assignment gives a vector bundle on S with a smooth Hermitian metric on the analytification. We sometimes write $\hat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ where $\mathcal{L}$ is the underlying line bundle on $\mathcal{X}$ and $\|\cdot\|$ is the norm associated with the smooth Hermitian metric (on the analytification), defined functorially.

Given an algebraic stack $\mathcal{X}$ which is flat and finite type over $\mathrm{Spec} \mathbb{Z}$ with smooth generic fiber, a *Hermitian vector bundle* on $\mathcal{X}$ is a vector bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with a smooth Hermitian metric on $\mathcal{E}|_{\mathcal{X}_{\mathbb{C}}}$ which is complex conjugation invariant.

Next, we discuss stacky degrees of Hermitian line bundles along the lines of [Che24, §4.1]. Assume that $\mathcal{Z}$ is a reduced 1-dimensional Deligne–Mumford stack which is proper and flat over $\mathrm{Spec} \mathbb{Z}$. Let $\widehat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a Hermitian line bundle on $\mathcal{Z}$. The Arakelov arithmetic degree $\widehat{\deg}(\widehat{\mathcal{L}})$ of $\widehat{\mathcal{L}}$ on $\mathcal{Z}$ is valued in $\mathbb{R}$, and may be described as follows. If $\mathcal{Z} = \mathrm{Spec} \mathcal{O}_E$ for a number field E , we have the standard definition

$$(9.1.1) \quad \widehat{\deg}(\widehat{\mathcal{L}}) := - \sum_v \log \|s\|_v \quad \text{with} \quad \|s\| := q_v^{-\mathrm{ord}_v(s)} \text{ if } v < \infty,$$

where the sum runs over all places v of E , the quantity q_v is the cardinality of the residue field at v , and s is any rational section of the line bundle $\mathcal{L}$.

If $\mathcal{Z}$ is integral (equivalently, reduced and irreducible by quasi-separatedness), there exists a finite surjection $\mathrm{Spec} \mathcal{O}_E \rightarrow \mathcal{Z}$ and set

$$(9.1.2) \quad \widehat{\deg}(\widehat{\mathcal{L}}) := \frac{1}{\deg(E/\mathbb{Z}_{\mathbb{Q}})} \widehat{\deg}(\mathcal{L}|_{\mathrm{Spec} \mathcal{O}_E})$$

where $\deg(E/\mathbb{Z}_{\mathbb{Q}})$ denotes the degree of the finite étale morphism $\mathrm{Spec} E \rightarrow \mathbb{Z}_K$. This definition does not depend on the choice of E or the morphism $\mathrm{Spec} \mathcal{O}_E \rightarrow \mathcal{Z}$, since any two such choices may be covered by finite surjections from a third such choice $\mathrm{Spec} \mathcal{O}_{E'}$ (and these finite surjections can be made compatible with the maps to $\mathcal{Z}$). If $\mathcal{Z}$ is not necessarily irreducible, we define $\widehat{\deg}(\widehat{\mathcal{L}})$ by summing over the irreducible components of $\mathcal{Z}$. If $\mathcal{X}$ a Deligne–Mumford stack which is flat, finite type, and quasi-separated over $\mathrm{Spec} \mathbb{Z}$, equipped with a Hermitian line bundle $\widehat{\mathcal{L}}$, and if Z is a cycle on $\mathcal{X}$ whose components satisfy the conditions imposed on $\mathcal{Z}$ from above, then we define $\widehat{\deg}(\widehat{\mathcal{L}}|_Z)$ by summing over the components of Z .

9.2. Green currents on $\mathcal{M}$. Now we are going to construct Green currents for $\mathcal{Z}(T)$ following [GS19]. For a positive definite matrix $y = a \cdot {}^t a \in \mathrm{Sym}_r(\mathbb{R})$ where $a \in \mathrm{GL}_r(\mathbb{R})$ and a complex number $\rho \in \mathbb{C}$ such that $\mathrm{Re}(\rho) > -1/2$, consider the following summation of currents on $\mathbb{D}$:

$$(9.2.1) \quad \mathfrak{g}^\circ(T, y, \rho) = \sum_{\mathbf{x} \in \Omega_T(V) \cap M_2(\mathbb{Z})^r} \mathfrak{g}_r^\circ(\mathbf{x} \cdot a, \rho).$$

The goal of this section is to prove the convergence and the meromorphic continuation property of the above summation. The summation (9.2.1) is closely related to another summation:

$$(9.2.2) \quad \nu^\circ(T, y) := \sum_{\mathbf{x} \in \Omega_T(V) \cap M_2(\mathbb{Z})^r} e^{2\pi i \mathrm{tr}(T\mathbf{y})} \nu_r(\mathbf{x} \cdot a),$$

the above summation (9.2.2) converges to a $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ -invariant smooth $(r-1, r-1)$ -form on $\mathbb{D}$ since ν_r is a $(r-1, r-1)$ smooth form on $\mathbb{D}$ with coefficients in the polynomial Fock space.

We have the following convergence lemma, as in [GS19, Proposition 4.3]. For completeness, we give a self-contained proof, with an explicit bound on $\mathrm{Re}(\rho)$, and also give the interchangeability of the $\mathbf{x}$ sum and t integral.

Lemma 9.2.1. *(a). The summation (9.2.1) converges for $\mathrm{Re}(\rho) > 0$ to a $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ -invariant locally integrable form on $\mathbb{D}$, and*

$$(9.2.3) \quad \mathfrak{g}^\circ(T, y, \rho) = \int_1^{+\infty} \nu^\circ(T, ty) \frac{dt}{t^{1+\rho}}.$$

(b). The summation (9.2.1) extends meromorphically⁶ to the right half-plane $\operatorname{Re}(\rho) > -1/2$ as a current on $\mathbb{D}$, which is holomorphic except for a possible simple pole at $\rho = 0$.

Proof. Let $U \subset \mathbb{D}$ be an open subset whose closure $\overline{U}$ is compact. The following set is finite by Lemma A.0.1:

$$S = \{\mathbf{x} \in M_2(\mathbb{Z})^r : T(\mathbf{x}) = T \text{ and } \mathbb{D}_{\mathbf{x}} \cap \overline{U} \neq \emptyset\}.$$

Let $S' = \Omega_T(V) \cap M_2(\mathbb{Z})^r \setminus S$, so that $S \sqcup S'$ indexes the non-zero terms appearing on the right hand side of (9.2.1). By Lemma A.0.1, there exists a number $C > 0$ such that for all $\mathbf{x} = (x_1, \dots, x_r) \in S'$,

$$(9.2.4) \quad R(\mathbf{x}, z) = \sum_{i=1}^r R(x_i, z) > C, \quad z \in \overline{U}.$$

Similar inequality also holds with $\mathbf{x}$ replaced by $\mathbf{x} \cdot a$ and C by another positive number C' .

Consider the following sums:

$$(9.2.5) \quad A = \sum_{\mathbf{x} \in S} \mathfrak{g}_r^\circ(\mathbf{x} \cdot a, \rho), \quad B = \sum_{\mathbf{x} \in S'} \mathfrak{g}_r^\circ(\mathbf{x} \cdot a, \rho)$$

For the summation A , it converges to a locally integrable form when $\operatorname{Re}(\rho) > 0$ and extends meromorphically to the right half-plane $\operatorname{Re}(\rho) > -1/2$ as a current by the finiteness of the set S and Lemma 7.6.3. Lemma 7.6.3 also implies that for $\operatorname{Re}(\rho) > 0$,

$$A = \sum_{\mathbf{x} \in S} \int_1^{+\infty} \nu^\circ(\sqrt{t}\mathbf{x} \cdot a) \frac{dt}{t^{1+\rho}} = \int_1^{+\infty} \sum_{\mathbf{x} \in S} \nu^\circ(\sqrt{t}\mathbf{x} \cdot a) \frac{dt}{t^{1+\rho}}$$

For the summation B , each term $\mathfrak{g}_r^\circ(\mathbf{x} \cdot a, \rho)$ is a smooth form over U (because $\mathbf{x} \notin S$) for all $\rho \in \mathbb{C}$ by Lemma 7.5.1. Let $z \in \overline{U}$ be a point, and fix a basis of the vector space $\bigwedge^{r-1, r-1} T_z^* \mathbb{D}$. The coordinates of the smooth forms $\mathfrak{g}_r^\circ(\mathbf{x} \cdot a, \rho)$ under this basis are bounded by finite linear combinations of integrals of the following type:

$$(9.2.6) \quad \int_1^{+\infty} \left(R(\sqrt{t}\mathbf{x} \cdot a, z) + t \cdot \operatorname{tr}(Ty) \right)^k \times e^{-2\pi t R(\mathbf{x} \cdot a, z)} \frac{dt}{t^{1+\rho}}, \quad k \in \mathbb{Z}_{>0}.$$

Now we try to estimate these coordinates by estimating (9.2.6). Firstly, we have $R(\mathbf{x}, z) > 0$ since $\mathbf{x} \in S'$, then each integral (9.2.6) converges for all $\rho \in \mathbb{C}$.

Moreover, by the compactness of $\overline{U}$, there exist positive numbers a_U and b_U such that

$$a_U \cdot \|\mathbf{x}\|^2 \leq R(\mathbf{x} \cdot a, z) + \operatorname{tr}(Ty) \leq b_U \cdot \|\mathbf{x}\|^2.$$

Then except finitely many $\mathbf{x}$, we have $R(\mathbf{x}, z) \geq a_U/2 \cdot \|\mathbf{x}\|^2$ for all the other $\mathbf{x} \in S'$. For those $\mathbf{x}$, the integral (9.2.6) is bounded in the following way:

$$\begin{aligned} & \int_1^{+\infty} t^k (R(\mathbf{x} \cdot a, z) + \operatorname{tr}(Ty))^k \times e^{-2\pi t R(\mathbf{x} \cdot a, z)} \frac{dt}{t^{1+\rho}} \leq \int_1^{+\infty} b_U^k \|\mathbf{x}\|^{2k} \times e^{-a_U \pi t \|\mathbf{x}\|^2} \frac{dt}{t^{1+\rho-k}} \\ & \leq C_{\rho, k} b_U^k \|\mathbf{x}\|^{2k} \int_1^{+\infty} e^{-a_U \pi t \|\mathbf{x}\|^2/2} dt = 2C_{\rho, k} b_U^k (a_U \pi)^{-1} \|\mathbf{x}\|^{2k-2} \cdot e^{-a_U \pi \|\mathbf{x}\|^2/2}, \end{aligned}$$

where $C_{\rho, k}$ is a positive number such that

$$\frac{1}{t^{1+\rho-k}} \leq C_{\rho, k} \cdot e^{a_U \pi t \|\mathbf{x}\|^2/2}, \quad \text{for all } t \geq 1.$$

⁶For the purpose of this paper, the notion of “meromorphic family of currents” is as in [GS19, Footnote 2].

Then we can find a positive number $C_{\rho,U}$ depending only on ρ and U , such that for all $\mathbf{x} \in S'$, the coordinates of $\mathbf{g}_r^\circ(\mathbf{x} \cdot a, \rho)$ are bounded by the following function

$$C_{\rho,U} \cdot e^{-\frac{\alpha U \pi}{4} \cdot \|\mathbf{x}\|^2}, \text{ for all } z \in \overline{U}.$$

Therefore the sum B converges absolutely for all $\rho \in \mathbb{C}$ to a smooth form on U by standard arguments for the convergence of theta series. Hence the following identity holds for all $\rho \in \mathbb{C}$,

$$B = \sum_{\mathbf{x} \in S'} \int_1^{+\infty} \nu^\circ(\sqrt{t}\mathbf{x} \cdot a) \frac{dt}{t^{1+\rho}} = \int_1^{+\infty} \sum_{\mathbf{x} \in S'} \nu^\circ(\sqrt{t}\mathbf{x} \cdot a) \frac{dt}{t^{1+\rho}}.$$

□

The summation (9.2.1) is independent of the choice of $a \in \mathrm{GL}_r(\mathbb{R})$ by the $\mathrm{O}_r(\mathbb{R})$ -invariance of the forms $\nu_r(\mathbf{x})$ ([GS19, Proposition 2.12(ii)]). It is also easy to verify that for $\gamma \in \mathrm{GL}_r(\mathbb{Z})$,

$$(9.2.7) \quad \mathbf{g}^\circ({}^t\gamma T \gamma, \mathbf{y}, \rho) = \mathbf{g}^\circ(T, \gamma \mathbf{y}^t \gamma, \rho).$$

The current $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$ descends to a current on $\mathcal{M}(\mathbb{C})$ by its invariance under $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$. This descended current will also be denoted $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$, and can be calculated using the following “quotient integral” procedure: any compactly supported smooth form $\eta \in A_c^{3-r, 3-r}(\mathcal{M}(\mathbb{C}))$ is specified by an $(\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}))$ -invariant form $\tilde{\eta} \in A^{3-r, 3-r}(\mathbb{D})$, which can be written as $\tilde{\eta} = \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})} \gamma^* \alpha$ for some $\alpha \in A_c^{3-r, 3-r}(\mathbb{D})$ via a partition of unity argument. We then have

$$(9.2.8) \quad \int_{\mathcal{M}(\mathbb{C})} \mathbf{g}^\circ(T, \mathbf{y}, \rho) \wedge \eta = \int_{\mathbb{D}} \mathbf{g}^\circ(T, \mathbf{y}, \rho) \wedge \alpha.$$

Equivalently, if $\mathcal{F} \subseteq \mathbb{D}$ is a fundamental domain (say, the self-product of the usual fundamental domain for $\mathbb{H}_1$), we have

$$(9.2.9) \quad \int_{\mathcal{M}(\mathbb{C})} \mathbf{g}^\circ(T, \mathbf{y}, \rho) \wedge \eta = \frac{1}{4} \int_{\mathcal{F}} \mathbf{g}^\circ(T, \mathbf{y}, \rho) \wedge \eta$$

for $\mathrm{Re}(\rho) > 0$, where $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$ is represented by a locally L^1 form on $\mathbb{D}$. The factor $\frac{1}{4}$ accounts for the trivial action of $\{\pm 1\} \times \{\pm 1\} \subseteq \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{D}$. For $\mathrm{Re}(\rho) > -1/2$, the current $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$ can then be calculated from (9.2.9) via unique meromorphic continuation in ρ .

By Lemma 9.2.1 and its proof, the constant term in the Laurent expansion

$$(9.2.10) \quad \mathbf{g}^\circ(T, \mathbf{y}) := \mathrm{CT}_{\rho=0} \mathbf{g}^\circ(T, \mathbf{y}, \rho)$$

descends to a current on $\mathcal{M}(\mathbb{C})$. The meaning of (9.2.10) is: for an element $\eta \in A_c^{3-r, 3-r}(\mathcal{M}(\mathbb{C}))$, the pairing $\int_{\mathcal{M}(\mathbb{C})} \mathbf{g}^\circ(T, \mathbf{y}, \rho) \wedge \eta$ defines a meromorphic function in ρ for $\mathrm{Re}(\rho) > -1/2$, and then

$$(9.2.11) \quad \int_{\mathcal{M}(\mathbb{C})} \mathbf{g}^\circ(T, \mathbf{y}) \wedge \eta := \mathrm{CT}_{\rho=0} \int_{\mathcal{M}(\mathbb{C})} \mathbf{g}^\circ(T, \mathbf{y}, \rho) \wedge \eta.$$

9.3. Extension to the compactification. We seek to extend the current $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$ on $\mathcal{M}(\mathbb{C})$ to a current on the compactification $\overline{\mathcal{M}}(\mathbb{C})$. At least for $\mathrm{Re}(\rho) > 0$, where $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$ is represented by a locally L^1 form on $\mathcal{M}(\mathbb{C})$, one can attempt to define the value of $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$ on $\eta \in A_c^{3-r, 3-r}(\overline{\mathcal{M}}(\mathbb{C}))$

by the integral

$$(9.3.1) \quad \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y, \rho) \wedge \eta = \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}^\circ(T, y, \rho) \wedge \eta|_{\mathcal{M}(\mathbb{C})}.$$

The convergence of this integral, however, is not automatic since the restriction $\eta|_{\mathcal{M}(\mathbb{C})}$ need not be compactly supported.

For $r = 3$, Theorem C.1.1 shows that $\mathfrak{g}^\circ(T, y, \rho)$ is in fact L^1 on $\mathcal{M}(\mathbb{C})$ at least if $\operatorname{Re}(\rho) > 2$, so that $\mathfrak{g}^\circ(T, y, \rho)$ defines a current on $\overline{\mathcal{M}}(\mathbb{C})$ for such ρ . That theorem also explains the unique meromorphic continuation of $\mathfrak{g}^\circ(T, y, \rho)$ to a family of currents on $\overline{\mathcal{M}}(\mathbb{C})$ for $\operatorname{Re}(\rho) > -1/2$, with at most a simple pole at $\rho = 0$. The constant term at $\rho = 0$ is then a current on $\overline{\mathcal{M}}(\mathbb{C})$ which extends the previously defined current $\mathfrak{g}^\circ(T, y)$ on $\mathcal{M}(\mathbb{C})$. We simplify notation and also write $\mathfrak{g}^\circ(T, y)$ for this extension to $\overline{\mathcal{M}}(\mathbb{C})$.

We expect an analogue of Theorem C.1.1 to hold for $r = 1$ and $r = 2$ as well. In particular, we expect a meromorphic extension of $\mathfrak{g}^\circ(T, y, \rho)$ on $\overline{\mathcal{M}}(\mathbb{C})$ to $\operatorname{Re}(\rho) > -1/2$, with a possible pole at $\rho = 0$. Since these cases are not needed for the present paper, we do not pursue them here.

However, in the special case where $r = 1$ and $T = (t) \neq \mathbf{0}_1$, the proof of [GS19, Proposition 4.3] and the calculation [GS19, Example 2.10] shows that $\mathfrak{g}^\circ(T, y) = \mathfrak{g}^\circ(T, y, 0)$ is represented by a locally L^1 function on $\mathcal{M}(\mathbb{C})$. In this case, the pairing (9.3.1) already converges absolutely at $\rho = 0$, as follows from [BK12b, Theorem 4.2]. Indeed, the function $\mathfrak{g}^\circ(T, y)$ is two times the Kudla's Green function studied by Berndt and Kühn. This function is nonnegative, since each summand in its defining series is an integral of a nonnegative Gaussian. Hence [BK12b, Theorem 4.2] implies

$$(9.3.2) \quad \int_{\mathcal{M}(\mathbb{C})} |\mathfrak{g}^\circ(T, y)| \Omega^2 < \infty.$$

The absolute convergence of $\int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}^\circ(T, y) \wedge \eta$ for any continuous $(2, 2)$ -form η on $\overline{\mathcal{M}}(\mathbb{C})$ then follows from the following lemma.

Lemma 9.3.1. *Let η be a continuous $(2, 2)$ -form on $\overline{\mathcal{M}}(\mathbb{C})$. Write $\eta|_{\mathcal{M}(\mathbb{C})} = f \cdot \Omega^2$ for some continuous function f on $\overline{\mathcal{M}}(\mathbb{C})$. Then f is bounded.*

Proof. Points on the boundary $\overline{\mathcal{M}}(\mathbb{C}) \setminus \mathcal{M}(\mathbb{C})$ are of the form (∞, z_2) or (z_1, ∞) or (∞, ∞) for z_1, z_2 in $\mathcal{E}ll(\mathbb{C})$. By compactness of $\overline{\mathcal{M}}(\mathbb{C})$, it suffices to show that the coefficients of Ω^2 are bounded away from 0 on any orbifold chart near any given boundary point. For points on $\mathcal{E}ll(\mathbb{C})$, we use the usual orbifold chart $\mathbb{H}_1 \rightarrow \mathcal{E}ll(\mathbb{C})$ with the upper half-plane $\mathbb{H}_1$. Near ∞ , we use the usual coordinate $q = e^{2\pi iz}$, for $|q| \ll 1$, which gives an orbifold chart near ∞ for q in a complex disc of radius $\ll 1$ equipped with the trivial action by $\mathbb{Z}/2\mathbb{Z}$. See for instance [Hai14, §4] for this orbifold description near the cusp ∞ .

On the above orbifold chart near a boundary point (∞, z_2) for $z_2 \neq \infty$, the form Ω^2 is a constant multiple of $(|q_1| \log |q_1|)^{-2} y_2^{-2} dq_1 \wedge d\bar{q}_1 \wedge dx_2 \wedge dy_2$, where $q_1 = e^{2\pi iz_1}$ is the usual local coordinate near ∞ , and where $x_2 + iy_2$ are coordinates in the upper-half plane. This expression for Ω^2 is clearly bounded away from 0 near $q_1 = 0$ and near (a lift in $\mathbb{H}_1$ of) a fixed z_2 . A similar argument applies near a boundary point (z_1, ∞) . For the boundary point (∞, ∞) , the form Ω^2 is a constant multiple of $(|q_1| \log |q_1|)^{-2} (|q_1| \log |q_1|)^{-2} dq_1 \wedge d\bar{q}_1 \wedge dq_2 \wedge d\bar{q}_2$, which is also bounded away from 0 for small q_1, q_2 . $\square$

9.4. Theta series associated to currents. Let $1 \leq r \leq 3$ be an integer. Consider the theta series associated to a Schwartz function $\varphi \in \mathcal{S}(\mathbb{V}^r)$, with ω the Weil representation from

$$\theta(g, h, \varphi) = \sum_{\mathbf{x} \in V^r} (\omega(g)\varphi)(h^{-1}\mathbf{x}).$$

For $T \in \text{Sym}_r(\mathbb{Q})$, define $\theta_T(g, h, \varphi) = \sum_{\mathbf{x} \in \Omega_T(V)} (\omega(g)\varphi)(h^{-1}\mathbf{x})$.

Now assume that φ is $O_r(\mathbb{R})$ -invariant, where $O_r(\mathbb{R}) \subseteq K_{r,\infty}$ is the diagonally embedded orthogonal group for the Gram matrix 1_r , with $K_{r,\infty} \subseteq \text{Sp}_{2r}(\mathbb{R})$ the standard maximal compact as in (3.1.3). For $g_z \in \text{Sp}(\mathbb{W}_r)(\mathbb{A})$ as in (3.1.4) where $z = \mathbf{x} + i\mathbf{y} \in \mathbb{H}_r$ and $\mathbf{y} = a \cdot {}^t a$, we define $\theta(z, h, \varphi) := \det(\mathbf{y})^{-1} \cdot \theta(g_z, h, \varphi)$ and for all $T \in \text{Sym}_r(\mathbb{Q})$, set

$$C_{\theta,T}(\mathbf{y}, h, \varphi) := \det(\mathbf{y})^{-1} \cdot \theta_T(g_z, h, \varphi) \cdot q^{-T}, \quad q^T = e^{2\pi i \text{tr}(Tz)}.$$

The $O_r(\mathbb{R})$ -invariance ensures that this does not depend on the choice of g_z . In particular, if φ is moreover factorizable as $\varphi = \varphi_\infty \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^r} \in \mathcal{S}(\mathbb{V}^r)$, then for $h = (h_\infty, e_f)$, we have

$$(9.4.1) \quad C_{\theta,T}(\mathbf{y}, h, \varphi) = \sum_{\mathbf{x} \in \Omega_T(V) \cap M_2(\mathbb{Z})^r} e^{2\pi i \text{tr}(T\mathbf{y})} \varphi_\infty(h_\infty^{-1}\mathbf{x} \cdot a).$$

For a matrix $T \in \text{Sym}_r(\mathbb{Q})$, a matrix $\mathbf{y} \in \text{Sym}_r(\mathbb{R})_{>0}$, a point $z \in \mathbb{D}$, and $\rho \in \mathbb{C}$ with $\text{Re}(\rho) > 0$, using the formula (9.2.3), definitions (9.2.2) and (7.6.3), and the $O(V)(\mathbb{R})$ -invariance of ν_r from [GS19, Proposition 2.6(d)] gives

$$(9.4.2) \quad \begin{aligned} (\mathfrak{g}^\circ(T, \mathbf{y}, \rho) \wedge \Omega^{3-r})_z &= \int_1^{+\infty} \nu^\circ(T, t\mathbf{y})_z \wedge \Omega_z^{3-r} \frac{dt}{t^{1+\rho}} \\ &= \int_1^{+\infty} C_{\theta,T}(t\mathbf{y}, h_z, \tilde{\nu}_r \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^r}) \frac{dt}{t^{1+\rho}} \cdot \Omega_z^2 \end{aligned}$$

for any $h_z \in O(V)(\mathbb{R})$ with $h_z \cdot (i, i) = z$. Note that the Schwartz function $\tilde{\nu}_r \in \mathcal{S}(V_\infty^r)$ is $O_r(\mathbb{R})$ -invariant by [GS19, Proposition 2.6(f)], and is $\text{Stab}_{O(V)(\mathbb{R})}(i, i)$ -invariant by [GS19, Proposition 2.6(d)].

9.5. Green current equation. Define

$$(9.5.1) \quad \varphi_{\text{KM}}^\circ(T, \mathbf{y}) = e^{2\pi i \text{tr}(T\mathbf{y})} \sum_{\mathbf{x} \in \Omega_T(V) \cap M_2(\mathbb{Z})^r} \varphi_{\text{KM}}(\mathbf{x} \cdot a).$$

If $T = (t)$ is a 1×1 matrix where $t \in \mathbb{Q}^\times$, it is proved that we have the following identity of currents on $M(\mathbb{C})$ [BK12a, Proposition 2.4]:

$$(9.5.2) \quad \frac{1}{2} \text{dd}^c \mathfrak{g}^\circ(T, \mathbf{y})_{\text{coarse}} + \delta_{T_t} = \frac{1}{2} \varphi_{\text{KM}}^\circ(T, \mathbf{y})_{\text{coarse}},$$

Where $\varphi_{\text{KM}}^\circ(T, \mathbf{y})_{\text{coarse}}$ is the current on the $M(\mathbb{C})$ defined by (9.2.9) without the $1/4$ factor. Similarly, $\text{dd}^c \mathfrak{g}^\circ(T, \mathbf{y})_{\text{coarse}}$ is defined on invariant forms on $\mathbb{D}$ with compact support modulo $\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$ as 4 times the stacky count defined in loc. cit. Recall that $T_t \subseteq M$ denotes the (reduced, scheme-theoretic) image of $\mathcal{T}_t \subseteq \mathcal{M}$ under the coarse moduli map $\mathcal{M} \rightarrow M$. This T_t is nonempty if and only if $t \in \mathbb{Z}_{>0}$. Acting on invariant forms on $\mathbb{D}$, we have $\delta_{T_t} = 4\delta_{\mathcal{T}_t} = 2\delta_{\mathcal{Z}(t)}$.

The formula (9.5.2) thus implies the following current identity on $\mathcal{M}(\mathbb{C})$:

$$(9.5.3) \quad \text{dd}^c \mathfrak{g}^\circ(T, \mathbf{y}) + \delta_{\mathcal{Z}(T)} = \varphi_{\text{KM}}^\circ(T, \mathbf{y}).$$

García and Sankaran give a much more general form of (9.5.3). We record the case relevant for this paper.

Lemma 9.5.1. *For $1 \leq r \leq 3$, let $T \in \text{Sym}_r(\mathbb{Q})$ be any matrix. The following equality of currents holds on $\mathcal{M}(\mathbb{C})$:*

$$(9.5.4) \quad \text{dd}^c \mathbf{g}^\circ(T, \mathbf{y}) + \delta_{\mathcal{Z}(T)} \wedge \Omega^{r-\text{rk}T} = \varphi_{\text{KM}}^\circ(T, \mathbf{y}).$$

Proof. The claimed equality of currents on $\mathcal{M}(\mathbb{C})$ can be checked locally on the orbifold chart $\mathbb{D} \rightarrow \mathcal{M}(\mathbb{C})$, via partitions of unity. Using the complex uniformization of $\mathcal{Z}(T)$ from (8.4.6), the claimed current identity now follows from the corresponding Green current identity on $\mathbb{D}$, which appears in (7.6.8). See also the proof of [GS19, Proposition 4.4]. $\square$

9.6. Arithmetic special cycle classes. For $1 \leq r \leq 3$, let $T \in \text{Sym}_r(\mathbb{Q})$ be a matrix and $\mathbf{y} \in \text{Sym}_r(\mathbb{R})$ be a positive definite matrix. Given such a pair $(T, \mathbf{y})$, we define a number $\det(\mathbf{y}_1) \in \mathbb{R}_{>0}$ as follows. Let $\alpha \in \text{GL}_r(\mathbb{Z})$ be a matrix such that

$$(9.6.1) \quad {}^t\alpha T \alpha = \begin{pmatrix} \mathbf{0}_{r-\text{rk}(T)} & \\ & T^\flat \end{pmatrix}$$

is block diagonal, where $T^\flat \in \text{Sym}_{\text{rk}(T)}(\mathbb{Q})$ is a nonsingular matrix. Such α are unique up to right multiplication by elements of the form

$$(9.6.2) \quad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

with block diagonal entries in $\text{GL}_{r-\text{rk}(T)}(\mathbb{Z})$ and $\text{GL}_{\text{rk}(T)}(\mathbb{Z})$, respectively. Therefore the following number is well-defined:

$$(9.6.3) \quad \det' T = \det T^\flat.$$

As elements in $Z^{\text{rk}(T)}(\mathcal{M})$, we also have

$$(9.6.4) \quad \mathcal{Z}(T) = \mathcal{Z}(T^\flat)$$

By Gram–Schmidt, we may decompose

$$(9.6.5) \quad \alpha^{-1} \mathbf{y}^t \alpha^{-1} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 & 0 \\ 0 & \mathbf{y}^\flat \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {}^t c & 1 \end{pmatrix}$$

for $c \in \text{M}_{r-\text{rk}(T), \text{rk}(T)}(\mathbb{R})$, and where $\mathbf{y}_1 \in \text{Sym}_{r-\text{rk}(T)}(\mathbb{R})$ and $\mathbf{y}^\flat \in \text{Sym}_{\text{rk}(T)}(\mathbb{R})$. The matrix c is uniquely determined by $\alpha^{-1} \mathbf{y}^t \alpha^{-1}$, and both $\mathbf{y}_1$ and $\mathbf{y}^\flat$ must be positive definite. The above discussion on uniqueness of α up to right translation by block upper triangular matrices shows that $\det(\mathbf{y}_1)$ and $\det(\mathbf{y}^\flat)$ are both independent of the choice of α . Given $(T, \mathbf{y})$, the $\text{GL}_{\text{rk}(T)}(\mathbb{Z})$ -orbit of the pair $(T^\flat, \mathbf{y}^\flat)$ is also well-defined, where $\gamma \in \text{GL}_{\text{rk}(T)}(\mathbb{Z})$ acts as $\gamma \cdot (T^\flat, \mathbf{y}^\flat) = ({}^t\gamma^{-1} T^\flat \gamma^{-1}, \gamma \mathbf{y}^\flat ({}^t\gamma))$.

By construction, we find that the quantities $\det(\mathbf{y}_1)$ and $\det(\mathbf{y}^\flat)$ do not change if the pair $(T, \mathbf{y})$ is replaced by $({}^t\alpha T \alpha, \alpha^{-1} \mathbf{y}^t \alpha^{-1})$ for any $\alpha \in \text{GL}_r(\mathbb{Z})$. The $\text{GL}_{\text{rk}(T)}(\mathbb{Z})$ -orbit of $(T^\flat, \mathbf{y}^\flat)$ also does not change. Moreover, the Fourier coefficient is also unchanged under the above operations.

Lemma 9.6.1. *Let $T \in \text{Sym}_r(\mathbb{Q})$ be a matrix and $y \in \text{Sym}_r(\mathbb{R})$ be a positive definite matrix. Let $\tilde{T}$ be the right hand side of (9.6.1). Let $\tilde{y} = \text{diag}\{y_1, y^b\}$ be the middle matrix on the right hand side of (9.6.5),*

$$(9.6.6) \quad C_T^r(y, s) = C_{\tilde{T}}^r(\tilde{y}, s)$$

Proof. It's easy to show that $C_T^r(y, s) = C_{\tilde{T}}^r(\alpha^{-1}y^t\alpha^{-1}, s)$, so it suffices to prove

$$C_{\tilde{T}}^r(\alpha^{-1}y^t\alpha^{-1}, s) = C_{\tilde{T}}^r(\tilde{y}, s).$$

By (9.6.5), an easy calculation implies that for the matrix $c \in M_{r-\text{rk}(T), \text{rk}(T)}(\mathbb{R})$,

$$g_{i\alpha^{-1}y^t\alpha^{-1}} = m(\theta)g_{i\tilde{y}}, \quad \theta = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \in \text{GL}_r(\mathbb{A}).$$

By [GS19, (5.56)], for all $\Phi \in I_r(s)$,

$$E_{\tilde{T}}(m(\theta)g, s, \Phi) = E_{\tilde{T}}(g, s, \Phi).$$

By the definition of the coefficient C_T , we have

$$C_{\tilde{T}}^r(\alpha^{-1}y^t\alpha^{-1}, s)e^{-2\pi\text{tr}(\tilde{T}\alpha^{-1}y^t\alpha^{-1})} = C_{\tilde{T}}^r(\tilde{y}, s)e^{-2\pi\text{tr}(\tilde{T}\tilde{y})}.$$

It's easy to verify that $\text{tr}(\tilde{T}\alpha^{-1}y^t\alpha^{-1}) = \text{tr}(\tilde{T}\tilde{y})$. Then (9.6.6) follows. $\square$

Definition 9.6.2. For $1 \leq r \leq 3$, let $T \in \text{Sym}_r(\mathbb{Q})$ be a matrix and $y \in \text{Sym}_r(\mathbb{R})$ be a positive definite matrix. Define the following current on $\mathcal{M}(\mathbb{C})$

$$(9.6.7) \quad \mathfrak{g}(T, y) := \begin{cases} \mathfrak{g}^\circ(T, y), & \text{if } \det(T) \neq 0; \\ \mathfrak{g}^\circ(T, y) - \log(\det y_1)\delta_{\mathcal{Z}(T)} \wedge \Omega^{r-\text{rk}(T)-1}, & \text{if } \det(T) = 0. \end{cases}$$

where $\det(y_1) \in \mathbb{R}_{>0}$ is associated to the pair (T, y) as in (9.6.5).

Remark 9.6.3. There is a similar modification of $\mathfrak{g}^\circ(T, y)$ which appears in [GS19, Definition 4.5], but our definition is not identical to theirs. Our modification of $\mathfrak{g}^\circ(T, y)$ is closer to the construction of [Che24, §12.4], but without the complications of loc. cit. involving class number issues.

Remark 9.6.4. The current $\delta_{\mathcal{Z}(T)} \wedge \Omega^{r-\text{rk}(T)-1}$ extends naturally to the compactification as follows. For $\eta \in A_c^{3-r, 3-r}(\overline{\mathcal{M}}(\mathbb{C}))$, define

$$\int_{\overline{\mathcal{M}}(\mathbb{C})} \left(\delta_{\mathcal{Z}(T)} \wedge \Omega^{r-\text{rk}(T)-1} \right) \wedge \eta := \int_{\mathcal{Z}(T)(\mathbb{C})} (\eta \wedge \Omega^{r-\text{rk}(T)-1}) \Big|_{\mathcal{Z}(T)(\mathbb{C})}.$$

The integral on the right is finite. Indeed, if T is positive semidefinite of rank 1, it is an integral over a finite union of modular curves, and convergence follows from their finite volume. If T is positive semidefinite of rank 2, it is a finite sum of values of the function η at CM point.

Therefore for $r = 3$, the current $\mathfrak{g}(T, y)$ extends to a current on $\overline{\mathcal{M}}(\mathbb{C})$ by our discussion in §9.3.

In the situation of Definition 9.6.2, the above discussion shows that we also have $\mathfrak{g}(^t\gamma T \gamma, y) = \mathfrak{g}(T, \gamma y^t \gamma)$ for any $\gamma \in \text{GL}_r(\mathbb{Z})$. Note that $\mathfrak{g}(T, y) = \mathfrak{g}^\circ(T, y)$ if $\det T \neq 0$. Moreover, the following green current equation holds on $\mathcal{M}(\mathbb{C})$,

$$(9.6.8) \quad \text{dd}^c \mathfrak{g}(T, y) + \delta_{\mathcal{Z}(T)} \wedge \Omega^{r-\text{rk}(T)} = \varphi_{\text{KM}}^\circ(T, y).$$

With notation as in Section 9.2, we define an arithmetic special cycle class $[\widehat{\mathcal{Z}}(T, y)] \in \widehat{\text{Ch}}^3(\mathcal{M})$ by

(9.6.9)

$$[\widehat{\mathcal{Z}}(T, y)] := \begin{cases} (\mathcal{Z}(T^b), \mathfrak{g}(T^b, y^b)) \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{r-\text{rk}(T)} + (0, \mathfrak{g}(T, y) - \mathfrak{g}(T^b, y^b) \wedge \Omega^{r-\text{rk}(T)}), & \text{if } T \neq \mathbf{0}_r \\ \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^r + (0, \mathfrak{g}(\mathbf{0}_r, y)), & \text{if } T = \mathbf{0}_r, \end{cases}$$

for the pair (T, y) . This construction (9.6.9) is essentially [GS19, (5.158)], except we have the additional complication that $\mathcal{M}$ is not proper, and also an additional complication from stackiness of $\mathcal{M}$. There is also the discrepancy from Remark 9.6.3. Due to the local complete intersection property for the special cycles $\mathcal{Z}(T)$ as explained in Lemma 8.2.2 and its proof, we do not make the additional derived modifications discussed in [Che24, §4.4].

To remedy the non-properness, it should be preferable to work with $\overline{\mathcal{M}}$ instead of $\mathcal{M}$, and to replace the Gillet–Soulé arithmetic intersection theory with a version of the theory from [BGKK07]. On the other hand, one can formulate the arithmetic intersection numbers more directly, as we explain next. We take this simpler approach, as the technical issues in defining the arithmetic cycle classes on the compactified stack $\overline{\mathcal{M}}$ are largely separate from the main ideas of this paper.

9.7. Arithmetic degrees of special cycles. For regular proper Deligne–Mumford stacks, there is a well-defined notion of arithmetic degree on the top-codimension arithmetic Chow group [Gil09]. However, the stack $\mathcal{M}$ is not proper over $\mathbb{Z}$. Although the line bundle underlying the Hermitian line bundle $\widehat{\mathcal{E}}^\vee$ extends to $\overline{\mathcal{M}}$, its metric has logarithmic singularities along the boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$, while its first Chern form has log-log growth [BGKK07, (7.60)]. Thus it seems natural to expect that the arithmetic special cycles should define classes in arithmetic Chow groups allowing log-log singularities along the boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$, though there is a technical complication due to stackiness. Moreover, by the analysis in [BK12a, Proposition 2.16], Kudla’s Green function $\mathfrak{g}^\circ(T, y)$ also fails to be a Green function with log-log singularities in the sense of Burgos–Kramer–Kühn [BGKK07].

Nonetheless, one can make a precise definition for the expected arithmetic degrees of the cycles $\widehat{\mathcal{Z}}(T, y)$, for example as in [Che24, §4.7].

If $\text{rk}(T) = 3$, the special cycle $\mathcal{Z}(T)$ is an Artinian Deligne–Mumford stack supported over at most one prime. Let’s say $\mathcal{Z}(T)$ is supported over p , define

$$\widehat{\deg} \left(\left(\widehat{\mathcal{E}}^\vee \right)^0 \Big|_{\mathcal{Z}(T)} \right) = \log p \cdot (\text{the stacky length of the Artinian Deligne–Mumford stack } \mathcal{Z}(T))$$

More precisely, choose a finite étale cover $Z \rightarrow \mathcal{Z}(T)$ of constant degree d , where Z is a scheme. The stacky length is the $\mathbb{Z}$ -length of Z divided by d , and the arithmetic degree is obtained by multiplying this quantity by $\log p$.

Next consider the case $\text{rk}(T) = 2$. By Lemma 8.2.2, we know that $\mathcal{Z}(T) \rightarrow \text{Spec } \mathbb{Z}$ is proper quasi-finite flat with étale generic fiber. The number $\widehat{\deg}((\widehat{\mathcal{E}}^\vee)|_{\mathcal{Z}(T)})$ was defined in Section 9.1.

If $\text{rk}(T) = 1$, let $T^b = (t)$ be the matrix obtained in (9.6.1). In [BGKK07, §7.8], the authors showed that the metrized tautological bundle on the coarse space M (in the sense of loc. cit.) extends to a Hermitian $\mathbb{Q}$ -line bundle with *good metric* (we refer to [BGKK07, Definition 7.4] for

the notion of *good metric*) along $\overline{M} \setminus M$. Denote by $\widehat{\mathcal{E}}_{\overline{M}}^{\vee}$ this metrized $\mathbb{Q}$ -line bundle on $\overline{M}$. Define

$$(9.7.1) \quad \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^{\vee} \right)^2 \big|_{\mathcal{Z}(T)} \right) = 2^{-1} \cdot \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}}^{\vee} \right)^2 \big|_{\overline{T}_t} \right).$$

The factor 2^{-1} is inserted because we divide by 4 for stacky considerations, and then multiply by 2 since $\mathcal{Z}(T) \rightarrow \mathcal{T}_t$ has degree 2 as in Example 8.3.1.

Finally if $\text{rk}(T) = 0$, we have $\mathcal{Z}(T) = \mathcal{M}$. Define

$$(9.7.2) \quad \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^{\vee} \right)^3 \big|_{\mathcal{Z}(\mathbf{0}_r)} \right) = 4^{-1} \cdot \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}}^{\vee} \right)^3 \big|_{\overline{M}} \right).$$

The factor 4^{-1} is inserted for stacky considerations.

Motivated by the (9.6.9) on the expected arithmetic special cycle class $\widehat{\mathcal{Z}}(T, y)$, define

$$(9.7.3) \quad \widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] := \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^{\vee} \right)^{3-\text{rk}(T)} \big|_{\mathcal{Z}(T)} \right) + \frac{1}{2} \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}(T, y) \wedge \Omega^{3-r}.$$

Since $\mathcal{M}(\mathbb{C})$ is noncompact, convergence of the integral is not automatic. For $r = 3$, however, convergence follows from the estimates in Appendix C, as discussed in §9.3. Thus the arithmetic degree in (9.7.3) is well defined in this case. Our main theorem explains that these arithmetic degrees appear in the Fourier coefficients of the classical genus 3 Siegel Eisenstein series.

Theorem 9.7.1. *For every $T \in \text{Sym}_3(\mathbb{Q})$ and $y \in \text{Sym}_3(\mathbb{R})_{>0}$, we have*

$$(9.7.4) \quad \widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = -(32\pi^2)^{-1} (C_T^3)'(y, 0).$$

The proof will be completed in Section 14.

10. FALTINGS HEIGHTS OF CORANK 1 CYCLES

For a matrix T of rank 1 or 3, the height of the corresponding cycle $\mathcal{Z}(T)$ is more or less known by [BGKK07, Theorem 7.61] (rank 1) and [GK93] (rank 3). In this paper, we handle the rank 2 case, i.e. when $\mathcal{Z}(T)$ is a 1-cycle.

10.1. Faltings heights of CM elliptic curves. For a rank 2 matrix T , recall (Lemma 8.2.2) that every object in $\mathcal{Z}(T)(\overline{\mathbb{Q}})$ consists of a pair of CM elliptic curves (and additional data). The right-hand side of (9.7.3) involves the Faltings heights of these two elliptic curves, which have CM by possibly non-maximal orders. This section contains some preparatory material for describing these heights. The procedure from [Che24, Part 3] for studying “height differences” can be simplified for the case of Faltings heights of elliptic curves, as we explain below.

If E is any elliptic curve over a number field K , and if E has everywhere semi-abelian reduction, recall that the stable Faltings height

$$(10.1.1) \quad h_{\text{Fal}}(E) := \frac{1}{[K : \mathbb{Q}]} \widehat{\deg}(\widehat{\omega}_{E, \text{Pet}}) \in \mathbb{R}$$

is a normalized arithmetic degree of the Hodge bundle $\widehat{\omega}_{E, \text{Pet}} := e^* \Omega_{\tilde{E}/\mathcal{O}_K}^1$ for the Néron model $\tilde{E}$ of E over $\text{Spec } \mathcal{O}_K$, with e being the identity section. Here we normalize our Archimedean metrics using the $\|-\|_{\text{Pet}}$ normalization from (8.1.1). The quantity $h_{\text{Fal}}(E)$ does not change if we extend the base field K . We always consider stable Faltings height.

Remark 10.1.1. It will be convenient notation to define the (stable) Faltings height $h_{\text{Fal}}(E)$ for elliptic curves E over arbitrary fields κ of characteristic zero, subject to the condition that $E_{\bar{\kappa}}$ descends to $\bar{\mathbb{Q}}$. Indeed, we can pick any descent of $E_{\bar{\kappa}}$ to a number field (sufficiently large to ensure everywhere semi-abelian reduction) and take its Faltings height. This will not depend on the choice of descent. In particular, this applies to any CM elliptic curve in characteristic zero, which always descends to $\bar{\mathbb{Q}}$ in the sense above.

Next, let $F/\mathbb{Q}$ be an arbitrary imaginary quadratic field. If E is an elliptic curve with CM by the maximal order $\mathcal{O}_F$, then we have

$$(10.1.2) \quad h_{\text{Fal}}(E) = h_{\text{Fal},F}^{\text{CM}} := \frac{1}{2} \frac{L'(1, \eta_F)}{L(1, \eta_F)} + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)} + \frac{1}{4} \log |D| - \frac{1}{2} \log(2\pi)$$

from the Chowla–Selberg formula (compare [KRY04, Proposition 10.10]), where η_F is the quadratic character associated to $F/\mathbb{Q}$, and D is the discriminant of $\mathcal{O}_F$.

If E instead has CM by a possibly non-maximal order, the Faltings height was largely computed in the work of Nakkajima and Taguchi [NT91]. We give a reformulation of their result in a more convenient form for us. The height will be given in terms of (10.1.2) and some explicit local quantities as in Lemma 10.1.4.

We first recall the following “height difference” formula, which is a (not difficult) reformulation of [Fal83, Lemma 5]. Given any isogeny $\phi: E_1 \rightarrow E_2$ of arbitrary elliptic curves with semi-abelian reduction over a number field K , we have

$$(10.1.3) \quad h_{\text{Fal}}(E_2) - h_{\text{Fal}}(E_1) = \frac{1}{2} \log \deg \phi - \frac{1}{[K : \mathbb{Q}]} \log |\text{coker}(\phi_*: \text{Lie } \tilde{E}_1 \rightarrow \text{Lie } \tilde{E}_2)|$$

where $\tilde{E}_i$ are the respective Néron models over $\text{Spec } \mathcal{O}_K$, and where $|\cdot|$ means cardinality of the finite $\mathcal{O}_K$ -module (and this is also valid for abelian varieties).

In particular, if E is any elliptic curve with CM by a (possibly non-maximal) order in F , we have

$$(10.1.4) \quad h_{\text{Fal}}(E) - h_{\text{Fal},F}^{\text{CM}} \in \sum_{\text{primes } p} \mathbb{Q} \cdot \log p$$

because any two elliptic curves with CM by F are isogenous over an algebraic closure. Note also that $\text{End}(E)$ does not change with extension of the base field, if E has CM.

Although the quantity $\deg \phi$ appears as a measure of “Archimedean ramification”, we will eventually view the quantities $\log \deg \phi$ as contributing to “non-Archimedean terms” which can be calculated by p -divisible groups as p varies.

In light of (10.1.4), we work locally, one prime at a time. We set $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for orders $\mathcal{O} \subseteq \mathcal{O}_F$, and also set $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Lemma 10.1.2. *Let E and E' be elliptic curves in characteristic zero, with CM by orders $\mathcal{O}$ and $\mathcal{O}'$ in the same imaginary quadratic field F . If p is a prime such that $\mathcal{O}_p \cong \mathcal{O}'_p$, then we have*

$$(10.1.5) \quad h_{\text{Fal}}(E) - h_{\text{Fal}}(E') \in \sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell.$$

In particular, if $\mathcal{O} \cong \mathcal{O}'$, we have $h_{\text{Fal}}(E) = h_{\text{Fal}}(E')$.

Proof. By (10.1.3), it is enough to show the existence of a prime-to- p isogeny $E \rightarrow E'$. Without loss of generality, we may assume that E and E' are given over $\mathbb{C}$, as any isogeny over $\mathbb{C}$ will descend to $\overline{\mathbb{Q}}$ (for any descents of E and E').⁷

After picking appropriate embeddings $F \rightarrow \text{End}^\circ(E)$ and $F \rightarrow \text{End}^\circ(E')$, we may assume that E and E' have the same CM type. We can assume $E \cong \mathbb{C}/\mathfrak{a}$ and $E' \cong \mathbb{C}/\mathfrak{a}'$ as complex tori, where $\mathfrak{a}, \mathfrak{a}' \subseteq F$ are $\mathbb{Z}$ -lattices of rank 2 which satisfy $\text{End}_F(\mathfrak{a}) = \mathcal{O}$ and $\text{End}_F(\mathfrak{a}') = \mathcal{O}'$. Here we set e.g. $\text{End}_F(\mathfrak{a}) := \text{End}_{\mathbb{Z}}(\mathfrak{a}) \cap \text{End}_F(\mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Q})$.

Set $\mathfrak{a}_p := \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathfrak{a}'_p := \mathfrak{a}' \otimes_{\mathbb{Z}} \mathbb{Z}_p$. One can show that both $\mathfrak{a}_p$ and $\mathfrak{a}'_p$ are free of rank one as $\mathcal{O}_p$ -modules. Briefly, one can find an element $u \in F_p^\times$ such that $u \cdot \mathfrak{a}_p \subseteq \mathcal{O}_{F,p}$ and $1 \in u \cdot \mathfrak{a}_p$, and then if $s \in \mathbb{Z}_{\geq 0}$ is the integer such that $[\mathcal{O}_{F,p} : \mathcal{O}_p] = p^s$, we must have $u \cdot \mathfrak{a}_p = \mathbb{Z}_p + p^s \mathcal{O}_{F,p}$ being equal to the unique order of index p^s (and the same applies to $\mathfrak{a}'_p$).

In other words, we can find some $b \in F_p^\times$ such that $b \cdot \mathfrak{a}_p = \mathfrak{a}'_p$, as submodules of F_p . Since $F^\times$ is dense in $F_p^\times$, we can moreover assume $b \in F^\times$. Then multiplication by b gives a prime-to- p quasi-isogeny $E \rightarrow E'$ (which can be assumed an isogeny if one prefers, by clearing denominators away from p). $\square$

Remark 10.1.3. For the reader's convenience, we remark that the condition $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ from Lemma 10.1.2 is equivalent to the condition $v_p([\mathcal{O}_F : \mathcal{O}]) = v_p([\mathcal{O}_F : \mathcal{O}'])$, because there is a unique order of index p^s inside $\mathcal{O}_{F,p}$ for each integer $s \geq 0$, namely $\mathbb{Z}_p + p^s \mathcal{O}_{F,p}$, and these are all of the orders.

Next, we use the notation

$$(10.1.6) \quad \eta_F^*(p) := \begin{cases} 1 & \text{if } p \text{ is split in } \mathcal{O}_F \\ 0 & \text{if } p \text{ is ramified in } \mathcal{O}_F \\ -1 & \text{if } p \text{ is inert in } \mathcal{O}_F \end{cases}$$

and set

$$(10.1.7) \quad \delta_{\text{Fal}, F_p}(s) := \frac{1}{2} \left(s - \frac{(1 - p^{-s})(1 - \eta_F^*(p))}{(1 - p^{-1})(p - \eta_F^*(p))} \right)$$

for integers $s \geq 0$. Note that this only depends on the isomorphism class of the $\mathbb{Q}_p$ -algebra $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Lemma 10.1.4. *Let E and E_0 be elliptic curves in characteristic zero, with CM by the same imaginary quadratic field F . Write $\mathcal{O}$ for the endomorphism ring of E , and assume E_0 has CM by the maximal order $\mathcal{O}_F$.*

Fix any prime p , and let $s \in \mathbb{Z}_{\geq 0}$ be the integer such that $\mathcal{O}_p$ is the unique order of index p^s inside $\mathcal{O}_{F,p}$. We then have

$$(10.1.8) \quad h_{\text{Fal}}(E) - h_{\text{Fal}}(E_0) = \delta_{\text{Fal}, F_p}(s) \cdot \log p \pmod{\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell}.$$

⁷This type of argument also shows that any characteristic zero abelian varieties A and A' with CM by the same maximal order in a finite étale $\mathbb{Q}$ -algebra $\tilde{F}$ (not necessarily a product of CM fields, but necessarily totally complex) and the same CM type (for fixed embeddings $\tilde{F} \rightarrow \text{End}^0(A)$ and $\tilde{F} \rightarrow \text{End}^0(A')$) must have the same (stable) Faltings height, i.e. there exist prime-to- p isogenies (which can be chosen to be $\tilde{F}$ -linear) between A and A' for every prime p .

Proof. By Lemma 10.1.2, we may assume that E and E_0 are the complex elliptic curves $\mathbb{C}/\mathcal{O}$ and $\mathbb{C}/\mathcal{O}_F$ respectively, for some embedding $F \rightarrow \mathbb{C}$. The inclusion $\mathcal{O} \rightarrow \mathcal{O}_F$ gives an isogeny $E \rightarrow E_0$ whose degree has p -part being p^s . Its dual $E_0 \rightarrow E$ is an isogeny degree whose p -part is p^s . Descend the data of $E_0 \rightarrow E$ and the actions $\mathcal{O}_F \subset E_0$ and $\mathcal{O} \subset E$ to some number field K , and we may assume that E_0 and E have everywhere good reduction over K .

The computation of Nakkajima and Taguchi [NT91] (see also [KRY04, Proposition 10.3] and its proof) can be formulated as follows: for any isogeny $\phi: E_{0,\check{K}} \rightarrow E_{\check{K}}$ over a finite extension $\check{K}$ of $\mathbb{Q}_p$, if the degree of ϕ has p -part p^s , then

$$(10.1.9) \quad \frac{1}{[\check{K} : \mathbb{Q}_p]} \text{length}_{\mathcal{O}_{\check{K}}}(\text{coker}(\phi_*: \text{Lie } \tilde{E}_{0,\text{Spec } \mathcal{O}_{\check{K}}} \rightarrow \text{Lie } \tilde{E}_{\text{Spec } \mathcal{O}_{\check{K}}})) = \frac{1}{2} \frac{(1 - p^{-s})(1 - \eta_F^*(p))}{(1 - p^{-1})(p - \eta_F^*(p))}$$

for the Néron models $\tilde{E}_0$ and $\tilde{E}$ over $\mathcal{O}_{\check{K}}$. See also [Che24, (7.2.6)], stated in terms of p -divisible groups. Upon summing over the p -adic places of K , it follows that any isogeny $\phi: E_0 \rightarrow E$ (over K) with $\deg \phi$ having p -part p^s has $\frac{1}{[K:\mathbb{Q}]} v_p(|\text{coker}(\phi_*: \text{Lie } \tilde{E}_0 \rightarrow \text{Lie } \tilde{E})|)$ equal to the right-hand side of (10.1.9); then apply (10.1.3). $\square$

The following lemma will help detect CM orders using p -divisible groups (after using the equivalence between étale p -divisible groups and Tate modules). The point is that we will apply the lemma in situations where the base field is not a number field.

Lemma 10.1.5. *Let E be an elliptic curve over a field of characteristic 0. Assume E has CM by an imaginary quadratic field F , and set $\mathcal{O} := \text{End}(E)$.*

Fix a prime p . Suppose we have $\dim_{\mathbb{Z}_p} \text{End}(T_p(E)) = 2$, where $\text{End}(T_p(E))$ denotes Galois equivariant endomorphisms of the p -adic Tate module.

Then we have $\mathcal{O}_p \cong \text{End}(T_p(E))$.

Proof. Recall that the canonical map $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow \text{End}(T_p(E))$ is injective [Mum85, Theorem 3, page 176]. Dimension considerations then show that the map $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}_p \rightarrow \text{End}(T_p(E)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism. We find that an element $\phi \in \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ maps to $\text{End}(T_p(E))$ (equivalently, preserves the p -adic Tate module) if and only if $\phi \in \text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, as can be shown by clearing denominators away from p . Since $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ has dense image in $\text{End}(T_p(E)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, we conclude that $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \rightarrow \text{End}(T_p(E))$ also has dense image, so $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \text{End}(T_p(E))$ is closed (by compactness) and dense and hence the inclusion is an equality. $\square$

10.2. Decomposition of heights. In Section 10.2, we consider $T \in \text{Sym}_r(\mathbb{Q})$ with $\text{rk}(T) = 2$ (so either $r = 2$ or $r = 3$). We decompose the term $\widehat{\deg}(\hat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)})$ over the prime numbers. For the reader's convenience, we recall that the metric $\|\cdot\|$ on $\mathcal{E}^\vee$ defined in Section 8.1 is different from Petersson metric $\|\cdot\|_{\text{Pet}}$ also defined in loc. cit.: the two metrics on $\mathcal{E}^\vee$ are related by

$$(10.2.1) \quad \|\cdot\|^2 = 4\pi e^\gamma \|\cdot\|_{\text{Pet}}^2.$$

Recall the definition $\hat{\mathcal{E}}_{\text{Pet}} = p_1^* \hat{\omega}_{\mathcal{A},\text{Pet}} \otimes p_2^* \hat{\omega}_{\mathcal{A},\text{Pet}}$ from Section 8.1. The (stacky) arithmetic degree $\widehat{\deg}(\hat{\mathcal{E}}_{\text{Pet}}^\vee|_{\mathcal{Z}(T)})$ of the Hermitian line bundle $\hat{\mathcal{E}}^\vee$ on $\mathcal{Z}(T)$ (as defined in [Che24, Section 4.1]) may be

calculated by the formula

$$(10.2.2) \quad \widehat{\deg}(\widehat{\mathcal{E}}_{\text{Pet}}^{\vee}|_{\mathcal{Z}(T)}) = - \sum_{\substack{\alpha \in |\mathcal{Z}(T)(\kappa)| \\ \alpha = (E, E')}} \frac{h_{\text{Fal}}(E) + h_{\text{Fal}}(E')}{\# \text{Aut}(\alpha)}$$

for any algebraically closed field κ of characteristic zero, where the sum runs over the set of isomorphism classes in the groupoid $\mathcal{Z}(T)(\kappa)$. The formula (10.2.2) may be verified by covering $\mathcal{Z}(T)$ by a disjoint union of finite morphisms from schemes of the form $\text{Spec } \mathcal{O}_K$ for various number fields K . The well-definedness of the heights $h_{\text{Fal}}(E)$ and $h_{\text{Fal}}(E')$ even if $\kappa \neq \overline{\mathbb{Q}}$ follows because Lemma 8.2.2 states that the condition $(E, E', \mathbf{x}) \in \mathcal{Z}(T)(\kappa)$ implies that E and E' both have CM by the imaginary quadratic field $\mathbb{Q}(\sqrt{-\det' T})$; then recall that any CM elliptic curve descends to $\overline{\mathbb{Q}}$ and apply the discussion of Remark 10.1.1. The use of $\kappa \neq \overline{\mathbb{Q}}$ will largely be a matter of convenient bookkeeping later. For the rest of Section 10.2, we set $F := \mathbb{Q}(\sqrt{-\det' T})$.

Recall that we consider the stacky degree $\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}})$ of the generic fiber $\mathcal{Z}(T)_{\mathbb{Q}} \rightarrow \text{Spec } \mathbb{Q}$, then we must have

$$(10.2.3) \quad \widehat{\deg}(\widehat{\mathcal{E}}_{\text{Pet}}^{\vee}|_{\mathcal{Z}(T)}) + 2(\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}}))(h_{\text{Fal}, F}^{\text{CM}}) \in \sum_{\text{primes } p} \mathbb{Q} \cdot \log p$$

via (10.1.4). We define

$$(10.2.4) \quad \widehat{\deg}_{\widehat{\mathcal{E}}^{\vee}}(\mathcal{Z}(T))_p \in \mathbb{Q} \cdot \log p$$

to be the “ p -part” of (10.2.3), defined by linear independence of $\log p$ for varying primes p . Given $\alpha \in \mathcal{Z}(T)(\kappa)$ as in (10.2.2), we define $\delta_{\text{Fal}, p}(\alpha) \in \mathbb{Q} \cdot \log p$ by the formulas

$$(10.2.5) \quad h_{\text{Fal}}(E) + h_{\text{Fal}}(E') - 2h_{\text{Fal}, F}^{\text{CM}} = \sum_p \delta_{\text{Fal}, p}(\alpha) \quad \text{with} \quad \delta_{\text{Fal}, p}(\alpha) \in \mathbb{Q} \cdot \log p$$

and linear independence of $\log p$ for varying p . We then have

$$(10.2.6) \quad \widehat{\deg}_{\widehat{\mathcal{E}}^{\vee}}(\mathcal{Z}(T))_p = - \sum_{\substack{\alpha \in |\mathcal{Z}(T)(\kappa)| \\ \alpha = (E, E')}} \frac{\delta_{\text{Fal}, p}(\alpha)}{\# \text{Aut}(\alpha)}.$$

The point is that $\widehat{\deg}_{\widehat{\mathcal{E}}^{\vee}}(\mathcal{Z}(T))_p$ will be calculated via a local analogue (Definition 11.6.1) in combination with Rapoport–Zink uniformization (Section 12).

11. LOCAL SPECIAL CYCLES

We fix an arbitrary prime p throughout this section, allowing $p = 2$. Let $\mathbb{X}^s$ (resp. $\mathbb{X}^o$) be a supersingular (resp. ordinary) p -divisible group of dimension 1 and height 2 over $\mathbb{F}$. Given a choice of $*$ in $\{s, o\}$, the object $\mathbb{X}^*$ is unique up to isomorphism.

We set

$$(11.0.1) \quad \mathbb{B}^* := \text{Hom}^{\circ}(\mathbb{X}^*, \mathbb{X}^*).$$

Fix any principal polarization λ on $\mathbb{X}^*$ (such λ is unique up to $\mathbb{Z}_p^{\times}$ -scalar). There is an induced Rosati involution $\dagger$ on $\mathbb{B}^*$ given by $x^{\dagger} := \lambda^{-1} x^{\vee} \lambda$ (which does not depend on the choice of λ). This induces a quadratic form $q: \mathbb{B}^* \rightarrow \mathbb{Q}_p$ given by $q(x) := x^{\dagger} x$, with associated $\mathbb{Q}_p$ -bilinear pairing

$(-, -)$ on $\mathbb{B}^*$ given by the formulas

$$(11.0.2) \quad (x, y) := \mathrm{tr}^\circ(x^\dagger \circ y) \quad \mathrm{tr}^\circ(a) := a^\dagger + a$$

for $x, y, a \in \mathbb{B}^*$. More explicitly, the pair $(\mathbb{B}^*, \dagger)$ is isomorphic (as a $\mathbb{Q}_p$ -algebra with involution) to the unique (4-dimensional) quaternion division algebra over $\mathbb{Q}_p$ with its unique standard involution if $* = s$, and is isomorphic to $\mathbb{Q}_p \times \mathbb{Q}_p$ with its unique nontrivial involution if $* = o$.

For each $* \in \{s, o\}$, we have a canonical full rank $\mathbb{Z}_p$ -lattice

$$(11.0.3) \quad \mathrm{Hom}(\mathbb{X}^*, \mathbb{X}^*) \subseteq \mathrm{Hom}^\circ(\mathbb{X}^*, \mathbb{X}^*)$$

which we denote $\mathbb{L}^* \subseteq \mathbb{B}^*$.

We find it useful to consider “schemification” of locally Noetherian formal schemes which are finite relative schemes over e.g. $\mathrm{Spf} \check{\mathbb{Z}}_p$ (compare [Che24, Appendix B]), where $\check{\mathbb{Z}}_p$ is the discrete valuation ring given by the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$. That is, the association $\mathrm{Spec} B \mapsto \mathrm{Spf} B$ gives an equivalence of categories of finite relative schemes over $\mathrm{Spec} \check{\mathbb{Z}}_p$ and $\mathrm{Spf} \check{\mathbb{Z}}_p$; we write $(\mathrm{Spf} B)^{\mathrm{sch}} := \mathrm{Spec} B$ and call this the “schemification” of $\mathrm{Spf} B$.

11.1. Rapoport–Zink spaces. For $* \in \{s, o\}$, we first consider the functor $\mathcal{N}_0^*: (\mathrm{Sch}/\mathrm{Spf} \check{\mathbb{Z}}_p)^{\mathrm{op}} \rightarrow \mathrm{Set}$ defined as follows: given a scheme S over $\mathrm{Spf} \check{\mathbb{Z}}_p$, the set $\mathcal{N}_0^*(S)$ parameterizes data (X, ρ) where X/S is a p -divisible group and $\rho: X \times_S \bar{S} \rightarrow \mathbb{X}^* \times_{\mathbb{F}} \bar{S}$ is a quasi-isogeny which is an isomorphism in geometric fibers, with $\bar{S} := S \times_{\mathrm{Spf} \check{\mathbb{Z}}_p} \mathbb{F}$. It is a standard fact that $\mathcal{N}_0^*$ is (non-canonically) represented by $\mathrm{Spf} \check{\mathbb{Z}}_p[[t]]$ (take any lift to $\mathrm{Spf} \check{\mathbb{Z}}_p$, e.g. a suitable canonical lifting in the sense of Gross, and then apply Grothendieck–Messing theory).

We next consider some Rapoport–Zink spaces

$$(11.1.1) \quad \mathcal{N}^{*,0} \subseteq \mathcal{N}^* \subseteq \tilde{\mathcal{N}}^*$$

which are slight variants of one another, and whose uses will be explained shortly. Each of the spaces in (11.1.1) will be (non-canonically) a countable disjoint union of copies of $\mathrm{Spf} \check{\mathbb{Z}}_p[[t, t']]$.

We first define $\tilde{\mathcal{N}}^*: (\mathrm{Sch}/\mathrm{Spf} \check{\mathbb{Z}}_p)^{\mathrm{op}} \rightarrow \mathrm{Set}$ to parameterize tuples (X, ρ, X', ρ') with (X, ρ) and (X', ρ') data as above, but where ρ and ρ' are no longer required to be isomorphisms in geometric fibers, and we instead require $\deg(\rho) = \deg(\rho')$.

We let $\mathcal{N}^* \subseteq \tilde{\mathcal{N}}^*$ be the open and closed subfunctor consisting of tuples (X, ρ, X', ρ') with ρ being an isomorphism in geometric fibers.

We let $\mathcal{N}^{*,0} \subseteq \tilde{\mathcal{N}}^*$ be the open and closed subfunctor consisting of tuples (X, ρ, X', ρ') with both ρ and ρ' being isomorphisms in geometric fibers. There is a canonical isomorphism $\mathcal{N}^{*,0} \rightarrow \mathcal{N}_0^* \times_{\mathrm{Spf} \check{\mathbb{Z}}_p} \mathcal{N}_0^*$.

We find that the (disconnected) space $\tilde{\mathcal{N}}^*$ arises naturally in Rapoport–Zink uniformization. On the other hand, the natural local special cycles on $\tilde{\mathcal{N}}^*$ (Definition 11.2.1) are not quasi-compact in general, so we find it convenient to restrict to $\mathcal{N}^*$ for counting purposes. We have $\mathcal{N}^{*,0} = \mathcal{N}^*$ in the supersingular case, but not in the ordinary case. The connected space $\mathcal{N}^{*,0}$ seems to be “too small” for our counting purposes in the ordinary case, though we still find it useful in some arguments.

Consider groups

$$(11.1.2) \quad H'(\mathbb{Q}_p) := \{(h, h') \in \text{End}^\circ(\mathbb{X}^*)^\times \times \text{End}^\circ(\mathbb{X}^*)^\times : \deg(h) = \deg(h')\}$$

$$(11.1.3) \quad H'_0(\mathbb{Q}_p) := \{(h, h') \in H'(\mathbb{Q}_p) : h \in \text{End}(\mathbb{X}^*)^\times\}$$

$$(11.1.4) \quad H'(\mathbb{Z}_p) := \{(h, h') \in H'(\mathbb{Q}_p) : h \in \text{End}(\mathbb{X}^*)^\times \text{ and } h' \in \text{End}(\mathbb{X}^*)^\times\}$$

depending on the choice of $*$ in $\{s, o\}$, which we suppress from notation.

There is an action $H'(\mathbb{Q}_p) \curvearrowright \tilde{\mathcal{N}}^*$, where $(h, h') \in H'(\mathbb{Q}_p)$ acts as $(X, \rho, X', \rho') \mapsto (X, h\rho, X', h'\rho')$. The subgroups $H'(\mathbb{Z}_p) \subseteq H'(\mathbb{Q}_p)$ and $H'_0(\mathbb{Q}_p) \subseteq H'(\mathbb{Q}_p)$ are normal, and the action $H'(\mathbb{Q}_p) \curvearrowright \tilde{\mathcal{N}}^*$ induces a simply transitive action $H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p) \curvearrowright \tilde{\mathcal{N}}^*(\mathbb{F})$. We pick the identification $H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p) \cong \tilde{\mathcal{N}}^*(\mathbb{F})$ sending $1 \in H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p)$ to $\mathcal{N}^{*,0}(\mathbb{F})$.

We have an open and closed disjoint union decomposition

$$(11.1.5) \quad \tilde{\mathcal{N}}^* = \coprod_{i \in H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p)} \tilde{\mathcal{N}}_i^*$$

where $\tilde{\mathcal{N}}_i^* \subseteq \tilde{\mathcal{N}}^*$ is the open and closed subfunctor with $\tilde{\mathcal{N}}_i^*(\mathbb{F}) \subseteq \tilde{\mathcal{N}}^*(\mathbb{F}) \cong H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p)$ given by the $H'_0(\mathbb{Q}_p)$ -orbit corresponding to i . When $i = 1$, we have $\tilde{\mathcal{N}}_i^* = \mathcal{N}^*$.

We have

$$(11.1.6) \quad \begin{aligned} H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p) &\cong \begin{cases} \mathbb{Z} & \text{in the supersingular case} \\ \mathbb{Z}^3 & \text{in the ordinary case} \end{cases} & H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p) &\cong \mathbb{Z}^2 \\ H'_0(\mathbb{Q}_p)/H'(\mathbb{Z}_p) &\cong \begin{cases} \{0\} & \text{in the supersingular case} \\ \mathbb{Z} & \text{in the ordinary case.} \end{cases} \end{aligned}$$

11.2. Cycles.

Definition 11.2.1. Given any subset $H \subset \mathbb{B}^*$, we define the *special cycle* $\tilde{\mathcal{Z}}^*(H) \subset \tilde{\mathcal{N}}^*$ to be the subfunctor cut out by the condition

$$(\rho')^{-1} \circ x \circ \rho \in \text{Hom}(X, X')$$

for all $x \in H$, for tuples $(X, \rho, X', \rho') \in \tilde{\mathcal{N}}^*(S)$ for schemes S over $\text{Spf } \check{\mathbb{Z}}_p$. We then define $\mathcal{Z}^*(H) := \tilde{\mathcal{Z}}^*(H) \cap \mathcal{N}^*$ and $\mathcal{Z}^{*,0}(H) := \tilde{\mathcal{Z}}^*(H) \cap \mathcal{N}^{*,0}$.

Several standard remarks apply to Definition 11.2.1, i.e. $(\rho')^{-1} \circ x \circ \rho : X \rightarrow X'$ is viewed as a quasi-homomorphism by Drinfeld rigidity, and the subfunctor $\tilde{\mathcal{Z}}^*(H) \subset \tilde{\mathcal{N}}^*$ is represented by a closed formal subscheme by (the proof of) [RZ96, Proposition 2.9]. The involution of $\tilde{\mathcal{N}}^*$ given by swapping (X, ρ) and (X', ρ') takes $\tilde{\mathcal{Z}}^*(H)$ isomorphically to $\tilde{\mathcal{Z}}^*(H^\dagger)$. To see this, we use the fact that given any $(X, \rho) \in \mathcal{N}_0^*(S)$, the fixed principal polarization on $\mathbb{X}^*$ lifts (uniquely) to a principal polarization on X (e.g. [Che24, Lemma 5.6.1]).

Given a tuple $\mathbf{x} = (x_1, \dots, x_r)$ of elements in $\mathbb{B}^*$, we can forget information to view $\mathbf{x}$ as a subset of $\mathbb{B}^*$, so we abuse notation to write e.g. $\tilde{\mathcal{Z}}^*(\mathbf{x})$ for the associated special cycle. Consider the temporary notation $\tilde{\varphi}_p := \mathbf{1}_{\mathbb{L}^*}^{\otimes r} \in C_c^\infty((\mathbb{B}^*)^r)$ for an understood $r \geq 1$. If $\mathbf{x} \in (\mathbb{B}^*)^r$ is an r -tuple, we have

$$(11.2.1) \quad \tilde{\mathcal{Z}}^*(\mathbf{x})(\mathbb{F}) = \{h \in H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p) : h \in \text{supp}(h \mapsto \tilde{\varphi}_p(h^{-1}\mathbf{x}))\}$$

as a subset $\tilde{\mathcal{N}}^*(\mathbb{F}) = H'(\mathbb{Q}_p)/H'(\mathbb{Z}_p)$. In particular, we have

$$(11.2.2) \quad \tilde{\mathcal{Z}}^*(\mathbf{x})(\mathbb{F}) = \begin{cases} \tilde{\mathcal{N}}^*(\mathbb{F}) & \text{if } \mathbf{x} \subseteq \mathbb{L}^* \\ \emptyset & \text{if } \mathbf{x} \not\subseteq \mathbb{L}^* \end{cases}$$

in the supersingular case (because $H'(\mathbb{Q}_p)$ preserves $\mathbb{L}^*$ in the supersingular case).

There is an action $H'(\mathbb{Q}_p) \curvearrowright \mathbb{B}^*$ where $(h, h') \in H'(\mathbb{Q}_p)$ acts as $x \mapsto h'xh^{-1}$. This preserves quadratic forms up to $\mathbb{Z}_p^\times$ -scalar. For $\tilde{h} \in H'(\mathbb{Q}_p)$ and any subset $H \subseteq \mathbb{B}^*$, we then have

$$(11.2.3) \quad \tilde{h} \cdot \tilde{\mathcal{Z}}^*(H) = \tilde{\mathcal{Z}}^*(\tilde{h} \cdot H).$$

Considering the decomposition from (11.1.5) and defining $\tilde{\mathcal{Z}}^*(\mathbf{x})_i := \tilde{\mathcal{Z}}^*(\mathbf{x}) \cap \tilde{\mathcal{N}}_i^*$ gives an open and closed decomposition

$$(11.2.4) \quad \tilde{\mathcal{Z}}^*(\mathbf{x}) = \coprod_{i \in H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p)} \tilde{\mathcal{Z}}^*(\mathbf{x})_i.$$

11.3. Cycles and lattices of rank 2. The rest of Section 11 is largely devoted to studying the cycles $\mathcal{Z}^*(M)$ for $\mathbb{Z}_p$ -lattices $M \subseteq \mathbb{B}^*$ of rank 2. Note that any such M is automatically non-degenerate: if $*$ = s, then $\mathbb{B}^*$ is anisotropic, and if $*$ = o, then M spans all of $\mathbb{B}^*$.

Given any $\mathbb{Z}_p$ -lattice M of rank 2 equipped with a non-degenerate quadratic form, we consider its Gross–Keating invariants $\text{GK}(M) = (a, b)$ for integers $a \leq b$ (compare [GK93, §4]), as recalled in Section B. We set

$$(11.3.1) \quad \nu(M) := a + b \quad d(M) := b - a.$$

The lattice M is integral if and only if $a \geq 0$.

Given a rank 2 lattice $M \subseteq \mathbb{B}^*$ which is not integral, we have $\tilde{\mathcal{Z}}^*(M) = \emptyset$. Indeed, by the transitive action $H'(\mathbb{Q}_p) \curvearrowright \tilde{\mathcal{N}}^*(\mathbb{F})$, it is enough to check $\mathcal{Z}^{*,0}(M) = \emptyset$. Then note that for any $(X, \rho, X', \rho') \in \mathcal{Z}^{*,0}(M)(\mathbb{F})$, and every $x \in M$, we must have $x^\dagger x \in \text{End}(X)$, i.e. $x^\dagger x \in \mathbb{Z}_p$.

We know that $\mathbb{Q}_p[X]/(X^2 - \text{disc}(M))$ is a ramified extension of $\mathbb{Q}_p$ if and only if $\nu(M)$ is odd (e.g. follows from Lemma B.1.2). If M is integral, we define the type $t(M)$ of M to be

$$(11.3.2) \quad t(M) = \begin{cases} 0, & \text{if } a = b = 0; \\ 1, & \text{if } a = 0 \text{ and } b > 0; \\ 2, & \text{if } a > 0. \end{cases}$$

We then have $\nu(M) = d(M)$ if and only if $t(M) \leq 1$.

Given a finite étale $\mathbb{Q}_p$ -algebra F of degree 2, and given an integer $s \geq 0$, we write $\mathcal{O}_{F,s} := \mathbb{Z}_p + p^s \mathcal{O}_F$ for the unique order of index p^s inside $\mathcal{O}_F$.

Lemma 11.3.1. *Two $\mathbb{Z}_p$ -lattices $M, M' \subseteq \mathbb{B}^*$ of rank 2 lie in the same $H'_0(\mathbb{Q}_p)$ -orbit if and only if $\text{GK}(M) = \text{GK}(M')$ and $\text{disc}(M_{\mathbb{Q}_p}) = \text{disc}(M'_{\mathbb{Q}_p})$. This is also equivalent to lying in the same $H'(\mathbb{Q}_p)$ -orbit.*

Proof. Since the $H'(\mathbb{Q}_p)$ -action preserves quadratic forms up to $\mathbb{Z}_p^\times$ -scalar, it certainly preserves the invariants in the lemma statement.

Suppose instead that M and M' have the same invariants as in the lemma statement. Set $D := \text{disc}(M_{\mathbb{Q}_p}) = \text{disc}(M'_{\mathbb{Q}_p})$ and $F := \mathbb{Q}_p[X]/(X^2 - D)$ (we only use this $\mathbb{Q}_p$ -algebra up to isomorphism).

We first claim that the condition on discriminants implies that $M_{\mathbb{Q}_p}$ and $M'_{\mathbb{Q}_p}$ lie in the same $H'_0(\mathbb{Q}_p)$ -orbit. In the ordinary case, there is nothing to show as $M_{\mathbb{Q}_p} = M'_{\mathbb{Q}_p} = \mathbb{B}^\circ$. In the supersingular case, Lemma B.1.1 implies that we can find nonzero $x \in M$ and $x' \in M'$ with $v_p(q(x)) = v_p(q(x'))$. It is enough to show that $x^{-1}M_{\mathbb{Q}_p}$ and $x'^{-1}M'_{\mathbb{Q}_p}$ lie in the same $H'(\mathbb{Q}_p)$ -orbit. But $\mathbb{Q}_p \subseteq x^{-1}M_{\mathbb{Q}_p}$ and $\mathbb{Q}_p \subseteq x'^{-1}M'_{\mathbb{Q}_p}$. This implies that $x^{-1}M_{\mathbb{Q}_p} \subseteq \mathbb{B}^s$ and $x'^{-1}M'_{\mathbb{Q}_p} \subseteq \mathbb{B}^s$ are both $\mathbb{Q}_p$ -subalgebras, not just $\mathbb{Q}_p$ -submodules. These $\mathbb{Q}_p$ -subalgebras are both isomorphic to F . Skolem–Noether then implies that $x^{-1}M_{\mathbb{Q}_p}$ and $x'^{-1}M'_{\mathbb{Q}_p}$ are both conjugate inside $\mathbb{B}^s$, but we would like to show that they are conjugate by an element of $\mathcal{O}_{\mathbb{B}^s}^\times$, i.e. a unit in the maximal order of $\mathbb{B}^s$. It suffices to note the existence of $u \in \mathbb{B}^s$ with $v_p(q(u)) = 1$ which normalizes the $\mathbb{Q}_p$ -subalgebra $x^{-1}M_{\mathbb{Q}_p}$. If $F/\mathbb{Q}_p$ is ramified, we can take $u \in x^{-1}M_{\mathbb{Q}_p}$. If $F/\mathbb{Q}_p$ is inert, there exists u for which conjugation induces the nontrivial Galois involution on $x^{-1}M_{\mathbb{Q}_p}$. This finishes the proof that $M_{\mathbb{Q}_p}$ and $M'_{\mathbb{Q}_p}$ lie in the same $H'_0(\mathbb{Q}_p)$ -orbit.

To prove the lemma, it thus suffices to verify the following claim. View F as the 2-dimensional $\mathbb{Q}_p$ quadratic space with quadratic form $N_{F/\mathbb{Q}_p}(-)$, and let $M, M' \subseteq F$ be $\mathbb{Z}_p$ -lattices of rank 2 with the same Gross–Keating invariants. Then we claim that there exists $b \in F^\times$ with $v_p(N_{F/\mathbb{Q}_p}(b)) = 0$ such that $bM = M'$.

First, select any $a, a' \in F^\times$ such that $M = a \cdot \mathcal{O}_{F,s}$ and $M' = a' \cdot \mathcal{O}_{F,s'}$ for some integers $s, s' \geq 0$ (compare with the proof of Lemma 10.1.2, where we argued that $\mathfrak{a}_p$ and $\mathfrak{a}'_p$ were both free of rank 1 over an appropriate order).

If $e := v_p(N_{F/\mathbb{Q}_p}(a))$ and $e' := v_p(N_{F/\mathbb{Q}_p}(a'))$, then we have

$$(11.3.3) \quad \text{GK}(M) = \begin{cases} (e, e + 2s) & \text{if } F/\mathbb{Q}_p \text{ is unramified} \\ (e, e + 2s + 1) & \text{if } F/\mathbb{Q}_p \text{ is ramified} \end{cases}$$

$$(11.3.4) \quad \text{GK}(M') = \begin{cases} (e', e' + 2s') & \text{if } F/\mathbb{Q}_p \text{ is unramified} \\ (e', e' + 2s' + 1) & \text{if } F/\mathbb{Q}_p \text{ is ramified} \end{cases}$$

(e.g. as in Lemma 11.4.1). In particular, $\text{GK}(M) = \text{GK}(M')$ if and only if $e' = e$ and $s = s'$, in which case we can take $b := a'/a$. $\square$

Lemma 11.3.2. *Let $M \subseteq \mathbb{B}^*$ be an integral $\mathbb{Z}_p$ -lattice of rank 2. Set $F := \mathbb{Q}_p[X]/(X^2 - \text{disc}(M))$.*

The two $\mathbb{Z}_p$ -subalgebras of $\text{End}^\circ(\mathbb{X}^*)$ generated by elements of the form $y^\dagger x$ (resp. $xy^\dagger$) for $x, y \in M$ are both (non-canonically) isomorphic to $\mathcal{O}_{F, [\nu(M)/2]}$. Given any $\mathbb{Z}_p$ -basis $\{x, y\}$ for M , these $\mathbb{Z}_p$ -subalgebras are equal to $\mathbb{Z}_p[y^\dagger x]$ and $\mathbb{Z}_p[xy^\dagger]$, respectively. In particular, F is a field (resp. split) if and only if $\mathbb{X}^*$ is supersingular (resp. ordinary).

Proof. The supersingular case should be contained in the work of Gross–Keating (see [Rap07, Lemma 3.1]). We give a uniform explanation for the supersingular and ordinary cases.

The claim $y^\dagger x \in \text{End}(\mathbb{X}^*)$ follows because $(y^\dagger x)^\dagger (y^\dagger x) \in \mathbb{Z}_p$, and $y^\dagger x + x^\dagger y \in \mathbb{Z}_p$, using integrality of M . Note that we also conclude $x^\dagger y \in \text{End}(\mathbb{X}^*)$. It suffices to prove the lemma for $y^\dagger x$: in the

supersingular case, we have $xy^\dagger = x(y^\dagger x)x^{-1}$ (and recall that $\text{End}^\circ(\mathbb{X}^*)^\times$ normalizes $\text{End}(\mathbb{X}^*)$), and in the ordinary case, we can use commutativity of $\text{End}^\circ(\mathbb{X}^*)$.

We compute $\text{disc}(M) = \frac{1}{4}(y^\dagger x - x^\dagger y)^2$. In particular, we find that the $\mathbb{Q}_p$ -subalgebra of $\text{End}^\circ(\mathbb{X}^*)$ generated by $y^\dagger x$ is isomorphic to F . We must have $y^\dagger x \in \mathcal{O}_F$ (because $y^\dagger x \in \text{End}(\mathbb{X}^*)$), and also $y^\dagger x \notin \mathbb{Q}_p$ because $\text{disc}(M) \neq 0$.

Recall (Lemma B.1.2) that we have

$$(11.3.5) \quad [\nu(M)/2] = \begin{cases} \lfloor \frac{v_p(4 \text{disc}(M))}{2} \rfloor - 1 & \text{if } p = 2 \text{ and } F/\mathbb{Q}_p \text{ is ramified} \\ \lfloor \frac{v_p(4 \text{disc}(M))}{2} \rfloor & \text{else.} \end{cases}$$

On the other hand, for any $a \in \mathcal{O}_F$ with $a \notin \mathbb{Z}_p$, some casework shows that the order $\mathbb{Z}_p[a] \subseteq \mathcal{O}_F$ has index $p^{\lfloor (v_p((a-a^\sigma)^2))/2 \rfloor - 1}$ if $p = 2$ and $F/\mathbb{Q}_p$ is ramified, and has index $p^{\lfloor (v_p((a-a^\sigma)^2))/2 \rfloor}$ otherwise, where $\sigma: F \rightarrow F$ is the unique nontrivial involution. Since $\dagger$ induces σ on $\mathbb{Q}_p[y^\dagger x] \cong F$, the lemma now follows from the formulas in (11.3.5) and the expression $\text{disc}(M) = \frac{1}{4}(y^\dagger x - x^\dagger y)^2$. $\square$

Corollary 11.3.3. *Let M and F be as in the statement of Lemma 11.3.2. Pick any $\mathbb{Z}_p$ -basis $\{x, y\}$ for M , and let F_M (resp. F'_M) be the $\mathbb{Q}_p$ -subalgebras of $\text{End}^\circ(\mathbb{X}^*)$ given by $\mathbb{Q}_p[y^\dagger x]$ and $\mathbb{Q}_p[xy^\dagger]$, respectively. Pick any isomorphisms $\iota_F: F \rightarrow F_M$ and $\iota'_F: F \rightarrow F'_M$.*

Let S be any scheme over $\text{Spf } \check{\mathbb{Z}}_p$, and consider any $(X, \rho, X', \rho') \in \tilde{\mathcal{Z}}^*(M)(S)$. Then the map $\iota_F: \mathcal{O}_{F, [\nu(M)/2]} \rightarrow \text{End}(\mathbb{X}^*)$ lifts via ρ to an embedding $\mathcal{O}_{F, [\nu(M)/2]} \hookrightarrow \text{End}(X)$, and similarly for ι'_F, ρ' , and X' .

Proof. This follows from Lemma 11.3.2, i.e. $y^\dagger x$ lifts via ρ to $\text{End}(X)$, and $xy^\dagger$ lifts via ρ' to $\text{End}(X')$. We are using the standard lifting of principal polarizations, as explained after Definition 11.2.1. $\square$

Lemma 11.3.4. *Let $M \subseteq \mathbb{B}^*$ be any $\mathbb{Z}_p$ -lattice of rank 2. Then $\tilde{\mathcal{Z}}^*(M)$ is a countable disjoint union of connected components, each of which is adic finite flat over $\text{Spf } \check{\mathbb{Z}}_p$, and with schemification having étale generic fiber.*

Proof. By the transitive action of $H'(\mathbb{Q}_p)$ on the connected components of $\tilde{\mathcal{N}}^*$, it is enough to check that $\mathcal{Z}^{*,0}(M) \rightarrow \text{Spf } \check{\mathbb{Z}}_p$ is adic finite flat over $\text{Spf } \check{\mathbb{Z}}_p$, with schemification having étale generic fiber.

The claim should then be standard in the supersingular case after [GK93, §5] and [Rap07, §3]. We explain this in our language (treating the ordinary case at the same time), for the reader's convenience.

Given $\gamma \in \text{End}^\circ(\mathbb{X}^*)$, we obtain a local special cycle $\mathcal{Z}_0^*(\gamma) \subseteq \mathcal{N}_0^*$ defined similarly as in Definition 11.2.1, but with (X', ρ') replaced with (X, ρ) and H replaced by the singleton set $\{\gamma\}$. For $\gamma \notin \mathbb{Q}_p$, it is known that $\mathcal{Z}_0^*(\gamma) \subseteq \mathcal{N}_0^*$ is an effective Cartier divisor which is adic finite flat over $\text{Spf } \check{\mathbb{Z}}_p$ with schemification having étale generic fiber; this is explained e.g. in [CLZ] (both ordinary and supersingular cases are needed).

Pick any $\mathbb{Z}_p$ -basis $\{x_1, x_2\}$ for M . Set $\gamma_1 = x_2^\dagger x_1$ and $\gamma_2 = x_1 x_2^\dagger$. Then the closed immersion $\mathcal{Z}^{*,0}(M) \rightarrow \mathcal{N}^{*,0}$ factors through $\mathcal{Z}_0^*(\gamma_1) \times_{\text{Spf } \check{\mathbb{Z}}_p} \mathcal{Z}_0^*(\gamma_2)$, by Corollary 11.3.3. From this, it immediately follows that $\mathcal{Z}^{*,0}(M) \rightarrow \text{Spf } \check{\mathbb{Z}}_p$ is adic finite and with schemification having étale generic fiber.

It remains to show that $\mathcal{Z}^{*,0}(M) \rightarrow \text{Spf } \check{\mathbb{Z}}_p$ is flat. For this, first note $\mathcal{Z}^{*,0}(M) = \mathcal{Z}^{*,0}(x_1) \cap \mathcal{Z}^{*,0}(x_2)$. For each $i = 1, 2$, we also know that $\mathcal{Z}^{*,0}(x_i) \subseteq \mathcal{N}^{*,0}$ is either an effective Cartier divisor

or equal to all of $\mathcal{N}^{*,0}$ by a standard method⁸ which can be explained as follows: a deformation theory argument (e.g. Grothendieck–Messing deformation theory) shows that on the square-zero thickening of $\mathcal{Z}^{*,0}(x_i)$, the closed formal subscheme $\mathcal{Z}^{*,0}(x_i)$ is the zero locus of a canonical section (depending on x_i) of a certain line bundle coming from Hodge filtrations; then apply Nakayama’s lemma. We already know that $\mathcal{Z}^{*,0}(M) \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p$ is finite adic with $\dim \mathcal{Z}^{*,0}(M)^{\mathrm{sch}} \leq 1$, so it follows that $\mathcal{Z}^{*,0}(M)^{\mathrm{sch}}$ is Cohen–Macaulay, i.e. because after fixing an identification $\mathcal{N}^{*,0} \cong \mathrm{Spf} \check{\mathbb{Z}}_p[[t, t']]$, the equations defining $\mathcal{Z}^{*,0}(x_i)$ and $\mathcal{Z}^{*,0}(x_i)$ must give a regular sequence in $\mathrm{Spf} \check{\mathbb{Z}}_p[[t, t']]$ at all points in the image of $\mathcal{Z}^{*,0}(M)^{\mathrm{sch}} \rightarrow \mathrm{Spec} \check{\mathbb{Z}}_p[[t, t']]$. This reasoning also shows that $\mathcal{Z}^{*,0}(M)^{\mathrm{sch}}$ is of pure dimension 1, so we conclude that $\mathcal{Z}^{*,0}(M)^{\mathrm{sch}} \rightarrow \mathrm{Spec} \check{\mathbb{Z}}_p$ is flat by miracle flatness. $\square$

11.4. Quasi-canonical liftings. Let F be a finite étale $\mathbb{Q}_p$ -algebra of degree 2. Then either F is a quadratic field extension of $\mathbb{Q}_p$, or we have $F \simeq \mathbb{Q}_p \times \mathbb{Q}_p$. We say that F is non-split if $F/\mathbb{Q}_p$ is a quadratic field extension, and that F is split if $F \simeq \mathbb{Q}_p \times \mathbb{Q}_p$. We let $\mathcal{O}_F \subseteq F$ be the integral closure of $\mathbb{Z}_p$, i.e. $\mathcal{O}_F$ is the ring of integers of F if F is a field, while we have $\mathcal{O}_F = \mathbb{Z}_p \times \mathbb{Z}_p$ in the split case. Given any integer $s \geq 0$, we set

$$(11.4.1) \quad \mathcal{O}_{F,s} = \mathbb{Z}_p + p^s \cdot \mathcal{O}_F.$$

for the unique order of index p^s inside $\mathcal{O}_F$.

Denote by $\check{F}$ the field $\check{\mathbb{Q}}_p$ when F is split (equipped with a morphism $F \rightarrow \check{\mathbb{Q}}_p$), and otherwise write $\check{F}$ for the completion of the maximal unramified extension of F , equipped with a canonical map $F \rightarrow \check{F}$.

Consider an embedding $\iota_F: F \rightarrow \mathrm{End}^\circ(\mathbb{X}^*)$. This necessarily means $* = s$ if F is nonsplit and $* = o$ if F is split.

We recall the notion of quasi-canonical lifting from [Gro86] (when F is non-split) and [Meu07] (when F is split), slightly modified to suit our current purposes. For $s \geq 0$ and $* \in \{s, o\}$, a *quasi-canonical lifting* (and canonical lifting when $s = 0$) for the pair $(\mathbb{X}^*, \iota_F)$ is an object $(\mathfrak{X}_{F,s}^*, \rho) \in \mathcal{N}_0^*(\mathrm{Spf} \mathcal{O}_{\check{K}})$ for some finite extension $\check{K}/\check{\mathbb{Q}}_p$, with the property that the action $\iota_F: \mathcal{O}_{F,s} \rightarrow \mathrm{End}(\mathbb{X}^*)$ lifts to $\mathfrak{X}_{F,s}^*$, and that moreover $\mathrm{End}(\mathfrak{X}_{F,s}^*) \cong \mathcal{O}_{F,s}$. Quasi-canonical liftings are unique up to isomorphism after forgetting the framing ρ .

In the non-split case (resp. the split case), the subgroup $\mathcal{O}_{F,s}^\times \subset \mathcal{O}_F^\times$ (resp. $(1 + p^s \mathbb{Z}_p)^\times \subset \mathbb{Z}_p^\times$) has an associated finite totally ramified abelian extension $\check{F}_s$ of $\check{F}$ with ring of integers $\mathcal{O}_{\check{F}_s}$, by local class field theory. The extension degree is

$$(11.4.2) \quad [\check{F}_s : \check{F}] = \begin{cases} p^s(1 - \eta^*(p)p^{-1}), & \text{if } s \geq 1; \\ 1, & \text{if } s = 0. \end{cases}$$

where $\eta^*(p) := -1, 0, 1$ in the inert, ramified, and split cases respectively. In the split case, we have $\check{F}_s = \check{F}[\zeta_{p^s}]$ where ζ_{p^s} is a primitive p^s -th root of unity. By [Gro86, Proposition 5.3] (the non-split case) and [Meu07, Proposition 3.5] (the split case), the ring of definition of $\mathfrak{X}_{F,s}^*$ is $\mathcal{O}_{\check{F}_s}$.

If $(\mathfrak{X}_{F,s}^*, \rho)$ is a level s quasi-canonical lifting for $(\mathbb{X}^*, \iota_F)$ over $\mathrm{Spf} \mathcal{O}_{\check{F}_s}$, then the induced morphism $\mathrm{Spf} \mathcal{O}_{\check{F}_s} \rightarrow \mathcal{N}_0^*$ is a closed immersion. Any quasi-canonical lifting of level s for $(\mathbb{X}^*, \iota_F)$ over some

⁸This method at least goes back to Katz–Mazur in [KM85, Proposition 6.8.6]; see also e.g. [Wew07b, Proposition 5.1].

$\mathrm{Spf} \mathcal{O}_{\check{K}}$ is the pullback of $(\mathfrak{X}_{F,s}^*, \rho)$ from the previous sentence along a unique morphism $\mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}_s}$.

Lemma 11.4.1. *Let r, s be nonnegative integers. Fix embeddings $\iota_F: F \rightarrow \mathrm{End}^\circ(\mathbb{X}^*)$ and $\iota'_F: F \rightarrow \mathrm{End}^\circ(\mathbb{X}^*)$. Let $\check{K}/\check{F}$ be any finite degree field extension, and $(\mathfrak{X}_{F,r}^*, \rho)$ and $(\mathfrak{X}_{F,s}^*, \rho')$ be quasi-canonical liftings over $\mathrm{Spf} \mathcal{O}_{\check{K}}$ of level r and s , for the pairs $(\mathbb{X}^*, \iota_F)$ and $(\mathbb{X}^*, \iota'_F)$ respectively. Form the $\mathbb{Z}_p$ -submodule*

$$(11.4.3) \quad M := \mathrm{Hom}(\mathfrak{X}_{F,r}^*, \mathfrak{X}_{F,s}^*) \subseteq \mathrm{Hom}^\circ(\mathbb{X}^*, \mathbb{X}^*) = \mathbb{B}^*.$$

If $r \leq s$, then the following conclusions hold.

- (1) *The space M is a free $\mathcal{O}_{F,r}$ -module of rank one by pre-composition, and any generator ψ satisfies $q(\psi) = \deg \psi = p^{|s-r|}$. The quadratic form on M is given by $q(\psi \cdot a) = N_{F/\mathbb{Q}_p}(a)q(\psi)$. In the ordinary case, any such $\psi \in \mathrm{Hom}(\mathbb{X}^*, \mathbb{X}^*)$ is an isomorphism on the connected part.*
- (2) *We have $F \cong \mathbb{Q}_p[X]/(X^2 - \mathrm{disc}(M))$ (non-canonically).*
- (3) *The lattice M has Gross–Keating invariants (a, b) which are characterized by the expression*

$$(11.4.4) \quad (r, s) = (\lfloor d(M)/2 \rfloor, \lfloor \nu(M)/2 \rfloor). \quad d(M) := b - a \quad \nu(M) := a + b$$

and the additional data of whether $F/\mathbb{Q}_p$ is ramified, i.e.

$$(11.4.5) \quad (a, b) = (|s - r|, r + s) \quad \text{if } F/\mathbb{Q}_p \text{ is unramified}$$

$$(11.4.6) \quad (a, b) = (|s - r|, r + s + 1) \quad \text{if } F/\mathbb{Q}_p \text{ is ramified.}$$

- (4) *Given $f \in \mathrm{End}^\circ(\mathbb{X}^*)$, we have $M \cdot f \subseteq M$ if and only if $f \in \iota_F(\mathcal{O}_{F,r})$.*

If $r \geq s$, the above claims hold with the following modifications. Conclusion (1) holds but with $\mathrm{Hom}(\mathfrak{X}_{F,r}^*, \mathfrak{X}_{F,s}^*)$ instead being free of rank one over $\mathcal{O}_{F,s}$ by post-composition, and any generator ψ is instead an isomorphism on the étale part in the ordinary case. Conclusion (2) holds verbatim. Conclusion (3) also holds after swapping r and s in (11.4.4). Conclusion (4) holds upon considering $f \cdot M$ instead of $M \cdot f$, and replacing $\iota_F(\mathcal{O}_{F,r})$ with $\iota'_F(\mathcal{O}_{F,s})$.

Proof. Note that the $r \leq s$ and $r \geq s$ cases are equivalent upon applying the Rosati involution (with respect to the principal polarizations lifting the fixed principal polarization on $\mathbb{X}^*$). Recall also that $\nu(M)$ is odd if and only if $\mathbb{Q}_p[X]/(X^2 - \mathrm{disc}(M))$ is a ramified extension of $\mathbb{Q}_p$, as follows from Lemma B.1.2. Also note that claim (2) follows from claim (1), e.g. as in the proof of Lemma 11.3.2. Note that claim (1) implies that the quadratic form on M is integral.

We now explain how claim (4) follows from claims (1) and (2). It is enough to check the $r \leq s$ case. The embedding ι'_F induces an embedding $F \rightarrow \mathrm{End}_{\iota_F(F)}(M_{\mathbb{Q}_p})$, where $\iota_F(F) \subset M_{\mathbb{Q}_p}$ is given by pre-composition and $\iota'_F(F) \subset M_{\mathbb{Q}_p}$ is given by post-composition. But since $M_{\mathbb{Q}_p}$ is free of rank one as a $\iota_F(F)$ -module, we conclude that it is also free of rank one as a $\iota'_F(F)$ -module, and that $\mathrm{End}_{\iota_F(F)}(M_{\mathbb{Q}_p}) = \mathrm{End}_{\iota'_F(F)}(M_{\mathbb{Q}_p})$. In particular, if $f \in \mathrm{End}^\circ(\mathbb{X}^*)$ satisfies $M \cdot f \subseteq M$, then we have $f \in \mathrm{End}_{\iota'_F(F)}(M_{\mathbb{Q}_p}) = \iota_F(F)$. But since M is free of rank one as an $\iota_F(\mathcal{O}_{F,r})$ -module, we must moreover have $f \in \iota_F(\mathcal{O}_{F,r})$.

Given the above remarks, the supersingular case of the lemma is proved in [Wew07a, Proposition 1.1]. We thus restrict to the ordinary case. We fix an identification $\mathbb{X}^\circ \cong \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$, which fixes an identification $\mathbb{B}^\circ \cong \mathbb{Q}_p \times \mathbb{Q}_p$. Given this fixed data, the lifts $(\mathfrak{X}_{F,r}^\circ, \rho)$ and $(\mathfrak{X}_{F,s}^\circ, \rho')$ correspond to

choices of primitive p^r -th and p^s -th roots of unity $\zeta_{p^r}, \zeta_{p^s} \in \mathcal{O}_{\check{K}}$, respectively (see the discussion on lifting theory for ordinary p -divisible groups in [Che24, §6.1]). By e.g. [Che24, (6.1.5)], an element $(f_1, f_2) \in \mathbb{B}^\circ$ lies in M if and only if

$$(11.4.7) \quad (\zeta_{p^r})^{f_1} = (\zeta_{p^s})^{f_2}.$$

Without loss of generality, we may assume $r \leq s$. Select $a \in \mathbb{Z}_p$ satisfying $(\zeta_{p^s})^a = \zeta_{p^r}$, and note $v_p(a) = |s - r|$. Then (f_1, f_2) lies in M if and only if $f_2 - af_1 \equiv 0 \pmod{p^s}$. In other words, every element of M can be written in the form

$$(11.4.8) \quad \alpha \cdot (1, a) + \beta p^r (0, 1) \cdot (1, a)$$

for unique values $\alpha, \beta \in \mathbb{Z}_p$, i.e. M is free of rank one over $\mathcal{O}_{F,r}$ with generator $(1, a)$. This proves claims (1) and (2). It remains to prove (3) in the ordinary case, i.e. to show that M has Gross–Keating invariants $(|s - r|, s + r)$. It is enough to show that the $\mathcal{O}_{F,r}$ viewed as a $\mathbb{Z}_p$ -lattice (of rank 2) with quadratic form $N_{F/\mathbb{Q}_p}(-)$ has Gross–Keating invariants $(0, 2r)$. For the $\mathbb{Z}_p$ -basis $\{(1, 1), (0, p^r)\}$ of $\mathcal{O}_{F,r}$, the associated quadratic form is given by $x^2 + p^r xy$. Inspecting the definition of Gross–Keating invariants from (B.1.2) shows that the invariants are indeed $(0, 2r)$ in this case. $\square$

Remark 11.4.2. The isomorphism class of the quadratic lattice M from Lemma 11.4.1 can in general depend on the other data, and not just r, s . In both the supersingular and ordinary cases, we can rescale the quadratic form on M by an arbitrary element of $\mathbb{Z}_p^\times$ by changing the pair (ι'_F, ρ') , and this can even change the isomorphism class of the rational quadratic space $M_{\mathbb{Q}_p}$ (consider the Hasse invariant).

11.5. Primitive cycles. Consider any $\mathbb{Z}_p$ -lattice $M \subseteq \mathbb{B}^*$ of rank 2.

Definition 11.5.1. The *primitive cycle* $\mathcal{Z}^*(M)^\circ \subseteq \mathcal{Z}^*(M)$ is the closed formal subscheme given by the (scheme-theoretic) union of the components which are not contained in $\mathcal{Z}^*(M')$ for any rank 2 lattice $M' \supsetneq M$.

Lemma 11.3.4 implies that $\mathcal{Z}^*(M) \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p$ is a countable disjoint union of formal schemes which are adic finite flat over $\mathrm{Spf} \check{\mathbb{Z}}_p$, and where the schemification of $\mathcal{Z}^*(M)$ has étale generic fiber. The “components” in Definition 11.5.1 mean the closed formal subschemes associated to the irreducible components of the schemification. Note that $\mathcal{Z}^*(M)^\circ = \emptyset$ if M is not integral (because $\mathcal{Z}^*(M) = \emptyset$ if M is not integral).

Lemma 11.5.2. *For any $\mathbb{Z}_p$ -lattice $L \subseteq \mathbb{B}^*$ of rank 2, we have a scheme-theoretic union⁹ decomposition*

$$(11.5.1) \quad \mathcal{Z}^*(L) = \bigcup_{\substack{L \subseteq M \\ M \text{ integral}}} \mathcal{Z}^*(M)^\circ$$

of closed formal subschemes of $\mathcal{N}^*$, where the union runs over $\mathbb{Z}_p$ -lattices $M \subseteq L_{\mathbb{Q}_p}$ of rank 2.

Proof. This follows from the definition of $\mathcal{Z}^*(M)^\circ$. For fixed L , there are only finitely many integral $\mathbb{Z}_p$ -lattices $M \subseteq L_{\mathbb{Q}_p}$ satisfying $M \supsetneq L$. $\square$

⁹We mean to intersect ideal sheaves.

Lemma 11.5.3. *Let $\check{K}$ be any finite degree field extension of $\check{\mathbb{Q}}_p$. Given any object $(X, \rho, X', \rho') \in \mathcal{N}^*(\mathrm{Spf} \mathcal{O}_{\check{K}})$, the corresponding morphism $\mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathcal{N}^*$ factors through $\mathcal{Z}^*(M)^\circ$ if and only if*

$$(11.5.2) \quad \mathrm{Hom}(X, X') = M$$

as $\mathbb{Z}_p$ -submodules of $\mathbb{B}^*$.

Proof. We are using the usual Drinfeld rigidity to view $\mathrm{Hom}(X, X')$ as a submodule of $\mathbb{B}^*$ (using ρ and ρ'). If $\mathrm{Hom}(X, X') = M$, then the morphism $\mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathcal{N}^*$ certainly factors through $\mathcal{Z}^*(M)$, and it also cannot factor through $\mathcal{Z}^*(M')$ for any rank 2 lattice $M' \supsetneq M$, i.e. $\mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathcal{N}^*$ factors through $\mathcal{Z}^*(M)^\circ$.

Conversely, consider $(X, \rho, X', \rho') \in \mathcal{Z}^*(M)^\circ(\mathrm{Spf} \mathcal{O}_{\check{K}})$. Set $F := \mathbb{Q}_p[X]/(X^2 - \mathrm{disc}(M))$. Corollary 11.3.3 implies that both X and X' admit faithful actions by $\mathcal{O}_{F, [\nu(M)/2]}$. We claim that $\mathrm{End}^\circ(X) \cong \mathrm{End}^\circ(X') \cong F$ (non-canonically). It is enough to check that $\mathrm{End}^\circ(X)$ and $\mathrm{End}^\circ(X')$ are both commutative, since any embedding $F \hookrightarrow \mathrm{End}^\circ(\mathbb{X}^*)$ has image which is self-centralizing. In the ordinary case, this follows because $\mathrm{End}^\circ(\mathbb{X}^*) \cong \mathbb{Q}_p \times \mathbb{Q}_p$ is commutative. In the supersingular case, this follows because there exists a morphism $\mathrm{End}^\circ(X') \rightarrow \check{K}$ given by the action $\mathrm{End}(X') \rightarrow \mathrm{End}(\mathrm{Lie} X') \cong \mathcal{O}_{\check{K}}$ (then use the fact that any $\mathbb{Q}_p$ -subalgebra of $\mathrm{End}^\circ(\mathbb{X}^*)$ is simple). It now follows that there exists $(h, h') \in H'_0(\mathbb{Q}_p)$ such that $(X, h\rho)$ and $(X', h'\rho')$ are both quasi-canonical liftings of levels $s, s' \leq [\nu(M)/2]$, with respect to certain embeddings $\iota_F, \iota'_F: F \rightarrow \mathrm{End}^\circ(\mathbb{X}^*)$, which we will not need to describe explicitly. It suffices to take any $(h, h') \in H'_0(\mathbb{Q}_p)$ such that $h\rho$ and $h'\rho'$ are both isomorphisms in geometric fibers; this is automatic in the supersingular case.

Then Lemma 11.4.1 implies that $M' := \mathrm{Hom}(X, X') \subseteq \mathbb{B}^*$ is a $\mathbb{Z}_p$ -sublattice of rank 2, where we make the identification $\mathrm{Hom}(X, X') \subseteq \mathbb{B}^*$ using (ρ, ρ') . We must have $M' \supseteq M$, by construction. The morphism $\mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathcal{N}^*$ associated to (X, ρ, X', ρ') factors through $\mathcal{Z}^*(M')$. The construction of $\mathcal{Z}^*(M)^\circ$ now implies $M' = M$. $\square$

Corollary 11.5.4. *Given distinct $\mathbb{Z}_p$ -lattices $M, M' \subseteq \mathbb{B}^*$ of rank 2, the closed formal subschemes $\mathcal{Z}^*(M)^\circ \subseteq \mathcal{N}^*$ and $\mathcal{Z}^*(M')^\circ \subseteq \mathcal{N}^*$ do not share any components.*

Proof. This follows immediately from Lemma 11.5.3, i.e. the existence of such a common component implies the existence of $(X, \rho, X', \rho') \in \mathcal{N}^*(\mathrm{Spf} \mathcal{O}_{\check{K}})$ for some finite degree field extension $\check{K}/\check{\mathbb{Q}}_p$ such that $\mathrm{Hom}(X, X') = M$ and $\mathrm{Hom}(X, X') = M'$. We are again implicitly using the fact that $\mathcal{Z}^*(M)^\circ$ and $\mathcal{Z}^*(M')^\circ$ are disjoint unions of formal schemes which are adic finite flat over $\mathrm{Spf} \check{\mathbb{Z}}_p$ with étale generic fiber, in order to work with components. $\square$

In the next lemma, we again set $F := \mathbb{Q}_p[X]/(X^2 - \mathrm{disc}(M))$.

Lemma 11.5.5. *Let $M \subseteq \mathbb{B}^*$ be any integral $\mathbb{Z}_p$ -lattice of rank 2. There exists a scheme-theoretic union decomposition*

$$(11.5.3) \quad \mathcal{Z}^*(M)^\circ = \mathcal{Z}^*(M)_+^\circ \cup \mathcal{Z}^*(M)_-^\circ$$

of closed formal subschemes, characterized by the following requirements.

- (1) *We have $\mathcal{Z}^*(M)_+^\circ \cong \mathrm{Spf} \mathcal{O}_{\check{F}_{[\nu(M)/2]}}$ and $\mathcal{Z}^*(M)_-^\circ \cong \mathrm{Spf} \mathcal{O}_{\check{F}'_{[\nu(M)/2]}}$ as formal schemes over $\mathrm{Spf} \check{\mathbb{Z}}_p$.*

(2) The universal pairs of p -divisible groups over $\mathcal{Z}^*(M)_+^\circ$ and $\mathcal{Z}^*(M)_-^\circ$ (after forgetting framings ρ and ρ') are of the form

$$(11.5.4) \quad (\mathfrak{X}_{F,[d(M)/2]}^*, \mathfrak{X}_{F,[\nu(M)/2]}^*) \quad \text{and} \quad (\mathfrak{X}_{F,[\nu(M)/2]}^*, \mathfrak{X}_{F,[d(M)/2]}^*)$$

respectively, where each $\mathfrak{X}_{F,s}^*$ denotes any p -divisible group¹⁰ of height 2 dimension 1 such that $\text{End}(\mathfrak{X}_{F,s}^*) \cong \mathcal{O}_{F,s}$.

If $t(M) \leq 1$, then we have $\mathcal{Z}^*(M)_+^\circ = \mathcal{Z}^*(M)_-^\circ$. If $t(M) = 2$, then we have $\mathcal{Z}^*(M)_+^\circ \neq \mathcal{Z}^*(M)_-^\circ$. The formal scheme $\mathcal{Z}^*(M)^\circ$ is disconnected in the ordinary case if $t(M) = 2$, and is connected in all other cases.

Proof. Let $\mathfrak{X}_{F,[d(M)/2]}^*$ and $\mathfrak{X}_{F,[\nu(M)/2]}^*$ be p -divisible groups over $\text{Spf } \mathcal{O}_{\check{F}[\nu(M)/2]}$ as in the lemma statement. Recall that these p -divisible groups are unique up to isomorphism (from the theory of quasi-canonical liftings, as recalled in Section 11.4).

Select framings ρ and ρ' to obtain $(\mathfrak{X}_{F,[d(M)/2]}^*, \rho, \mathfrak{X}_{F,[\nu(M)/2]}^*, \rho') \in \mathcal{N}^{*,0}(\text{Spf } \mathcal{O}_{\check{F}[\nu(M)/2]})$. Any ρ and ρ' which are isomorphisms in geometric fibers will suffice. We claim that the associated morphism $\text{Spf } \mathcal{O}_{\check{F}[\nu(M)/2]} \rightarrow \mathcal{N}^{*,0}$ is a closed immersion. Indeed, the theory of quasi-canonical liftings (recalled in Section 11.4) implies that it is a closed immersion after projection to the second factor in $\mathcal{N}^{*,0} \cong \mathcal{N}_0^* \times \mathcal{N}_0^*$, so $\text{Spf } \mathcal{O}_{\check{F}[\nu(M)/2]} \rightarrow \mathcal{N}^{*,0}$ being a closed immersion follows from separatedness of that projection morphism (and the usual graph argument).

By Lemma 11.4.1, the data (ρ, ρ') identifies $\text{Hom}(\mathfrak{X}_{F,[d(M)/2]}^*, \mathfrak{X}_{F,[\nu(M)/2]}^*)$ with a $\mathbb{Z}_p$ -lattice $M' \subseteq \mathbb{B}^*$ of rank 2, with $\text{disc}(M') = \text{disc}(M)$ modulo $(\mathbb{Q}_p^\times)^2$ and with $\text{GK}(M') = \text{GK}(M)$. By Lemma 11.3.1, we conclude that M and M' lie in the same $H'_0(\mathbb{Q}_p)$ -orbit. That is, we can find $(h, h') \in H'_0(\mathbb{Q}_p)$ such that

$$(11.5.5) \quad (\mathfrak{X}_{F,[d(M)/2]}^*, h\rho, \mathfrak{X}_{F,[\nu(M)/2]}^*, h'\rho') \in \mathcal{N}^*(\text{Spf } \mathcal{O}_{\check{F}[\nu(M)/2]})$$

is such that the pair $(h\rho, h'\rho')$ induces $\text{Hom}(\mathfrak{X}_{F,[d(M)/2]}^*, \mathfrak{X}_{F,[\nu(M)/2]}^*) = M$ as sublattices of $\mathbb{B}^*$. Fix one such choice of (h, h') . The morphism $\text{Spf } \mathcal{O}_{\check{F}[\nu(M)/2]} \rightarrow \mathcal{N}^*$ associated to (11.5.5) is also a closed immersion, because it is a $H'_0(\mathbb{Q}_p)$ -translate of the previous morphism. We define $\mathcal{Z}^*(M)_+^\circ \subseteq \mathcal{N}^*$ to be the closed subfunctor defined by the closed immersion corresponding to (11.5.5). We define $\mathcal{Z}^*(M)_-^\circ \subseteq \mathcal{N}^*$ by a similar procedure, where we swap $\mathfrak{X}_{F,[d(M)/2]}^*$ and $\mathfrak{X}_{F,[\nu(M)/2]}^*$. If $t(M) = 2$, we must have $\mathcal{Z}^*(M)_+^\circ \neq \mathcal{Z}^*(M)_-^\circ$ as the two universal pairs of p -divisible groups over $\mathcal{Z}^*(M)_+^\circ$ and $\mathcal{Z}^*(M)_-^\circ$ are non-isomorphic even after forgetting the framings.

Next, consider any finite degree field extension $\check{K}/\check{\mathbb{Q}}_p$, and consider any object

$$(11.5.6) \quad (X, \rho, X', \rho') \in \mathcal{Z}^*(M)^\circ(\text{Spf } \mathcal{O}_{\check{K}})$$

corresponding to a morphism $\text{Spf } \mathcal{O}_{\check{K}} \rightarrow \mathcal{Z}^*(M)^\circ$. As in the proof of Lemma 11.5.3, we have $\text{End}(X) \cong \mathcal{O}_{F,s}$ and $\text{End}(X') \cong \mathcal{O}_{F,s'}$ for some integers $0 \leq s, s' \leq [\nu(M)/2]$. The pair (ρ, ρ') identifies $\text{Hom}(X, X') = M$ as sublattices of $\mathbb{B}^*$. Inspecting the formulas for Gross–Keating invariants in Lemma 11.4.1(3) (and also the $H'_0(\mathbb{Q}_p)$ -invariance of Gross–Keating invariants from

¹⁰Note that the condition $\text{End}(\mathfrak{X}_{F,s}^*) \cong \mathcal{O}_{F,s}$ forces $\mathfrak{X}_{F,s}^*$ to be supersingular (resp. ordinary) if F is nonsplit (resp. split).

Lemma 11.3.1) implies that the only possibilities are $(s, s') = ([d(M)/2], [\nu(M)/2])$ or $(s, s') = ([\nu(M)/2], [d(M)/2])$.

It suffices to show that $\mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathcal{Z}^*(M)^\circ$ factors through $\mathcal{Z}^*(M)_+^\circ$ in the first case, and through $\mathcal{Z}^*(M)_-^\circ$ in the second case. Note that this will also show that $\mathcal{Z}^*(M)_+^\circ = \mathcal{Z}^*(M)_-^\circ$ if $t(M) \leq 1$, which is equivalent to the condition $\nu(M) = d(M)$ (also recall that $d(M) \equiv \nu(M) \pmod{2}$).

Suppose $(s, s') = ([d(M)/2], [\nu(M)/2])$. As $\mathfrak{X}_{F, [d(M)/2]}^*$ and $\mathfrak{X}_{F, [\nu(M)/2]}^*$ are characterized up to isomorphism by their endomorphism rings (and this holds also over $\mathrm{Spf} \mathcal{O}_{\check{K}}$; see Section 11.4 again), we may write

$$(11.5.7) \quad (X, \rho, X', \rho') = (\mathfrak{X}_{F, [d(M)/2]}^*, g\rho, \mathfrak{X}_{F, [\nu(M)/2]}^*, g'\rho')$$

for an element $(g, g') \in H'_0(\mathbb{Q}_p)$, and where $\mathfrak{X}_{F, [d(M)/2]}^*$ and $\mathfrak{X}_{F, [\nu(M)/2]}^*$ are defined by pullback of the p -divisible groups fixed above (abusing notation and recycling the same symbols) along some morphism $\phi_0: \mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}, [\nu(M)/2]}$.

We first claim that $(g, g') = (h, h')$ modulo $H'(\mathbb{Z}_p)$ with (h, h') as in the above construction of $\mathcal{Z}^*(M)_+^\circ$. This is automatic in the supersingular case, since $H'_0(\mathbb{Q}_p)/H'(\mathbb{Z}_p) = \{0\}$ in that case. Assume instead that X and X' are ordinary. By Lemma 11.4.1, (and the above construction of $\mathcal{Z}^*(M)_+^\circ$) we know that the action $\mathcal{O}_{F, s} \subset \mathfrak{X}_{F, [d(M)/2]}^*$ gives M the structure of a free $\mathcal{O}_{F, s}$ -module of rank one. Pick any generator $\psi \in M$. Fix an identification $\mathbb{X}^\circ \cong \mu_{p^\infty} \times \underline{\mathbb{Q}_p}/\mathbb{Z}_p$ as in the proof of Lemma 11.4.1. By part (1) of that lemma, we must have $g'^{-1}\psi g = h'^{-1}\psi h = (1, p^{s'-s})$ modulo $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. This is enough to show $(g, g') = (h, h')$ modulo $H'(\mathbb{Z}_p)$, since $g' = h' = \psi \cdot (1, p^{s-s'})$ modulo $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$.

Pick any $\mathbb{Z}_p$ -basis $\{x, y\}$ for M . Let $F_M, F'_M \subseteq \mathrm{End}^\circ(\mathbb{X}^*)$ be the $\mathbb{Q}_p$ -subalgebras generated by $y^\dagger x$ and $xy^\dagger$, respectively. Fix any isomorphisms $\iota_F: F \rightarrow F_M$ and $\iota'_F: F \rightarrow F'_M$ (which exist by Lemma 11.3.2).

We claim that the pair $(\mathfrak{X}_{F, [\nu(M)/2]}^*, g'\rho')$ over $\mathrm{Spf} \mathcal{O}_{\check{K}}$ is (up to isomorphism compatible with framings) the pullback of $(\mathfrak{X}_{F, [\nu(M)/2]}^*, h'\rho')$ along a unique morphism $\phi: \mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}, [\nu(M)/2]}$ (which need not coincide with ϕ_0). Indeed, first note that $(\mathfrak{X}_{F, [\nu(M)/2]}^*, h'^{-1}g'\rho')$ over $\mathrm{Spf} \mathcal{O}_{\check{K}}$ is a quasi-canonical lifting (of level $[\nu(M)/2]$) for the pair $(\mathbb{X}^*, a \mapsto h'^{-1}\iota'_F(a)h')$ over $\mathrm{Spf} \mathcal{O}_{\check{K}}$, i.e. Corollary 11.3.3 implies that $\iota'_F(\mathcal{O}_{F, [\nu(M)/2]})$ lifts to $\mathrm{End}(\mathfrak{X}_{F, [\nu(M)/2]}^*)$ with respect to the framing $(g'\rho')$. We are also using $h'^{-1}g' \in \mathrm{End}(\mathbb{X}^*)$ here. We similarly know that $(\mathfrak{X}_{F, [\nu(M)/2]}^*, \rho')$ over $\mathrm{Spf} \mathcal{O}_{\check{F}, [\nu(M)/2]}$ is a quasi-canonical lifting for the pair $(\mathbb{X}^*, a \mapsto h'^{-1}\iota'_F(a)h')$. The discussion preceding Lemma 11.4.1 now shows that $(\mathfrak{X}_{F, [\nu(M)/2]}^*, h'^{-1}g'\rho')$ arises as pullback of $(\mathfrak{X}_{F, [\nu(M)/2]}^*, \rho')$ along a unique morphism $\phi: \mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}, [\nu(M)/2]}$.

Pulling back the entire tuple (11.5.5) along ϕ yields a tuple

$$(11.5.8) \quad (\mathfrak{X}_{F, [d(M)/2]}^*, f g\rho, \mathfrak{X}_{F, [\nu(M)/2]}^*, g'\rho') \in \mathcal{Z}^*(M)_+^\circ(\mathrm{Spf} \mathcal{O}_{\check{K}})$$

for some $f \in \mathrm{End}(\mathbb{X}^*)^\times$. Our task is to show that the pairs $(\mathfrak{X}_{F, [d(M)/2]}^*, f g\rho)$ and $(\mathfrak{X}_{F, [d(M)/2]}^*, g\rho)$ are isomorphic.

By construction, we have $\mathrm{Hom}(\mathfrak{X}_{F, [d(M)/2]}^*, \mathfrak{X}_{F, [\nu(M)/2]}^*) = M$ as sublattices of $\mathbb{B}^*$ with respect to both framings $(g\rho, g'\rho')$ and $(f g\rho, g'\rho')$. This implies $M \cdot f = M$. Applying Lemma 11.4.1(4), we conclude $f \in \iota_F(\mathcal{O}_{F, [d(M)/2]}^\times)$. But by construction, the embedding $\iota_F: \mathcal{O}_{F, [d(M)/2]} \rightarrow \mathrm{End}(\mathbb{X}^*)$

lifts via $g\rho$ to an action $\mathcal{O}_{F,[d(M)/2]} \subset \mathfrak{X}_{F,[d(M)/2]}^*$, which shows that the pairs $(\mathfrak{X}_{F,[d(M)/2]}^*, fg\rho)$ and $(\mathfrak{X}_{F,[d(M)/2]}^*, g\rho)$ are isomorphic.¹¹

It remains to consider the case $(s, s') = ([\nu(M)/2], [d(M)/2])$. This is largely similar to the case considered above, but we briefly indicate the argument. First write

$$(11.5.9) \quad (X, \rho, X', \rho') = (\mathfrak{X}_{F, [\nu(M)/2]}^*, g\rho, \mathfrak{X}_{F, [d(M)/2]}^*, g'\rho')$$

for some $(g, g') \in H'_0(\mathbb{Q}_p)$. Let $(h, h') \in H'_0(\mathbb{Q}_p)$ be as in the construction for $\mathcal{Z}^*(M)^\circ_-$ (analogously to the construction of $\mathcal{Z}^*(M)^\circ_+$). We can skip the step of showing $(g, g') = (h, h')$ modulo $H'(\mathbb{Z}_p)$, though it will also be true, upon replacing $(1, p^{s'-s})$ with $(p^{s-s'}, 1)$. Take the same ι_F and ι'_F as above. Arguing as above shows that the pair $(\mathfrak{X}_{F, [\nu(M)/2]}^*, g\rho)$ is the pullback of $(\mathfrak{X}_{F, [\nu(M)/2]}^*, h\rho)$ along a unique morphism $\phi: \mathrm{Spf} \mathcal{O}_{\check{K}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}, [\nu(M)/2]}$. We then obtain a point

$$(11.5.10) \quad (\mathfrak{X}_{F, [\nu(M)/2]}^*, g\rho, \mathfrak{X}_{F, [d(M)/2]}^*, fg'\rho') \in \mathcal{Z}^*(M)^\circ_-(\mathrm{Spf} \mathcal{O}_{\check{K}})$$

for some $f \in \mathrm{End}^\circ(\mathbb{X}^*)$, and would like to show $f \in \iota'_F(\mathcal{O}_{F, [d(M)/2]}^\times)$. Again, we find $f \cdot M = M$, which implies the claim via Lemma 11.4.1(4). This then shows that the pairs $(\mathfrak{X}_{F, [d(M)/2]}^*, fg'\rho')$ and $(\mathfrak{X}_{F, [d(M)/2]}^*, g'\rho')$ are isomorphic.

The last remaining claim of the lemma statement is to show that $\mathcal{Z}^*(M)^\circ$ is disconnected if $t(M) = 2$ in the ordinary case (as $\mathcal{Z}^*(M)^\circ$ is certainly connected in the other cases; recall that $\mathcal{N}^*(\mathbb{F})$ is a singleton in the supersingular case). Recall that we have a canonical identification $\mathcal{N}^*(\mathbb{F}) = H'_0(\mathbb{Q}_p)/H'(\mathbb{Z}_p)$ (Section 11.1). In the above notation, let $\psi \in M$ be generator for M as a free $\iota_F(\mathcal{O}_{F, [d(M)/2]})$ -module (the element ψ will also be a generator for M as a free $\iota'_F(\mathcal{O}_{F, [d(M)/2]})$ -module). Then we have

$$(11.5.11) \quad \mathcal{Z}^*(M)^\circ_+(\mathbb{F}) = \{(h, h') \in H'_0(\mathbb{Q}_p)/H'(\mathbb{Z}_p) : h' = \psi \cdot (1, p^{[d(M)/2] - [\nu(M)/2]}) \text{ modulo } \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times\}$$

$$(11.5.12) \quad \mathcal{Z}^*(M)^\circ_-(\mathbb{F}) = \{(h, h') \in H'_0(\mathbb{Q}_p)/H'(\mathbb{Z}_p) : h' = \psi \cdot (p^{[d(M)/2] - [\nu(M)/2]}, 1) \text{ modulo } \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times\}$$

(using the identification $\mathrm{End}^\circ(\mathbb{X}^*) \cong \mathbb{Q}_p \times \mathbb{Q}_p$ induced by the identification $\mathbb{X}^* \cong \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p$ from above). In particular, we have $[d(M)/2] \neq [\nu(M)/2]$ if $t(M) = 2$, and thus $\mathcal{Z}^*(M)^\circ_+(\mathbb{F}) \neq \mathcal{Z}^*(M)^\circ_-(\mathbb{F})$ in this case. $\square$

Corollary 11.5.6. *For $\mathbb{Z}_p$ -lattices $M \subseteq \mathbb{B}^*$ of rank 2, the quantities*

$$(11.5.13) \quad \deg(\mathcal{Z}^*(M) \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p) \quad \deg(\mathcal{Z}^*(M)^\circ \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p)$$

depend only on the data of $\mathrm{GK}(M)$ and $\mathrm{disc} M_{\mathbb{Q}_p}$.

Proof. For $\mathcal{Z}^*(M)^\circ$, this follows immediately from Lemma 11.5.5. For $\mathcal{Z}^*(M)$, this follows from the fact that the $H'(\mathbb{Q}_p)$ -orbit of M determines and is determined by $\mathrm{GK}(M)$ and $\mathrm{disc} M_{\mathbb{Q}_p}$ as an element of $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ (Lemma 11.3.1); one can then apply the claim for $\mathcal{Z}^*(M)^\circ$ in combination with Lemmas 11.5.2 and Corollary 11.5.4, as $H'(\mathbb{Q}_p) \subset \mathbb{B}^*$ preserves the quadratic form up to $\mathbb{Z}_p^\times$ -scalar. $\square$

¹¹In more detail: Corollary 11.3.3 shows that ι_F lifts via $g\rho$ to an embedding $\mathcal{O}_{F, [\nu(M)/2]} \hookrightarrow \mathrm{End}(\mathfrak{X}_{F, [d(M)/2]}^*)$, but since $\mathrm{End}(\mathfrak{X}_{F, [d(M)/2]}^*) \cong \mathcal{O}_{F, [d(M)/2]}$, this lift of ι_F must extend to the possibly larger ring $\mathcal{O}_{F, [d(M)/2]}$.

11.6. Local “height defect” of special cycles. We conclude this section with a key definition needed to formulate our main non-Archimedean local theorem (Theorem 13.0.1). Let L be a non-degenerate quadratic $\mathbb{Z}_p$ -lattice of rank 2, and set $F = \mathbb{Q}_p[X]/(X^2 - \text{disc } L)$. Pick an embedding of quadratic spaces $L \rightarrow \mathbb{B}^*$: this always exists, and we necessarily have $*$ = s if F is nonsplit and $*$ = o if F is split.

Pick any embedding $\check{\mathbb{Q}}_p \rightarrow \mathbb{C}_p$. Given $z \in \mathcal{Z}^*(L)^{\text{sch}}(\mathbb{C}_p)$, pick any descent of z to a morphism $\text{Spf } \mathcal{O}_{\check{L}} \rightarrow \mathcal{Z}(L)$ for some finite degree field extension $\check{L}$ of $\check{\mathbb{Q}}_p$. By Lemmas 11.5.5 and 11.5.2, pulling back the universal pair of p -divisible groups from $\mathcal{Z}^*(L)$ to $\text{Spf } \mathcal{O}_{\check{L}}$ (forgetting the extra data such as framings, etc.) gives a pair of p -divisible groups $\mathfrak{X}$ and $\mathfrak{X}'$ whose endomorphism rings are (up to isomorphism) the unique orders in $\mathcal{O}_F$ of index p^{s_z} and $p^{s'_z}$, for some $s_z, s'_z \in \mathbb{Z}_{\geq 0}$.

Definition 11.6.1. With notation as above, we set

$$(11.6.1) \quad \delta_{\text{Fal}}(L) := \sum_{z \in \mathcal{Z}^*(L)^{\text{sch}}(\mathbb{C}_p)} \delta_{\text{Fal}}(z) \quad \delta_{\text{Fal}}(z) := \delta_{\text{Fal},F}(s_z) + \delta_{\text{Fal},F}(s'_z)$$

with $\delta_{\text{Fal},F}(s) \in \mathbb{Q}$ as defined in (10.1.7).

By Lemmas 11.5.5 and 11.5.2 and Corollary 11.5.4, we equivalently have

$$(11.6.2) \quad \delta_{\text{Fal}}(L) = \sum_{\substack{L \subseteq M \\ M \text{ integral}}} \deg \mathcal{Z}^*(M)^\circ \cdot \left(\delta_{\text{Fal}} \left(\left\lfloor \frac{d(M)}{2} \right\rfloor \right) + \delta_{\text{Fal}} \left(\left\lfloor \frac{\nu(M)}{2} \right\rfloor \right) \right)$$

where the sum runs over lattices $M \subseteq L_{\mathbb{Q}_p}$. Using Lemma 11.3.1, we see that $\delta_{\text{Fal}}(L)$ depends only on $\text{GK}(L)$ and $\text{disc}(L_{\mathbb{Q}_p})$, and is independent of the choice of embedding $L \rightarrow \mathbb{B}^*$. We also see that the right-hand side of (11.6.1) does not change if $\mathcal{Z}^*(L)^{\text{sch}}$ is replaced by $\tilde{\mathcal{Z}}^*(L)_i^{\text{sch}}$, upon using the action $H'(\mathbb{Q}_p) \curvearrowright \tilde{\mathcal{N}}^*$, which is transitive on the decomposition in (11.1.5) since the action $H'(\mathbb{Q}_p) \curvearrowright \mathbb{B}^*$ which preserves the quadratic form up to scalar, and hence preserves discriminants and Gross–Keating invariants of rank 2 lattices (Lemma 11.3.1).

Next, consider $T \in \text{Sym}_r(\mathbb{Q}_p)$ of rank 2. Let L be the unique non-degenerate quadratic $\mathbb{Z}_p$ -lattice of rank 2 which admits a tuple $\mathbf{x} \in L^r$ with $\text{span}(\mathbf{x}) = L$ and with normalized Gram matrix $\frac{1}{2}(\mathbf{x}, \mathbf{x}) = T$. For any choice of embedding $L \rightarrow \mathbb{B}^*$, with “ $*$ ” uniquely determined by L , we set

$$(11.6.3) \quad \delta_{\text{Fal},p}(T) := \delta_{\text{Fal}}(L) \quad \deg_p(T) := \deg \mathcal{Z}^*(L).$$

Again the choice of embedding $L \rightarrow \mathbb{B}^*$ does not matter, and the right-hand sides moreover depend only on $\text{GK}(L)$ and $\text{disc}(L_{\mathbb{Q}_p})$.

12. HEIGHTS AND RAPOPORT–ZINK UNIFORMIZATION

Fix a prime p and write $\mathbb{F} := \overline{\mathbb{F}}_p$. In this section, we study ordinary and supersingular Rapoport–Zink uniformization. We treat the ordinary and supersingular cases in parallel. We use the notation $*$ $\in$ {s, o} where “s” and “o” mean “supersingular” and “ordinary”, respectively.

12.1. Base points. The Rapoport–Zink uniformization morphisms depend on a choice of base point, which plays a larger role in the ordinary case.

In the notation of Section 8.4, we take $\mathbf{S}$ to be all primes but p , and let $K \subseteq H'(\mathbb{A}_f)$ be any open compact subgroup contained in K° and good away from $\mathbf{S}$, i.e. “good at p ”. We are mostly interested

in $K = K^\circ$, but this does not play much of a role in Section 12. We will abuse notation and suppress K from notation in Section 12, e.g. writing $\mathcal{M}$ instead of $\mathcal{M}_K$. Recall that if S is a connected locally Noetherian scheme over $\mathrm{Spec} \mathbb{Z}_{(p)}$, we can express objects in $\mathcal{M}(S)$ as tuples $(E, E', \bar{\eta}, \bar{\eta}')$ where the pair $(\bar{\eta}, \bar{\eta}')$ is a K^p -orbit of trivializations of away-from- p Tate modules which preserve alternating pairings up to a common scalar (more precisely, see loc. cit.).

By a supersingular (resp. ordinary) *base point*, we mean an object

$$(12.1.1) \quad (\mathbf{E}, \mathbf{E}', \bar{\eta}, \bar{\eta}') \in \mathcal{M}(\bar{\mathbb{F}})$$

with $\mathbf{E}$ and $\mathbf{E}'$ both supersingular (resp. ordinary), equipped with two additional pieces of data.

(1) A pair of isomorphisms

$$(12.1.2) \quad \rho: \mathbb{X}^* \rightarrow \mathbf{E}[p^\infty] \quad \rho': \mathbb{X}^* \rightarrow \mathbf{E}'[p^\infty]$$

for $\mathbb{X}^*$ the framing p -divisible group used to define Rapoport–Zink spaces in Section 11 (with $* = s$ and $* = o$ in the supersingular and ordinary cases, respectively).

(2) A fixed representative (η, η') for $(\bar{\eta}, \bar{\eta}')$, (not just a K^p -orbit).

Given two base points

$$(12.1.3) \quad (\mathbf{E}_1, \mathbf{E}'_1) \quad (\mathbf{E}_2, \mathbf{E}'_2)$$

(additional data suppressed), a quasi-isogeny from the first to the second will mean a pair of quasi-isogenies $\mathbf{E}_1 \rightarrow \mathbf{E}_2$ and $\mathbf{E}'_1 \rightarrow \mathbf{E}'_2$ of the same degree. In this case, we say that the two base points are *isogenous* (and this is an equivalence relation). Note that the notion of isogenous base points does not depend on the additional data $(\rho, \rho', \eta, \eta')$.

In the supersingular case, there is a unique base point up to isogeny. As in the proof of [RZ96, Theorem 6.30], this can be viewed as largely a consequence of the group $H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ satisfying the Hasse principle. For our purposes, we can also explain this phenomenon as follows. If $\mathbf{E}$ is any supersingular elliptic curve over $\mathbb{F}$, then $\mathrm{End}^\circ(\mathbf{E})$ is the quaternion algebra over $\mathbb{Q}$ which is nonsplit only at p and ∞ , and so $\mathrm{End}^\circ(\mathbf{E})^\times$ (viewed as an algebraic group over $\mathbb{Q}$) is a form of GL_2 and hence satisfies the Hasse principle. From this, it follows that every positive rational number is a norm of an element in this quaternion algebra, i.e. there is a quasi-endomorphism of $\mathbf{E}$ of degree n for any positive integer n . An explanation of this Hasse principle for quaternion algebras is given in [Voi21, Theorem 14.7.4] and attributed to Hasse–Schilling.

In the ordinary case, base points exist but are no longer unique up to isogeny. This failure of uniqueness persists even if we consider only elliptic curves with CM by the same maximal order $\mathcal{O}_F$ in some imaginary quadratic field F . For example, let $F = \mathbb{Q}(\sqrt{-6})$ and choose p to be split in $\mathcal{O}_F$. Let $\mathfrak{a} \subseteq \mathcal{O}_F$ be the unique prime ideal lying over the prime 3. Consider the complex elliptic curves $E = \mathbb{C}/\mathcal{O}_F$ and $E' = \mathbb{C}/\mathfrak{a}$ (pick an embedding $F \rightarrow \mathbb{C}$). Descend E and E' to any number field, assume E and E' have good (ordinary) reduction at all primes over p , and form $\mathbf{E}$ and $\mathbf{E}'$ by base-change to $\mathbb{F}$ (pick any p -adic place of the number field and pick an embedding of the residue field into $\mathbb{F}$). We claim that the base points $(\mathbf{E}, \mathbf{E})$ and $(\mathbf{E}, \mathbf{E}')$ are not isogenous. Indeed, first note that $\mathrm{Hom}^\circ(\mathbf{E}, \mathbf{E})$ and $\mathrm{Hom}^\circ(\mathbf{E}, \mathbf{E}')$ are both free F -modules of rank one, but generated by isogenies of degree 1 and 3 respectively. Then note that the number 3 is not the norm of any element of F (since 6 is a norm but 2 is not a norm 3-adically, e.g. by local class field theory).

12.2. Uniformization. We use our usual notation $H = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2$ from Section 8.4. Consider the special cycle $\mathcal{Z}(T) \rightarrow \mathcal{M}$ from Section 8.2, and assume $\mathrm{rk}(T) = 2$ or $\mathrm{rk}(T) = 3$. This means $r = 2$ or $r = 3$ (we also allow the case of singular T with $\mathrm{rk}(T) = 2$ and $r = 3$). In these cases, recall (Lemma 8.2.2) that $\mathcal{Z}(T) \rightarrow \mathrm{Spec} \mathbb{Z}$ is proper quasi-finite. If $\mathrm{rk}(T) = 2$, we set $F := \mathbb{Q}(\sqrt{-\det' T})$, with notation as in loc. cit..

Consider the base change $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}$. For $* \in \{s, o\}$, we write $\check{\mathcal{Z}}^*(T)_p$ for the formal completion of $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}$ along its supersingular or ordinary locus,¹² depending on whether $* = s$ or $* = o$. Note that Lemma 8.2.2 implies that the supersingular and ordinary loci are always closed on $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}$, and either empty or the entire special fiber.

If $\mathrm{rk}(T) = 2$, Lemma 8.2.2 implies that we simply have

$$(12.2.1) \quad \check{\mathcal{Z}}^*(T)_p = \begin{cases} \mathcal{Z}(T)_{\mathrm{Spf} \check{\mathbb{Z}}_p} & \text{if } p \text{ is nonsplit in } F \text{ and } * = s \\ \mathcal{Z}(T)_{\mathrm{Spf} \check{\mathbb{Z}}_p} & \text{if } p \text{ is split in } F \text{ and } * = o \\ \emptyset & \text{else.} \end{cases}$$

If $\mathrm{rk}(T) = 3$, Lemma 8.2.2 implies that we have $\check{\mathcal{Z}}^*(T)_p = \emptyset$ unless $* = s$ and p is the unique prime specified in Lemma 8.2.2(d).

Fix a base point $(\mathbf{E}, \mathbf{E}', \boldsymbol{\rho}, \boldsymbol{\rho}', \boldsymbol{\eta}, \boldsymbol{\eta}')$ as in Section 12.1. If $* = o$ and $\mathrm{rk}(T) = 2$, we also assume that $\mathbf{E}$ and $\mathbf{E}'$ both have CM by F (so that p splits in F). One can construct uniformizations for other base points, but this is the case we will need.

Let $\mathbf{V} := \mathrm{Hom}^\circ(\mathbf{E}, \mathbf{E}')$. This is a $\mathbb{Q}$ -vector space and is equipped with a positive definite bilinear pairing $(-, -)$ given by

$$(12.2.2) \quad (x, y) := \mathrm{tr}^\circ(x^\vee \circ y) \quad \mathrm{tr}^\circ(a) := a^\vee + a$$

for $x, y \in \mathrm{Hom}^\circ(\mathbf{E}, \mathbf{E}')$ and $a \in \mathrm{End}^\circ(\mathbf{E})$.

Consider the group scheme $\mathbf{H}' \rightarrow \mathrm{Spec} \mathbb{Z}$ given by

$$(12.2.3) \quad \mathbf{H}'(R) := \{(\gamma, \gamma') \in (\mathrm{End}(\mathbf{E}) \otimes_{\mathbb{Z}} R)^\times \times (\mathrm{End}(\mathbf{E}') \otimes_{\mathbb{Z}} R)^\times : \gamma^\vee \circ \gamma = \gamma'^\vee \circ \gamma' \in R^\times\}$$

for $\mathbb{Z}$ -algebras R (where we make our usual abuse of notation to use the principal polarization to identify $\mathbf{E}$ with its dual, and similarly for $\mathbf{E}'$).

The datum $(\boldsymbol{\rho}, \boldsymbol{\rho}')$ induces maps

$$(12.2.4) \quad \mathbf{H}'(\mathbb{Q}_p) \rightarrow H'(\mathbb{Q}_p) \quad \mathbf{H}'(\mathbb{Z}_p) \rightarrow H'(\mathbb{Z}_p)$$

for $H'(\mathbb{Q}_p)$ and $H'(\mathbb{Z}_p)$ from Section 11.1. There are similar induced identifications of quadratic spaces $\mathbf{V}(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{B}^*$, where $\mathbb{B}^*$ is as in Section 11.

The datum $(\boldsymbol{\eta}, \boldsymbol{\eta}')$ induces a morphism of group schemes

$$(12.2.5) \quad \mathbf{H}'_{\mathbb{A}_f^p} \rightarrow H_{\mathbb{A}_f^p}.$$

¹²By this, we mean the set of points on (the underlying topological space of) $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}$ in positive characteristic which correspond to $(E, E', \mathbf{x})$ with E and E' both supersingular or both ordinary, depending on whether $* = s$ or $* = o$.

In the supersingular case, the $\ell \neq p$ case of [RZ96, Corollary 6.29]¹³ shows that this map is an isomorphism. The map need not be an isomorphism in the ordinary case. There is a similar induced map $\mathbf{V}(\mathbb{A}_f^p) \rightarrow V(\mathbb{A}_f^p)$ which preserves quadratic forms. Again, this is an isomorphism in the supersingular case, but need not be an isomorphism in the ordinary case.

Using the techniques from [RZ96, §6], one can uniformize $\check{\mathcal{Z}}^*(T)_p$ by local special cycles on Rapoport–Zink spaces, as follows. Our notation and presentation of ideas are similar to that of [Che24, §11].

We consider a stack $\check{\mathcal{Z}}^*(T)_{p,\text{framed}}$ given by the setoids

$$(12.2.6) \quad \check{\mathcal{Z}}^*(T)_{p,\text{framed}}(S) = \left\{ \begin{array}{l} (E, E', \bar{\eta}, \bar{\eta}', \mathbf{x}) \in \check{\mathcal{Z}}^*(T)_p(S) \\ \phi: E_{\bar{S}} \rightarrow \mathbf{E}_{\bar{S}} \text{ and } \phi': E'_{\bar{S}} \rightarrow \mathbf{E}'_{\bar{S}} \text{ quasi-isogenies of the same degree} \end{array} \right\}$$

for schemes S over $\text{Spf } \mathbb{Z}_p$, with $\bar{S} := S \times_{\text{Spf } \mathbb{Z}_p} \mathbb{F}$ as usual, and where the degrees of ϕ and ϕ' are locally constant functions on S . We call (ϕ, ϕ') a “framing”. After viewing $\check{\mathcal{Z}}^*(T)_{p,\text{framed}}$ as a sheaf of sets (passing to isomorphism classes), there is a left action $\mathbf{H}'(\mathbb{Q}) \curvearrowright \check{\mathcal{Z}}^*(T)_{p,\text{framed}}$ given by acting on the framing, i.e. $(h, h') \cdot (\phi, \phi') = (h\phi, h'\phi')$.

Recall the spaces V , W , and W' from Section 8.4. Set

$$(12.2.7) \quad \varphi^p := \mathbf{1}_{\text{Hom}(L, L') \otimes_{\mathbb{Z}} \mathbb{Z}^p}^{\otimes r} \in C_c^\infty(V(\mathbb{A}_f^p)^r).$$

Given $\mathbf{x}^p \in V(\mathbb{A}_f^p)^r$, we consider an “away-from- p local special cycle” defined as the discrete set

$$(12.2.8) \quad \mathcal{Z}(\mathbf{x}^p) = \{h \in H(\mathbb{A}_f^p)/K^p : \varphi^p(h^{-1}\mathbf{x}^p) = 1\}.$$

There is a canonical isomorphism (equivalence)

$$(12.2.9) \quad \check{\mathcal{Z}}^*(T)_{p,\text{framed}} \xrightarrow{\sim} \coprod_{\substack{\mathbf{x} \in \mathbf{V}(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} \check{\mathcal{Z}}^*(\mathbf{x}_p) \times \mathcal{Z}(\mathbf{x}^p)$$

where $\mathbf{x}_p$ and $\mathbf{x}^p$ denote the images of $\mathbf{x}$ in $\mathbf{V}(\mathbb{Q}_p)^r \cong (\mathbb{B}^*)^r$ and $V(\mathbb{A}_f^p)^r$ respectively, using the canonical map $\mathbf{V}(\mathbb{A}_f^p) \rightarrow V(\mathbb{A}_f^p)$, and where $\check{\mathcal{Z}}^*(\mathbf{x}_p) \subseteq \tilde{\mathcal{N}}^*$ is the local special cycle associated to the tuple $\mathbf{x}_p$, as in Section 11.2. The equivalence in (12.2.9) is constructed by sending an object of $\check{\mathcal{Z}}^*(T)_p$ to the following data.

- (1) We define $\mathbf{x} \in V(\mathbb{Q})^r$ by the commutative diagram

$$(12.2.10) \quad \begin{array}{ccc} E_{\bar{S}} & \longrightarrow & \mathbf{E}_{\bar{S}} \\ \downarrow \mathbf{x} & & \downarrow \mathbf{x} \\ E'_{\bar{S}} & \longrightarrow & \mathbf{E}'_{\bar{S}} \end{array}.$$

- (2) (At p) We define a point of $\check{\mathcal{Z}}^*(\mathbf{x}_p)$ by passing to underlying p -divisible groups, with framings induced by (ϕ, ϕ') and (ρ, ρ') .

¹³The statement of loc. cit. should be modified to additionally require that the isocrystals N and N' (associated to A and A' respectively) have $N_{\mathbf{P}}$ and $N'_{\mathbf{P}}$ with the same slope for all $\mathbf{P} \mid p$, in the notation of loc. cit.. This condition is automatic for us in the supersingular case, but not automatic in the ordinary case.

(3) (Away from p) We define a point $h = (h_1, h_2) \in \mathcal{Z}(\mathbf{x}^p)$ by the commutative diagrams

$$(12.2.11) \quad \begin{array}{ccc} W_{\mathbb{A}_f^p} & \xrightarrow{\eta} & T_{\mathbb{A}_f^p}(\mathbf{E})^\circ \\ h_1 \uparrow & & \phi_* \uparrow \\ W_{\mathbb{A}_f^p} & \xrightarrow{\bar{\eta}} & T_{\mathbb{A}_f^p}(E)^\circ \end{array} \quad \begin{array}{ccc} W'_{\mathbb{A}_f^p} & \xrightarrow{\eta'} & T_{\mathbb{A}_f^p}(\mathbf{E}')^\circ \\ h_2 \uparrow & & \phi'_* \uparrow \\ W'_{\mathbb{A}_f^p} & \xrightarrow{\bar{\eta}'} & T_{\mathbb{A}_f^p}(E')^\circ \end{array} .$$

The resulting map (12.2.9) is $\mathbf{H}'(\mathbb{Q})$ -equivariant, where the action of $\mathbf{H}'(\mathbb{Q})$ on the right-hand side is given by $\mathbf{x} \mapsto h\mathbf{x}$ and the maps

$$(12.2.12) \quad \tilde{\mathcal{Z}}^*(\mathbf{x}_p) \rightarrow \tilde{\mathcal{Z}}^*(h \cdot \mathbf{x}_p) \quad \mathcal{Z}(\mathbf{x}^p) \rightarrow \mathcal{Z}(h \cdot \mathbf{x}^p)$$

induced by the Rapoport–Zink action $\mathbf{H}'(\mathbb{Q}) \rightarrow H'(\mathbb{Q}_p) \subset \tilde{\mathcal{N}}^*$ (Section 11.1) and the left-translation action $\mathbf{H}'(\mathbb{Q}) \rightarrow H'(\mathbb{A}_f^p) \subset H(\mathbb{A}_f^p)$.

Thus, after choosing a (quasi-)inverse to (12.2.9) (unique up to unique 2-isomorphism), we compose with the forgetful map $\check{\mathcal{Z}}^*(T)_{p,\text{framed}} \rightarrow \check{\mathcal{Z}}^*(T)_p$ and obtain a *uniformization map*

$$(12.2.13) \quad \left(\coprod_{\substack{\mathbf{x} \in \mathbf{V}(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} \tilde{\mathcal{Z}}^*(\mathbf{x}_p) \times \mathcal{Z}(\mathbf{x}^p) \right) \rightarrow \check{\mathcal{Z}}^*(T)_p$$

which is canonical given the fixed choice of base point (up to the choice of inverse for (12.2.9)). The uniformization map induces a map from the stack quotient

$$(12.2.14) \quad \tilde{\Theta}: \left[\mathbf{H}'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{V}(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} \tilde{\mathcal{Z}}^*(\mathbf{x}_p) \times \mathcal{Z}(\mathbf{x}^p) \right) \right] \rightarrow \check{\mathcal{Z}}^*(T)_p.$$

This is a formally locally of finite type and formally étale monomorphism of locally Noetherian formal algebraic stacks (compare [Che24, Lemma 11.6.1]).

Lemma 12.2.1. *In the supersingular case, the map $\tilde{\Theta}$ is an isomorphism.*

Proof. This can be verified using the techniques from [RZ96, §6], i.e. one can show that (12.2.13) exhibits $\check{\mathcal{Z}}^*(T)_p$ as a stack quotient for the action of $H'(\mathbb{Q})$ on the left-hand side of (12.2.13). For more details on a similar uniformization for unitary groups, see [Che24, §11.6]. A key point is to check surjectivity on $\mathbb{F}$ -points, which follows from the uniqueness of base points up to quasi-isogeny (which came from a Hasse principle) as explained in Section 12.1. $\square$

Lemma 12.2.2. *In the ordinary case, the map $\tilde{\Theta}$ is an open and closed immersion.*

Proof. First note that $\tilde{\Theta}$ is an open and closed immersion on underlying topological spaces, since we already know $\tilde{\Theta}$ is a monomorphism (hence injective on underlying topological spaces), and since the source and target of $\tilde{\Theta}$ have underlying topological spaces which are both finite and discrete (where each point of the underlying topological spaces is an image of an $\mathbb{F}$ -point). For $\check{\mathcal{Z}}^*(T)_p$, here we are using the proper quasi-finiteness proved in Lemma 8.2.2, and for the source of $\tilde{\Theta}$, we are using the fact that each $\tilde{\mathcal{Z}}^*(\mathbf{x}_p)$ has discrete underlying topological space, where every point is the image of an $\mathbb{F}$ -point (follows because $\tilde{\mathcal{Z}}^*(\mathbf{x}_p) \subseteq \mathcal{N}^*$ is a closed subfunctor, and $\mathcal{N}^*$ is discrete by the discussion after (11.1.1)).

Via the same proof strategy as in the supersingular case (again compare [Che24, §11.6] for more details), we find that $\tilde{\Theta}$ is formally étale, surjective onto its image, a monomorphism, and proper on reduced substacks, which is enough to imply that $\tilde{\Theta}$ is an isomorphism onto its (open and closed) image. $\square$

The uniformization map $\tilde{\Theta}$ depends on a choice of base point. Given a base point b , we sometimes write $\tilde{\Theta}_b$ instead of $\tilde{\Theta}$ to emphasize the dependence on b .

Lemma 12.2.3. *Given two base points b and b' , the images of $\tilde{\Theta}_b$ and $\tilde{\Theta}_{b'}$ coincide (resp. are disjoint) if b and b' are isogenous (resp. not isogenous).*

Proof. This follows from the definition of isogenous base points and the definition of $\check{Z}^*(T)_{p,\text{framed}}$, i.e. an object of $\check{Z}^*(T)_p(S)$ admits framings to two different base points if and only if those two base points are isogenous. $\square$

Lemma 12.2.4. *We can cover $\check{Z}^*(T)_p$ by the images of finitely many uniformization maps $\tilde{\Theta}_{b_1}, \dots, \tilde{\Theta}_{b_n}$ (allowing $n = 0$) which can be chosen to have pairwise disjoint images. In the supersingular case, at most one uniformization map is needed.*

Proof. The supersingular case was already proved in Lemma 12.2.1. For the ordinary case, we can associate to any object $\beta = (E, E', \bar{\eta}, \bar{\eta}', x) \in \check{Z}^*(T)_p(\mathbb{F})$ an ordinary base point b by forgetting x and choosing additional data $(\rho, \rho', \eta, \eta')$ as in the definition of a base point (always possible).

Then β is contained in the image of the open and closed uniformization map $\tilde{\Theta}_b$, so we can cover $\check{Z}^*(T)_p$ by the images of finitely many uniformization maps by quasi-compactness of $\check{Z}^*(T)_p$. We can moreover assume that these uniformization maps have pairwise disjoint images, by Lemma 12.2.3. $\square$

12.3. Heights. In Section 12.3, we assume $\text{rk}(T) = 2$. We explain how to use Rapoport–Zink uniformization to compute the quantity $\widehat{\deg}_{\hat{e}_v}(\mathcal{Z}(T))_p$ from Section 10.2.

We consider uniformization along the supersingular (resp. ordinary) locus of $\mathcal{Z}(T)_{\text{Spec } \check{\mathbb{Z}}_p}$ if p is nonsplit (resp. split) in $F = \mathbb{Q}(\sqrt{-\det' T})$. Then Lemma 8.2.2 implies that we have $\check{Z}^*(T)_p = \mathcal{Z}(T)_{\text{Spf } \check{\mathbb{Z}}_p}$, for $* \in \{s, o\}$ chosen as in the previous sentence.

Form the uniformization map $\tilde{\Theta}_b$ as in (12.2.14), associated to a fixed base point b . Fix an embedding $\check{\mathbb{Q}}_p \rightarrow \mathbb{C}_p$. Recall from (10.2.6) that $\widehat{\deg}_{\hat{e}_v}(\mathcal{Z}(T))_p$ may be expressed as a weighted sum over the groupoid $\mathcal{Z}(T)_{\text{Spec } \check{\mathbb{Z}}_p}(\mathbb{C}_p)$. We parameterize this groupoid by a suitable “generic fiber” of the uniformization (12.2.14), expressed by schemification as follows.

Recall the discussion on schemification at the beginning of Section 11. More generally, we obtain an equivalence e.g. between the categories

- (1) Stacks in groupoids $\mathcal{X}$ over $(\text{Sch}/\text{Spec } \check{\mathbb{Z}}_p)_{fppf}$ admitting a collection of morphisms $\{U_i \rightarrow \mathcal{X}\}_i$ where each $U_i \rightarrow \mathcal{X}$ is representable by schemes and finite locally free, where the collection is jointly covering, and where each $U_i \rightarrow \text{Spec } \check{\mathbb{Z}}_p$ is representable by schemes and finite. Consider 1-morphisms only up to 2-isomorphism.
- (2) Stacks in groupoids $\mathcal{X}$ over $(\text{Sch}/\text{Spf } \check{\mathbb{Z}}_p)_{fppf}$ admitting a collection of morphisms $\{U_i \rightarrow \mathcal{X}\}_i$ where each $U_i \rightarrow \mathcal{X}$ is representable by schemes and finite locally free, where the collection

is jointly covering, and where each $U_i \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p$ is representable by schemes and finite. Consider 1-morphisms only up to 2-isomorphism.

The equivalence is given by $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{Spf} \check{\mathbb{Z}}_p}$, and admits a quasi-inverse by “schemification of presentations”. This stacky schemification is convenient bookkeeping for us, but it can be avoided if one prefers (Remark 12.3.1). A similar “schemification” appears in [CLZ].

The stacks in the uniformization (12.2.14) satisfy the property in (2) above. For example, $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}$ satisfies property (2) because it admits a finite étale cover by a scheme (add level structure away from p) and the structure map $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p} \rightarrow \mathrm{Spec} \check{\mathbb{Z}}_p$ is proper and quasi-finite (Lemma 8.2.2). Schemifying, we obtain a morphism (unique up to 2-isomorphism)

$$(12.3.1) \quad \left[\mathbf{H}'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{V}(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} \tilde{\mathcal{Z}}^*(\mathbf{x}_p)^{\mathrm{sch}} \times \mathcal{Z}(\mathbf{x}^p) \right) \right] \rightarrow \mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}$$

The morphism in (12.3.1) is an isomorphism in the supersingular case, and an open and closed immersion in the ordinary case (follows from the formal algebraic stack version).

We next recall the structure of the local special cycle $\tilde{\mathcal{Z}}^*(\mathbf{x}_p) \subseteq \tilde{\mathcal{N}}^*$ from Sections 11.1 and 11.2. Recall that there is an open and closed disjoint union decomposition $\tilde{\mathcal{Z}}^*(\mathbf{x}_p) = \coprod_i \tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i$ indexed over $i \in H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p)$, in the notation (11.2.4). The terms $\tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i$ are connected in the supersingular case, but not necessarily connected in the ordinary case. This decomposition is compatible with the action $H'(\mathbb{Q}_p) \curvearrowright \tilde{\mathcal{N}}^*$ and the action $H'(\mathbb{Q}_p) \curvearrowright H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p)$ by left-translation.

Using Corollary 11.5.6 and Lemma 11.3.1, we find that the structure maps $\tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p$ have the same degree as i varies, since the action $H'(\mathbb{Q}_p) \curvearrowright \tilde{\mathcal{Z}}^*$ is transitive on the terms in the decomposition (11.1.5) indexed over $i \in H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p)$. For any i , we have

$$(12.3.2) \quad \deg(\tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p) = \deg_p(T) = \deg(\mathcal{Z}^*(\mathbf{x}_p) \rightarrow \mathrm{Spf} \check{\mathbb{Z}}_p) = \sum_{x \in \mathcal{Z}^*(\mathbf{x}_p)^{\mathrm{sch}}(\mathbb{C}_p)} 1$$

with $\deg_p(T)$ defined in (11.6.3).

From the schemified uniformization (12.3.1), if $\deg_{\mathbb{Q}}(\mathcal{Z}(T)_{\mathbb{Q}})$ denotes the stacky degree of $\mathcal{Z}(T)_{\mathbb{Q}} \rightarrow \mathrm{Spec} \mathbb{Q}$ (denoted $\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T)_{\mathbb{Q}})$ in Section 8.5), it now follows formally that we have

$$(12.3.3) \quad \deg_{\mathbb{Q}} \mathcal{Z}(T)_{\mathbb{Q}} = \sum_{\alpha \in \mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}^{\mathcal{M}}(\mathbb{C}_p)} \frac{1}{\# \mathrm{Aut}(\alpha)}$$

$$(12.3.4) \quad = \deg_p(T) \cdot \# \left[\mathbf{H}'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{V}(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p) \times \mathcal{Z}(\mathbf{x}^p) \right) \right]$$

in the supersingular case, where the first equality is largely by definition (Section 8.5), where the sum runs over isomorphism classes of the groupoid $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}^{\mathcal{M}}(\mathbb{C}_p)$, and where $\#[\cdots]$ means groupoid cardinality. In the ordinary case, the second equality (12.3.4) still holds if we replace the sum over $\mathcal{Z}(T)_{\mathrm{Spec} \check{\mathbb{Z}}_p}^{\mathcal{M}}(\mathbb{C}_p)$ with the sum over only the open and closed substack uniformized by the image of $\tilde{\Theta}_b$.

We next make a similar-looking decomposition for heights. As explained before Definition 11.6.1, every object $z \in \tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i^{\text{sch}}(\mathbb{C}_p)$ has an associated “local height defect” $\delta_{\text{Fal}}(z) \in \mathbb{Q}$, defined in terms of the endomorphism rings of the pair of universal p -divisible groups over the descent of z to $\text{Spf } \mathcal{O}_{\check{L}}$ for some finite degree field extension $\check{L}$ of $\check{\mathbb{Q}}_p$. Given $\tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i^{\text{sch}}$ mapping to the source of the uniformization map $\tilde{\Theta}_b$ from (12.2.14), and if $\alpha \in \mathcal{Z}(T)_{\text{Spec } \check{\mathbb{Z}}_p}(\mathbb{C}_p)$ is the image of some $z \in \tilde{\mathcal{Z}}^*(\mathbf{x}_p)_i^{\text{sch}}(\mathbb{C}_p)$, then Lemmas 10.1.4 and 10.1.5 imply that we have

$$(12.3.5) \quad \delta_{\text{Fal}}(z) \cdot \log p = \delta_{\text{Fal},p}(\alpha)$$

where $\delta_{\text{Fal},p}(\alpha)$ was defined in Section 10.2. This is because $\delta_{\text{Fal},p}(\alpha)$ is defined in terms of the p -index of the endomorphism rings of the pair of universal elliptic curves, while $\delta_{\text{Fal}}(z)$ is defined similarly but using universal p -divisible groups, but Lemma 10.1.5 (and the equivalence between étale p -divisible groups and Tate modules, applied over $\text{Spec } \check{L}$, as well as fully-faithfulness of the generic fiber functor of p -divisible groups as in [Tat67, Theorem 4]) implies that these endomorphism rings may be identified, since Rapoport–Zink uniformization (by construction) is compatible with universal p -divisible groups and universal abelian varieties.

Upon considering the formula (10.2.6) for $\widehat{\deg}_{\hat{\mathcal{E}}^\vee}(\mathcal{Z}(T))_p$ and the formula (11.6.1) for $\delta_{\text{Fal}}(\text{span}(\mathbf{x}_p))$, it then follows formally from the above discussion and the schemified uniformization (12.3.1) that we have

$$(12.3.6) \quad -\widehat{\deg}_{\hat{\mathcal{E}}^\vee}(\mathcal{Z}(T))_p = \sum_{\alpha \in \mathcal{Z}(T)_{\text{Spec } \check{\mathbb{Z}}_p}(\mathbb{C}_p)} \frac{\delta_{\text{Fal},p}(\alpha)}{\# \text{Aut}(\alpha)}$$

$$(12.3.7) \quad = \delta_{\text{Fal},p}(T) \cdot \# \left[\mathbf{H}'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{V}(\mathbb{Q})^r \\ \frac{1}{2}(\mathbf{x}, \mathbf{x}) = T}} H'(\mathbb{Q}_p)/H'_0(\mathbb{Q}_p) \times \mathcal{Z}(\mathbf{x}^p) \right) \right] \cdot \log p$$

in the supersingular case, with $\delta_{\text{Fal},p}(T)$ defined in (11.6.3). In the ordinary case, the second equality (12.3.7) continues to hold if we replace the sum over $\mathcal{Z}(T)_{\text{Spec } \check{\mathbb{Z}}_p}(\mathbb{C}_p)$ with the sum over only the open and closed substack uniformized by the image of $\tilde{\Theta}_b$.

In all cases, supersingular or ordinary, we find that the “normalized height difference” is given by

$$(12.3.8) \quad \frac{\widehat{\deg}_{\hat{\mathcal{E}}^\vee}(\mathcal{Z}(T))_p}{\deg_{\mathcal{E}^\vee}(\mathcal{Z}(T)_{\mathbb{Q}})} = -\frac{\delta_{\text{Fal},p}(T)}{\deg_p(T)} \cdot \log p$$

whenever $\mathcal{Z}(T) \neq \emptyset$, which is equivalent to $\deg_{\mathbb{Q}}(\mathcal{Z}(T)_{\mathbb{Q}}) \neq 0$. Indeed, the supersingular case of (12.3.8) follows immediately from dividing (12.3.7) by (12.3.4). The ordinary case follows as well, after we range over the finitely many uniformization maps $\tilde{\Theta}_b$ for different choices of base point b , as in Lemma 12.2.4.

Remark 12.3.1. It is possible to prove (12.3.4), (12.3.7), and (12.3.8) while avoiding schemification for stacks, although we find that language to be convenient bookkeeping. In brief, one can add level structure away from p , take constant finite étale covers by formal schemes, and divide by the degrees of these covers; compare [CLZ].

Part 3. Arithmetic degrees of special cycles

13. PROOF OF NON-ARCHIMEDEAN LOCAL IDENTITIES

The following is the main non-Archimedean local theorem of this article, which is an identity between the “change of Faltings height along special cycles” and near-central derivatives of normalized local densities.

In the theorem below, the left-hand sides were defined in (11.6.3), and the right-hand sides were defined in (2.3.5).

Theorem 13.0.1. *Let $L \subset \mathbb{B}^*$ be a $\mathbb{Z}_p$ -lattice of rank 2 with $* \in \{s, o\}$. Then*

(a) *For the special cycle $\mathcal{Z}^*(L)$, we have*

$$(13.0.1) \quad \deg \mathcal{Z}^*(L) = \text{Den}^*(L).$$

(b) *For the change of Faltings height along the special cycle $\mathcal{Z}^*(L)$, we have*

$$(13.0.2) \quad \delta_{\text{Fal}}(L) = \partial \text{Den}^*(L).$$

The goal of this section is to prove Theorem 13.0.1. Fix a rank 2 lattice $L \subset \mathbb{B}^*$ where $* \in \{s, o\}$ with $\text{GK}(L) = (a, b)$. Recall that $\nu(L) = a + b$ and $d(L) = b - a$.

13.1. **Base cases.** We first consider the case that $\nu(L) \leq 2$.

13.1.1. $\nu(L) = 0$. In this case $a = b = 0$, the special cycle $\mathcal{Z}^*(L) = \mathcal{Z}^*(L)^\circ \simeq \text{Spf } \mathcal{O}_{\check{F}_0}$ by Lemma 11.5.5, hence $\deg \mathcal{Z}^*(L) = 1$ by (11.4.2), and $\delta_{\text{Fal}}(L) = 0$ by (11.6.2). By (B.2.2) and (B.2.3),

$$\text{Den}^*(X, L) = 1,$$

hence $\text{Den}^*(L) = 1 = \deg \mathcal{Z}^*(L)$ and $\partial \text{Den}^*(L) = 0 = \delta_{\text{Fal}}(L)$. Theorem 13.0.1 is true.

13.1.2. $\nu(L) = 1$. In this case we must have $a = 0$ and $b = 1$. Such a lattice L exists only if $* = s$, and the only integral lattice containing L is the lattice L itself. Hence the special cycle $\mathcal{Z}^*(L) = \mathcal{Z}^*(L)^\circ \simeq \text{Spf } \mathcal{O}_{\check{F}_0}$ by Lemma 11.5.5, and

$$\delta_{\text{Fal}}(L) = \deg \mathcal{Z}^s(L)^\circ \cdot (\delta_{\text{Fal}}(0) + \delta_{\text{Fal}}(0)) = 0.$$

By (B.2.1),

$$\text{Den}^*(X, L) = 1,$$

hence $\text{Den}^*(L) = 1 = \deg \mathcal{Z}^*(L)$ and $\partial \text{Den}^*(L) = 0 = \delta_{\text{Fal}}(L)$. Theorem 13.0.1 is true.

13.1.3. $\nu(L) = 2$. In this case $(a, b) = (0, 2)$ or $(1, 1)$. We first consider the case that $(a, b) = (0, 2)$. Let $M \subset \mathbb{B}^*$ be an integral rank 2 lattice such that $L \subsetneq M$. The Gross-Keating invariant of M must be $(0, 0)$ (this is obvious for the case $p > 2$, the case $p = 2$ can be proved using Lemma B.1.1 (c) and (d)) and there is only one such M , hence the contribution of M to $\deg \mathcal{Z}^*(L)$ is 1, while 0

to $\delta_{\text{Fal}}(L)$. By Lemma 11.5.5 and (11.4.2),

$$\begin{aligned} \deg \mathcal{Z}^*(L) &= \deg \mathcal{Z}^*(L)^\circ + \deg \mathcal{Z}^*(M)^\circ = \begin{cases} p+1+1 = p+2, & \text{if } * = \text{s}; \\ p-1+1 = p, & \text{if } * = \text{o}. \end{cases} \\ \delta_{\text{Fal}}(L) &= \deg \mathcal{Z}^*(L)^\circ \cdot (\delta_{\text{Fal}}(1) + \delta_{\text{Fal}}(1)) = \begin{cases} (p+1) \cdot \left(1 - \frac{2}{p+1}\right) = p-1, & \text{if } * = \text{s} \\ (p-1) \cdot (1-0), & \text{if } * = \text{o} \end{cases} = p-1. \end{aligned}$$

On the other hand, by (B.2.2) and (B.2.3),

$$\text{Den}^*(X, L) = \begin{cases} X^{-1} + 1 + pX, & \text{if } * = \text{s}; \\ X^{-1} - 1 + pX, & \text{if } * = \text{o}. \end{cases}$$

Therefore $\text{Den}^*(L) = \deg \mathcal{Z}^*(L)$ and $\partial \text{Den}^*(L) = p-1 = \delta_{\text{Fal}}(L)$. Theorem 13.0.1 is true.

Next we consider the case that $(a, b) = (1, 1)$. Let $M \subset \mathbb{B}^*$ be a rank 2 lattice such that $L \subsetneq M \subset M^\vee \subsetneq L^\vee$. Such a lattice M corresponds to an isotropic line in the 2-dimensional nondegenerate quadratic space $L^\vee/L \simeq \mathbb{F}_p^2$, where the quadratic form is given by $\bar{x} \in L^\vee/L \rightarrow \overline{p \cdot q(x)}$.

- If $* = \text{s}$, such M doesn't exist because $L^\vee/L$ is anisotropic (this fact is obvious for $p > 2$, the case $p = 2$ can be proved using Lemma B.1.1 (b)).
- If $* = \text{o}$, there are in total 2 such lattices M because $L^\vee/L$ is nondegenerate and isotropic. However, the Gross–Keating invariant of such M must be $(0, 0)$, hence the contribution of M to $\deg \mathcal{Z}^*(L)$ is 1, while 0 to $\delta_{\text{Fal}}(L)$.

By Lemma 11.5.5 and (11.4.2),

$$\begin{aligned} \deg \mathcal{Z}^*(L) &= \begin{cases} 2(p+1), & \text{if } * = \text{s}; \\ 2(p-1) + 2 = 2p, & \text{if } * = \text{o}, \end{cases} \\ \delta_{\text{Fal}}(L) &= \deg \mathcal{Z}^*(L)^\circ \cdot (\delta_{\text{Fal}}(0) + \delta_{\text{Fal}}(1)) = \begin{cases} 2(p+1) \cdot \left(\frac{1}{2} - \frac{1}{p+1}\right), & \text{if } * = \text{s}; \\ 2(p-1) \cdot \left(\frac{1}{2} - 0\right), & \text{if } * = \text{o}, \end{cases} = p-1. \end{aligned}$$

On the other hand, by (B.2.2) and (B.2.3),

$$\text{Den}^*(X, L) = \begin{cases} X^{-1} + p + 1 + pX, & \text{if } * = \text{s}; \\ X^{-1} + p - 1 + pX, & \text{if } * = \text{o}. \end{cases}$$

Therefore $\text{Den}^*(L) = \deg \mathcal{Z}^*(L)$ and $\partial \text{Den}^*(L) = p-1 = \delta_{\text{Fal}}(L)$. Theorem 13.0.1 is true.

13.2. Counting Lattices.

Lemma 13.2.1. *Let L be a rank 2 $\mathbb{Z}_p$ -lattice spanned by elements x and y , there is a one-to-one correspondence between the following two sets*

$$(13.2.1) \quad \iota_L^x : \{(k, c) : k \in \mathbb{Z}_{\geq 1}, c \in \mathbb{Z}_p/p^k\mathbb{Z}_p\} \longrightarrow \{M \supsetneq L : x \notin pM\}$$

$$(k, c) \longmapsto M = \mathbb{Z}_p\text{-span of } x \text{ and } p^{-k}(y + cx).$$

Proof. Since $x \notin pM$, the lattice M is spanned over $\mathbb{Z}_p$ by x and $y' = ay + c'x$. If $\nu_p(a) \geq 0$, the lattice M is spanned by x , y and $c'x$ since $L \subset M$. Since $x \notin pM$, we must have $c' \in \mathbb{Z}_p$, then

$L = M$, which is a contradiction. Hence $\nu_p(a) < 0$. Scaling by an element in $\mathbb{Z}_p^\times$, we can assume that $a = p^{-k}$ for some $k \geq 1$ and $y' = p^{-k}(y + cx)$ where $c' = p^{-k}c$. Therefore we have

$$y = p^k y' - cx,$$

hence we must have $c \in \mathbb{Z}_p$ since $L \subset M$ and x, y' is a $\mathbb{Z}_p$ -basis of M .

However, the element c is only determined up to $\mathbb{Z}_p/p^k\mathbb{Z}_p$ because we can change the element y' by adding $\mathbb{Z}_p$ -multiples of x . Therefore the map ι_L^x (13.2.1) is surjective.

Now let (k_1, c_1) and (k_2, c_2) be two elements of the set on the left hand side of (13.2.1). Let M_i be the lattice spanned by x and $y_i = p^{-k_i}(y + c'_i x)$ where $i = 1, 2$, here we take $c'_i \in \mathbb{Z}_p$ as a lift of the element $c_i \in \mathbb{Z}_p/p^{k_i}\mathbb{Z}_p$. It's easy to see that

$$[x, y_2] = [x, y_1] \cdot \begin{pmatrix} 1 & p^{-k_2}(c'_2 - c'_1) \\ 0 & p^{k_1 - k_2} \end{pmatrix}.$$

Therefore if $M_1 = M_2$, we have $k_1 = k_2$ and $p^{-k_2}(c'_2 - c'_1) \in \mathbb{Z}_p$, which is equivalent to saying that $(k_1, c_1) = (k_2, c_2)$. Hence the map ι_L^x (13.2.1) is injective. $\square$

13.3. Gross-Keating invariants of intermediate lattices. Let L be a nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$. In §B.3, we construct a lattice L' such that

- (a). $L \subsetneq L'$ and $L'/L \simeq \mathbb{F}_p$.
- (b). $\text{GK}(L')$ is either $(a, b-2)$, or $(b-2, a)$ if $b-2 < a$.

By (b), we can find a basis $\mathbf{x} = (x_1, x_2)$ of L such that $x_1, p^{-1}x_2$ is a basis of L' . In the following paragraphs, denote by $L_{k,c}$ the lattice $\iota_L^x((k, c))$ for $k \geq 1$ and $c \in \mathbb{Z}_p/p^k\mathbb{Z}_p$. We also define $L_{0,0}$ to be the lattice L itself.

Lemma 13.3.1. *Let L be a nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$. For $k \geq 0$ and $c \in \mathbb{Z}_p/p^k\mathbb{Z}_p$, if L is not in the Case 1 of §B.3, the Gross-Keating invariant of the lattice $L_{k,c}$ is $\text{GK}(L_{k,c}) = (a - 2k, b)$.*

Proof. Let $\text{GK}(L_{k,c}) = (\mu_1, \mu_2)$. By Lemma B.1.2, $\mu_1 + \mu_2 = a + b - 2k$, hence we only need to show that $\mu_1 = a - 2k$. We first prove the case that $p > 2$ (Case 2 in §B.3), notice that under the basis $\mathbf{x} = (x_1, x_2)$ of L , the quadratic form is diagonalized in the way that $q(xx_1 + yx_2) = u_1 p^a x^2 + u_2 p^b y^2$. Then

$$q\left(x \cdot p^{-k}(x_1 + cx_2) + yx_2\right) = \left(u_1 p^{a-2k} + u_2 c^2 p^{b-2k}\right) x^2 + 2u_2 c p^{b-k} xy + u_2 p^b y^2.$$

Under the assumption that we are not in the Case 1 of §B.3, the lattice L is either isotropic with $a < b$, or L is anisotropic. Therefore we have $\nu_p(u_1 p^{a-2k} + u_2 c^2 p^{b-2k}) = a - 2k < \min\{\nu_p(2u_2 c p^{b-k}), \nu_p(u_2 p^b) = b\}$. Therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a).

Now we focus on the case that $p = 2$. If the lattice L is not diagonalizable, we have $a = b$ and L is anisotropic by Lemma B.1.1 (c), so we are in Case 3 of §B.3. Under the basis $\mathbf{x} = (x_1, x_2)$ of L , the quadratic form is given by $q(xx_1 + yx_2) = 2^a(x^2 + xy + y^2)$. Then

$$q\left(x \cdot 2^{-k}(x_1 + cx_2) + yx_2\right) = 2^{a-2k}(1 + c + c^2)x^2 + 2^{a-k}(1 + 2c)xy + 2^a y^2.$$

Notice that $\nu_2(2^{a-2k}(1 + c + c^2)) = a - 2k < \min\{\nu_2(2^{a-k}(1 + 2c)) = a - k, \nu_2(2^a) = a\}$, therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a).

If the lattice is diagonalizable, following the notations in Case 4 of §B.3, let $\mathbf{e}$ be a basis of L as in Lemma B.1.1 (d). Then

$$q(xe_1 + y \cdot e_2) = u_1 2^a x^2 + u_2 2^{b-\delta} y^2,$$

where $\delta = 0, 1, 2$ according to (B.1.3).

Case 4-0 For $\delta = 0$, we have $a \leq b$ and $a \not\equiv b \pmod{2}$. Then $x_1 = e_1$, $x_2 = e_2$ and

$$q\left(x \cdot 2^{-k}(x_1 + cx_2) + yx_2\right) = \left(u_1 2^{a-2k} + c^2 u_2 2^{b-2k}\right) x^2 + cu_2 2^{b+1-k} xy + u_2 2^b y^2.$$

Notice that $\nu_2(u_1 2^{a-2k} + c^2 u_2 2^{b-2k}) = a - 2k$, therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a).

Case 4-1 For $\delta = 1$, we have $a \leq b - 1$ and $a \equiv b - 1 \pmod{2}$. If $a \leq b - 3$, we have $x_1 = e_1$, $x_2 = e_2$ and

$$q\left(x \cdot 2^{-k}(x_1 + cx_2) + yx_2\right) = \left(u_1 2^{a-2k} + c^2 u_2 2^{b-1-2k}\right) x^2 + cu_2 2^{b-k} xy + u_2 2^{b-1} y^2.$$

Notice that $\nu_2(u_1 2^{a-2k} + c^2 u_2 2^{b-1-2k}) = a - 2k$, therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a). If $a = b - 1$, we have $x_1 = e_1$, $x_2 = e_1 + e_2$ and

$$\begin{aligned} & q\left(x \cdot 2^{-k}(x_1 + cx_2) + yx_2\right) \\ &= \left((1+c)^2 u_1 2^{a-2k} + c^2 u_2 2^{a-2k}\right) x^2 + (u_1 + c(u_1 + u_2)) 2^{a+1-k} xy + (u_1 + u_2) 2^a y^2. \end{aligned}$$

Notice that $\nu_2((1+c)^2 u_1 2^{a-2k} + c^2 u_2 2^{a-2k}) = a - 2k$, therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a).

Case 4-2 For $\delta = 2$, we have $a \leq b - 2$ and $a \equiv b \pmod{2}$. If $a \leq b - 4$, we have $x_1 = e_1$, $x_2 = e_2$ and

$$q\left(x \cdot 2^{-k}(x_1 + cx_2) + yx_2\right) = \left(u_1 2^{a-2k} + c^2 u_2 2^{b-2-2k}\right) x^2 + cu_2 2^{b-1-k} xy + u_2 2^{b-2} y^2.$$

Notice that $\nu_2(u_1 2^{a-2k} + c^2 u_2 2^{b-2-2k}) = a - 2k$, therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a). If $a = b - 2$, we have $x_1 = e_1$, $x_2 = e_1 + e_2$ and

$$\begin{aligned} & q\left(x \cdot 2^{-k}(x_1 + cx_2) + yx_2\right) \\ &= \left((1+c)^2 u_1 2^{a-2k} + c^2 u_2 2^{a-2k}\right) x^2 + (u_1 + c(u_1 + u_2)) 2^{a+1-2k} xy + (u_1 + u_2) 2^a y^2. \end{aligned}$$

Notice that $\nu_2((1+c)^2 u_1 2^{a-2k} + c^2 u_2 2^{a-2k}) = a - 2k$, therefore $\mu_1 = a - 2k$ by Lemma B.1.1 (a).

□

Lemma 13.3.2. *Let L be an isotropic quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, a)$ for some $a \in \mathbb{Z}$, i.e., Case 1 of §B.3. For $k \geq 1$ and $c \in \mathbb{Z}_p/p^k \mathbb{Z}_p$,*

$$\text{GK}(L_{k,c}) = \begin{cases} (a - 2k + t, a - t), & \text{if } c \text{ or } 1 + c \in p^t \mathbb{Z}_p^* / p^k \mathbb{Z}_p \text{ for some } 1 \leq t \leq k; \\ (a - 2k, a), & \text{otherwise.} \end{cases}$$

Proof. Let $\text{GK}(L_{k,c}) = (\mu_1, \mu_2)$. By Lemma B.1.2, $\mu_1 + \mu_2 = 2a - 2k$, hence we only need to find μ_1 using Lemma B.1.1 (a). Using the notations in Case 1 of §B.3,

$$(13.3.1) \quad \begin{aligned} q\left(x \cdot p^{-k}(x_1 + cx_2) + y \cdot x_2\right) &= p^{a-2k} \left((1+c)x + p^k y\right) \left(cx + p^k y\right) \\ &= c(1+c)p^{a-2k}x^2 + (1+2c)p^{a-k}xy + p^a y^2. \end{aligned}$$

If c or $1+c \in p^t \mathbb{Z}_p^*/p^k \mathbb{Z}_p$ for some $1 \leq t \leq k-1$, notice that $\nu_p(q(\mathbf{x})) \geq a - 2k + t$. Moreover, $q(p^{-k}(x_1 + cx_2)) = p^{a-2k}c(1+c)$ has p -adic valuation $a - 2k + t$, hence $\mu_1 = a - 2k + t$ and $\mu_2 = a - t$.

If $c \equiv 0$ or $-1 \pmod{p^k}$, we can just let $c = 0$ or -1 since the lattice $L_{k,c}$ is determined up to $c \in \mathbb{Z}_p/p^k \mathbb{Z}_p$, then the quadratic form of $L_{k,c}$ is equivalent to $p^{a-k}xy$, hence $\text{GK}(L_{k,c}) = (a-k, a-k)$ by Lemma B.1.1.

Otherwise, notice that $\nu_p(q(\mathbf{x})) \geq a - 2k$ and $q(p^{-k}(x_1 + cx_2)) = p^{a-2k}c(1+c)$ has p -adic valuation $a - 2k$, hence $\mu_1 = a - 2k$ and $\mu_2 = a$. $\square$

Corollary 13.3.3. *Let L be a nondegenerate integral quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$.*

- *If L is not in the Case 1 of §B.3, the lattice $L_{k,c}$ is integral if and only if $1 \leq k \leq [a/2]$.*
- *If L is in the Case 1 of §B.3, the lattice $L_{k,c}$ is integral if and only if $1 \leq k \leq [a/2]$ or*

$$\left\lfloor \frac{a}{2} \right\rfloor + 1 \leq k \leq a \text{ and } c \text{ or } 1+c \in p^t \mathbb{Z}_p^*/p^k \mathbb{Z}_p \text{ for some } t \geq 2k - a.$$

Proof. A nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$ is integral if and only if $a \geq 0$. Then this corollary follows from Lemma 13.3.1 and Lemma 13.3.2. $\square$

13.4. Geometric difference formula.

Proposition 13.4.1. *Let L be a nondegenerate integral quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$ such that $b \geq 2$. Let L' be the lattice constructed in §B.3.*

- *If $a \not\equiv b \pmod{2}$,*

$$(13.4.1) \quad \deg \mathcal{Z}^*(L) - \deg \mathcal{Z}^*(L') = (a+1)p^{(a+b-1)/2},$$

$$(13.4.2) \quad \begin{aligned} \delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L') \\ = \frac{(a+1)(b-1)p^{(a+b+3)/2} - 2(a+1)bp^{(a+b+1)/2} + (a+1)(b+1)p^{(a+b-1)/2} + 2p^{a+1} - 2}{2(p-1)^2}. \end{aligned}$$

- *If $a \equiv b \pmod{2}$ and L is anisotropic,*

$$(13.4.3) \quad \deg \mathcal{Z}^*(L) - \deg \mathcal{Z}^*(L') = (a+1)p^{(a+b)/2-1}(p+1),$$

$$(13.4.4) \quad \begin{aligned} \delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L') \\ = \frac{1}{2(p-1)^2} \cdot \left((a+1)bp^{(a+b+4)/2} - (a+1)(b+4)p^{(a+b+2)/2} \right. \\ \left. - (a+1)(b-4)p^{(a+b)/2} + (a+1)bp^{(a+b-2)/2} + 4p^{a+1} - 4 \right). \end{aligned}$$

- If $a \equiv b \pmod{2}$ and L is isotropic,

$$(13.4.5) \quad \deg \mathcal{Z}^*(L) - \deg \mathcal{Z}^*(L') = \begin{cases} (a+1)p^{(a+b)/2-1}(p-1), & \text{if } a < b; \\ p^{a-1}((a+1)p - a + 1), & \text{if } a = b. \end{cases}$$

$$(13.4.6) \quad \delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L') = \frac{(a+1)b}{2} \cdot p^{(a+b-2)/2}(p-1).$$

Proof. By the decomposition of $\mathcal{Z}^*(L)$ in Lemma 11.5.2, and the definition of $\delta_{\text{Fal}}(L)$ in (11.6.2), the differences $\deg \mathcal{Z}^*(L) - \deg \mathcal{Z}^*(L')$ and $\delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L')$ are both summations ranging over integral lattices M such that $L \subset M$ and $x_2 \notin pM$, such lattices M and their Gross–Keating invariant have been described explicitly in Lemma 13.2.1, 13.3.1 and 13.3.2.

Notice that for all the prime numbers p (including $p = 2$), the field $\mathbb{Q}_{L,p} = \mathbb{Q}_p(\sqrt{\text{disc } L})$ is a ramified extension of $\mathbb{Q}_p$ if and only if $a \not\equiv b \pmod{2}$, an unramified extension of $\mathbb{Q}_p$ if and only if L is anisotropic, and $\mathbb{Q}_p \times \mathbb{Q}_p$ if and only if L is isotropic (we refer to the proof of Lemma B.1.2 and Remark B.1.3).

Let's first consider the case that L is not in the Case 1 of §B.3. For $1 \leq k \leq [a/2]$ and $c \in \mathbb{Z}_p/p^k\mathbb{Z}_p$, the Gross–Keating invariant of the lattice $L_{k,c}$ is $(a - 2k, b)$, therefore

$$\deg \mathcal{Z}^*(L) - \deg \mathcal{Z}^*(L') = \sum_{k=0}^{[a/2]} \sum_{c \in \mathbb{Z}_p/p^k\mathbb{Z}_p} \deg \mathcal{Z}^*(L_{k,c})^\circ = \sum_{k=0}^{[a/2]} p^k \cdot \deg \mathcal{Z}^*(L_{k,0})^\circ.$$

and

$$\begin{aligned} \delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L') &= \sum_{k=0}^{[a/2]} \sum_{c \in \mathbb{Z}_p/p^k\mathbb{Z}_p} \deg \mathcal{Z}^*(L_{k,c})^\circ \cdot \left(\delta_{\text{Fal}} \left(\left[\frac{b-a}{2} \right] + k \right) + \delta_{\text{Fal}} \left(\left[\frac{a+b}{2} \right] - k \right) \right) \\ &= \sum_{k=0}^{[a/2]} p^k \cdot \deg \mathcal{Z}^*(L_{k,0})^\circ \cdot \left(\delta_{\text{Fal}} \left(\left[\frac{b-a}{2} \right] + k \right) + \delta_{\text{Fal}} \left(\left[\frac{a+b}{2} \right] - k \right) \right). \end{aligned}$$

Then formulae (13.4.1)–(13.4.6) (excluding the Case 1) follow from explicit computations based on Lemma 11.5.5, (11.4.2) and (10.1.7).

If L is in the Case 1 of §B.3, by Corollary 13.3.3,

$$\begin{aligned} \deg \mathcal{Z}^o(L) - \deg \mathcal{Z}^o(L') &= \deg \mathcal{Z}^o(L)^\circ + \sum_{k=1}^{[a/2]} \left(\sum_{t=0}^k N(k, t) \cdot \deg \mathcal{Z}^o(L_{k,p^t})^\circ \right) \\ &\quad + \sum_{k=[a/2]+1}^a \left(\sum_{t=2k-a}^k N(k, t) \cdot \deg \mathcal{Z}^o(L_{k,p^t})^\circ \right), \end{aligned}$$

and

$$\begin{aligned} \delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L') &= \deg \mathcal{Z}^{\circ}(L)^{\circ} \cdot (\delta_{\text{Fal}}(0) + \delta_{\text{Fal}}(a)) \\ &+ \sum_{k=1}^{\lfloor a/2 \rfloor} \left(\sum_{t=0}^k N(k, t) \cdot \deg \mathcal{Z}^{\circ}(L_{k,p^t})^{\circ} \cdot (\delta_{\text{Fal}}(k-t) + \delta_{\text{Fal}}(a-k)) \right) \\ &+ \sum_{k=\lfloor a/2 \rfloor + 1}^a \left(\sum_{t=2k-a}^k N(k, t) \cdot \deg \mathcal{Z}^{\circ}(L_{k,p^t})^{\circ} \cdot (\delta_{\text{Fal}}(k-t) + \delta_{\text{Fal}}(a-k)) \right), \end{aligned}$$

where the number $N(k, t)$ counts how many $c \in \mathbb{Z}_p/p^k\mathbb{Z}_p$ satisfy certain conditions in Lemma 13.3.2, explicitly,

$$N(k, t) = \begin{cases} 2\varphi(p^{k-t}), & \text{if } t > 0; \\ (p^k - 2p^{k-1}), & \text{if } t = 0. \end{cases}$$

Then formulae (13.4.1)–(13.4.6) (in the Case 1) follow from explicit computations based on Lemma 11.5.5, (11.4.2) and (10.1.7). $\square$

13.5. Analytic difference formula.

Lemma 13.5.1. *Let L be a nondegenerate integral quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$ such that $b \geq 2$. Let L' be the lattice constructed in §B.3.*

- If $a \not\equiv b \pmod{2}$,

$$(13.5.1) \quad \text{Den}^*(L) - \text{Den}^*(L') = (a+1)p^{(a+b-1)/2},$$

$$\begin{aligned} (13.5.2) \quad \partial \text{Den}^*(L) - \partial \text{Den}^*(L') \\ = \frac{(a+1)(b-1)p^{(a+b+3)/2} - 2(a+1)bp^{(a+b+1)/2} + (a+1)(b+1)p^{(a+b-1)/2} + 2p^{a+1} - 2}{2(p-1)^2}. \end{aligned}$$

- If $a \equiv b \pmod{2}$ and L is anisotropic,

$$(13.5.3) \quad \text{Den}^*(L) - \text{Den}^*(L') = (a+1)p^{(a+b)/2-1}(p+1),$$

$$\begin{aligned} (13.5.4) \quad \partial \text{Den}^*(L) - \partial \text{Den}^*(L') \\ = \frac{1}{2(p-1)^2} \cdot \left((a+1)bp^{(a+b+4)/2} - (a+1)(b+4)p^{(a+b+2)/2} \right. \\ \left. - (a+1)(b-4)p^{(a+b)/2} + (a+1)bp^{(a+b-2)/2} + 4p^{a+1} - 4 \right). \end{aligned}$$

- If $a \equiv b \pmod{2}$ and L is isotropic,

$$(13.5.5) \quad \text{Den}^*(L) - \text{Den}^*(L') = \begin{cases} (a+1)p^{(a+b)/2-1}(p-1), & \text{if } a < b; \\ p^{a-1}((a+1)p - a + 1), & \text{if } a = b. \end{cases}$$

$$(13.5.6) \quad \partial \text{Den}^*(L) - \partial \text{Den}^*(L') = \frac{(a+1)b}{2} \cdot p^{(a+b-2)/2}(p-1).$$

Proof. The lemma follows from the straightforward computation based on the explicit computation of the polynomial $\text{Den}^*(X, L)$ in Lemma B.2.2 (since we know the Gross–Keating invariants of L and L'). $\square$

Proof of Theorem 13.0.1. If the lattice L is not integral, the numbers $\deg \mathcal{Z}^*(L)$, $\text{Den}^*(L)$, $\delta_{\text{Fal}}(L)$ and $\partial \text{Den}^*(L)$ are all 0, so there is nothing to prove. Therefore we assume that the lattice L is integral.

Comparing Proposition 13.4.1 and Lemma 13.5.1, we observe that for all the rank 2 integral $\mathbb{Z}_p$ -lattices $L \subset \mathbb{B}^*$ with $\text{GK}(L) = (a, b)$ where $b \geq 2$,

$$\deg \mathcal{Z}^*(L) - \deg \mathcal{Z}^*(L') = \text{Den}^*(L) - \text{Den}^*(L'), \quad \delta_{\text{Fal}}(L) - \delta_{\text{Fal}}(L') = \partial \text{Den}^*(L) - \partial \text{Den}^*(L'),$$

where L' is another sublattice of $\mathbb{B}^*$ constructed in §B.3 whose Gross–Keating invariant is either $(a, b-2)$, or $(b-2, a)$ if $b-2 < a$. By mathematical induction, we can reduce Theorem 13.0.1 to the cases that $\nu(L) = 0, 1$ or 2 , which we have already verified in §13.1. This concludes the proof of Theorem 13.0.1. $\square$

14. PROOF OF THE ARITHMETIC SIEGEL–WEIL FORMULA

14.1. Integrals of Green currents. In this subsection, we evaluate the integrals of the Green currents $\mathfrak{g}^\circ(T, y)$ for 3×3 matrices T . These integrals converge by the discussion in §9.3. We also recall Berndt and Kühn’s explicit formulas for the integrals of Kudla’s Green function [BK12b, Theorem 4.2], which will simplify our proof.

Lemma 14.1.1. *Let $T \in \text{Sym}_3(\mathbb{Q})$ and $y \in \text{Sym}_3(\mathbb{R})_{>0}$ be symmetric matrices. We then have*

$$(14.1.1) \quad \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) = (16\pi^2)^{-1} \text{CT}_{\rho=0} \int_1^{+\infty} \frac{d}{dt} \left((C_T^3)'(ty, 0) \right) \frac{dt}{t^\rho}.$$

Proof. By Theorem C.1.1 and the discussion in §9.3, we have

$$\int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) = \text{CT}_{\rho=0} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y, \rho) \stackrel{(9.4.2)}{=} \text{CT}_{\rho=0} \int_{\mathcal{M}(\mathbb{C})} \int_1^{+\infty} C_{\theta, T} \left(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right) \frac{dt}{t^{1+\rho}} \cdot \Omega_z^2.$$

By Theorem C.1.2, the order of the two integrals on the right hand side can be interchanged for $\text{Re}(\rho) \gg 0$:

$$\int_{\mathcal{M}(\mathbb{C})} \int_1^{+\infty} C_{\theta, T} \left(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right) \frac{dt}{t^{1+\rho}} \cdot \Omega_z^2 = \int_1^{+\infty} \int_{\mathcal{M}(\mathbb{C})} C_{\theta, T} \left(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right) \Omega_z^2 \cdot \frac{dt}{t^{1+\rho}}.$$

In Appendix §C, we denote the function $C_{\theta, T}(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3})$ by $K_T(t, y, z)$. Remark C.1.3 implies that the integral of $C_{\theta, T}(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3})$ converges absolutely. Therefore the Siegel–Weil formula in Lemma 6.4.1 implies

$$(14.1.2) \quad \int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} \theta_T \left(g, h, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right) dh = \frac{1}{2} \cdot E_T \left(g, 0, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right),$$

where the measure on $O(V)(\mathbb{A})$ is chosen such that the total volume of $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$ is 1. The integration on $O(V)(\mathbb{A})$ can be reduced to an integration on $SO(V)(\mathbb{A})$ with a factor of $1/2$ by [Kud03, (4.14)], and then reduced to an integration on $\mathcal{M}(\mathbb{C})$ by the invariance of $\tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3}$ under

the standard compact subgroup of $SO(V)(\mathbb{A})$. By complex uniformization of $\mathcal{M}(\mathbb{C})$, there exists a positive constant C such that for all functions F on $O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})$ which are $\text{Stab}_{O(V)(\mathbb{R})}(i, i) \times O(V)(\hat{\mathbb{Z}})$ -invariant,¹⁴ we have

$$\int_{O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A})} F(h) dh = C \cdot \int_{\mathcal{M}(\mathbb{C})} F(h_z) \Omega_z^2$$

whenever the integrals are absolutely convergent. Taking $F = 1$ shows $C = \text{vol}(\mathcal{M}(\mathbb{C}), \Omega^2)^{-1} = 288$. Therefore

$$(14.1.3) \quad \int_{\mathcal{M}(\mathbb{C})} C_{\theta, T} \left(ty, h_z, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right) \Omega_z^2 = 576^{-1} C_T \left(ty, 0, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right).$$

Therefore we have

$$\begin{aligned} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) &= 576^{-1} C_T \int_1^{+\infty} C_T \left(ty, 0, \tilde{\nu}_3 \otimes \mathbf{1}_{M_2(\hat{\mathbb{Z}})^3} \right) \frac{dt}{t^{1+\rho}} \\ &\stackrel{[\text{GS19}, (5.139)]}{=} 288^{-1} C_T \int_1^{+\infty} \frac{d}{dt} \left(C'_T(ty, 0, \Phi_3^2) \right) \frac{dt}{t^\rho} \\ &= (16\pi^2)^{-1} C_T \int_1^{+\infty} \frac{d}{dt} \left((C_T^3)'(ty, 0) \right) \frac{dt}{t^\rho}. \end{aligned}$$

□

In §9.3, we saw that when T is a 1×1 nonsingular matrix, the function $\mathfrak{g}(T, y) = \mathfrak{g}^\circ(T, y)$ is absolutely integrable against the volume form Ω^2 (see (9.3.2)). The integral is closely related to the derivative of the classical genus 1 Eisenstein series.

Lemma 14.1.2. *For all $T \in \text{Sym}_1(\mathbb{Q}) \setminus \{(0)\}$ and $y \in \text{Sym}_1(\mathbb{R})_{>0}$, we have*

$$(14.1.4) \quad \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}(T, y) \wedge \Omega^2 = (96\pi)^{-1} \lim_{t \rightarrow +\infty} \left((C_T^1)'(ty, 1) - (C_T^1)'(y, 1) \right).$$

Proof. Both sides are nonzero only if T is integral. Suppose that $T = (m)$ where $m \in \mathbb{Z} \setminus \{0\}$. For the left hand side, it has been calculated explicitly in [BK12b, Theorem 4.2] that

$$\begin{aligned} (14.1.5) \quad \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}(T, y) \wedge \Omega^2 &= \frac{1}{4} \int_{M(\mathbb{C})} \mathfrak{g}^\circ(T, y, 0) \wedge \Omega^2 \\ &= \frac{\sigma_1(|m|)}{96\pi|m|y} \begin{cases} 1, & m \in \mathbb{Z}_{>0}; \\ e^{4\pi my} - 4\pi my \text{Ei}(4\pi my), & m \in \mathbb{Z}_{<0}. \end{cases} \end{aligned}$$

Notice that the Green function $\Xi(y, z, m)$ on $M(\mathbb{C})$ ([BK12a, Definition 2.3]) is $1/2$ times our green function $\mathfrak{g}(T, y)$ descended to $M(\mathbb{C})$.

¹⁴Here $O(V)(\hat{\mathbb{Z}})$ is abuse of notation for the stabilizer of $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ where $L = M_2(\mathbb{Z}) \subseteq V$. The notation $\text{Stab}_{O(V)(\mathbb{R})}(i, i)$ denotes the stabilizer in $O(V)(\mathbb{R})$ of $(i, i) \in \mathbb{D}^\pm$ for the canonical action $O(V)(\mathbb{R}) \curvearrowright \mathbb{D}^\pm$.

Berndt and Kühn [BK12b, Definition 2.1] introduced an Eisenstein series $E_2(\mathbf{z}, s)$ of weight 2 with the T -th Fourier coefficient $a_T(\mathbf{y}, s)$. The relation of $E_2(\mathbf{z}, s)$ series to $E^1(\mathbf{z}, s)$ is given by

$$(14.1.6) \quad E^1(\mathbf{z}, 2s - 1) = -8\pi E_2(\mathbf{z}, s).$$

The derivative of its Fourier coefficient $a_T(\mathbf{y}, s)$ at $s = 1$ is computed explicitly in [BK12b, (2.2.3)]. By (14.1.6),

$$(14.1.7) \quad (C_T^1)'(\mathbf{y}, 1) = -4\pi a_T'(\mathbf{y}, 1) = -\frac{\sigma_1(|m|)}{|m|\mathbf{y}} \begin{cases} 1 + 4\pi m\mathbf{y} \frac{\sigma_1'(m)}{\sigma_1(m)}, & m \in \mathbb{Z}_{>0}; \\ e^{4\pi m\mathbf{y}} - 4\pi m\mathbf{y} \operatorname{Ei}(4\pi m\mathbf{y}), & m \in \mathbb{Z}_{<0}. \end{cases}$$

Here for a positive integer m , $\sigma_1(m) = \sum_{d>0, d|m} d$ and $\sigma_1'(m) := \sigma_1(m) \log m - 2 \sum_{d>0, d|m} d \log d$. Then

$$(14.1.8) \quad \lim_{t \rightarrow +\infty} (C_T^1)'(t\mathbf{y}, 1) = \begin{cases} -4\pi \sigma_1'(m), & m \in \mathbb{Z}_{>0}; \\ 0, & m \in \mathbb{Z}_{<0}. \end{cases}$$

Then (14.1.4) follows from (14.1.5), (14.1.7), and (14.1.8). $\square$

Remark 14.1.3. The explicit expression (14.1.7) implies that the following integral

$$\int_1^{+\infty} \frac{d}{dt} \left((C_T^1)'(t\mathbf{y}, 1) \right) \frac{dt}{t^\rho}$$

converges for $\operatorname{Re}(\rho) > -1$ and defines a holomorphic function in ρ in that range. In particular, it is regular at $\rho = 0$. In other words, (14.1.4) is equivalent to

$$(14.1.9) \quad \int_{\mathcal{M}(\mathbb{C})} \mathbf{g}(T, \mathbf{y}) \wedge \Omega^2 = (96\pi)^{-1} C_T \int_1^{+\infty} \frac{d}{dt} \left((C_T^1)'(t\mathbf{y}, 1) \right) \frac{dt}{t^\rho}.$$

14.2. The case of corank 0. Let $T \in \operatorname{Sym}_3(\mathbb{Q})$ be a nonsingular matrix. Lemma 8.2.2 (d) implies that the special cycle $\mathcal{Z}(T)$ is nonempty only if T is half-integral off the diagonal, integral on the diagonal, positive definite, and the set $\operatorname{Diff}(T)$ in (8.2.2) has cardinality 1. In that case, the special cycle $\mathcal{Z}(T)$ is supported over the supersingular locus of $\mathcal{M}$ at the only prime $p \in \operatorname{Diff}(T)$. Moreover, the special cycle $\mathcal{Z}(T)$ is an Artinian Deligne–Mumford stack. Let L be a quadratic lattice of rank 3 over $\mathbb{Z}$ with moment matrix T . The stacky length of $\mathcal{Z}(T)$ is given by the following formula by [GK93, Proposition 3.22] (see also [Gö07a, §4]),

$$\begin{aligned} \widehat{\deg} \left(\left(\hat{\mathcal{E}}^\vee \right)^0 \big|_{\mathcal{Z}(T)} \right) &= \sum_{(E, E') \in \mathcal{M}^{\text{ss}}(\mathbb{F}_p)} \frac{R_{\operatorname{Hom}(E, E')}(T)}{\#\operatorname{Aut}(E) \cdot \#\operatorname{Aut}(E')} \cdot (-\partial \operatorname{Den}^*(1, L \otimes \mathbb{Z}_p) \log p) \\ &= \left(\prod_{l \neq p} \operatorname{Den}^*(1, L \otimes \mathbb{Z}_l) \right) \cdot (-\partial \operatorname{Den}^*(1, L \otimes \mathbb{Z}_p) \log p) \\ &\stackrel{(3.5.8)}{=} 2^{3/2} \left(\prod_{l \neq p} \frac{W_{T, l}(e, 0, 1_{M_2(\mathbb{Z}_l)^3})}{(1 - l^{-2})} \right) \cdot \frac{W'_{T, p}(e, 0, 1_{M_2(\mathbb{Z}_p)^3})}{(1 - p^{-2})}. \end{aligned}$$

By [GS19, Proposition 3.2 (iii)] (see also [BY20, Proposition 4.3 (ii)]),

$$\det \mathbf{y}^{-1} W_T(g_{i\mathbf{y}}) e^{2\pi \operatorname{tr}(T\mathbf{y})} = -2^{11/2} \pi^4.$$

Therefore the stacky length of $\mathcal{Z}(T)$ is given by

$$(14.2.1) \quad \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^\vee \right)^0 \Big|_{\mathcal{Z}(T)} \right) = -(32\pi^2)^{-1} (C_T^3)'(y, 0).$$

Theorem 14.2.1. *Let $T \in \text{Sym}_3(\mathbb{Q})$ be a nonsingular matrix. For all $y \in \text{Sym}_3(\mathbb{R})_{>0}$,*

$$(14.2.2) \quad \widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = -(32\pi^2)^{-1} (C_T^3)'(y, 0).$$

Proof. We first consider the case that T is positive definite. In that case, the sum in (9.2.1) is empty so $\mathfrak{g}(T, y) = 0$, and we hence have

$$\int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y) = 0.$$

If $\mathcal{Z}(T) = \emptyset$, then both sides of (14.2.2) vanish. If $\mathcal{Z}(T) \neq \emptyset$, then (14.2.2) holds by (14.2.1). Therefore (14.2.2) is true for positive definite matrices T .

If T is not positive definite, there exists a positive number C by [GS19, Proposition 3.3] such that

$$(C_T^3)'(ty, 0) = O(e^{-Ct}) \text{ as } t \rightarrow \infty.$$

Therefore by Lemma 14.1.1, we have

$$\widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y) = -(32\pi^2)^{-1} (C_T^3)'(y, 0).$$

□

14.3. The case of corank 1. After Lemma 14.1.1, the term which remains to be evaluated in the corank 1 case is the Faltings height of the special cycle. We relate this height with the Eisenstein series Fourier coefficient by combining Theorem 13.0.1 with the Rapoport–Zink uniformization from §12.

Proposition 14.3.1. *Let $T^\flat \in \text{Sym}_2(\mathbb{Q})$ be a nondegenerate matrix such that $\text{disc}(T^\flat) \notin (\mathbb{Q}^\times)^2$. Then*

$$(14.3.1) \quad \widehat{\deg} \left(\widehat{\mathcal{E}}^\vee \Big|_{\mathcal{Z}(T^\flat)} \right) = W_{T^\flat, \infty}^*(y^\flat, 1/2) \cdot \frac{d}{ds} \Big|_{s=-1/2} \left(\Lambda^*(s+1/2, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^*(s) \right)$$

Proof. If $\text{disc}(T^\flat) \notin (\mathbb{Q}^\times)^2$ and $T^\flat$ is not positive definite, the left hand side of (14.3.1) is 0 because $\mathcal{Z}(T^\flat) = \emptyset$, while the right hand side of (14.3.1) is also 0 by (4.1.10) (notice that the function $\Lambda^*(s+1/2, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^*(s)$ has no pole at $s = -1/2$ because $\Lambda^*(s, \eta_{T^\flat})$ is not $\xi(s)$ by our assumption).

Next we assume that $T^\flat$ is positive definite. If $T^\flat$ does not define an integral quadratic form, then we have $\mathcal{Z}(T^\flat) = \emptyset$ and we also have $W_{T^\flat, p}^*(s) = 0$ whenever $T^\flat$ does not define an integral quadratic $\mathbb{Z}_p$ -lattice, as explained at the end of Section 4.2. In this case, both sides of (14.3.1) are 0 again. Conversely, if $T^\flat$ does define an integral quadratic form, Lemma 8.5.2 and (4.3.5) imply that

$$(14.3.2) \quad \Lambda^*(0, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^*(-1/2) = \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{Z}(T^\flat)_{\mathbb{Q}}).$$

In this case, we have $W_{T^\flat, p}^\star(1/2) \neq 0$ for all p , as follows e.g. from inspection of the explicit formulas in Lemma B.2.2, as the polynomials of loc. cit. are non-vanishing at $X = 1$. Alternatively, the non-vanishing $W_{T^\flat, p}^\star(1/2) \neq 0$ also follows from (13.0.1) and inspection of (11.5.2). The value $\Lambda^*(0, \eta_{T^\flat})$ is always finite and nonzero, since $\eta_{T^\flat}$ is nontrivial at the Archimedean place.

For simplicity, denote the right hand side of (14.3.1) by $F(T^\flat)$. We thus have

$$(14.3.3) \quad \frac{F(T^\flat)}{\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T^\flat)_{\mathbb{Q}})} = \frac{\Lambda^{*\prime}(0, \eta_{T^\flat})}{\Lambda^*(0, \eta_{T^\flat})} + \sum_p \frac{W_{T^\flat, p}^{*\prime}(-1/2)}{W_{T^\flat, p}^*(-1/2)}$$

where the sum has only finitely many nonzero terms. The formula (10.1.2) and the functional equation $\Lambda^*(1-s, \eta_{T^\flat}) = \Lambda^*(s, \eta_{T^\flat})$ gives

$$(14.3.4) \quad \frac{\Lambda^{*\prime}(0, \eta_{T^\flat})}{\Lambda^*(0, \eta_{T^\flat})} = -2 h_{\text{Fal}, F}^{\text{CM}} - \frac{1}{2} \log(4\pi e^\gamma)$$

for $F = \mathbb{Q}[X]/(X^2 - \text{disc } T^\flat)$, and for γ the Euler–Mascheroni constant. Our non-Archimedean main theorem for corank 1 (Theorem 13.0.1) gives

$$(14.3.5) \quad \frac{W_{T^\flat, p}^{*\prime}(-1/2)}{W_{T^\flat, p}^*(-1/2)} = -\frac{\delta_{\text{Fal}, p}(T^\flat)}{\deg_p(T^\flat)} \cdot \log p$$

for all primes p , using also the comparison between $\text{Den}^\star$ and $W^\star$ and (4.2.1).

Recalling the local-to-global relation from (12.3.8), we find

$$(14.3.6) \quad \frac{F(T^\flat)}{\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T^\flat)_{\mathbb{Q}})} = -2 h_{\text{Fal}, F}^{\text{CM}} - \frac{1}{2} \log(4\pi e^\gamma) + \sum_p \frac{\widehat{\deg}_{\mathcal{E}^\vee}(\mathcal{Z}(T^\flat))_p}{\deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T^\flat)_{\mathbb{Q}})}.$$

By definition of $\widehat{\deg}_{\mathcal{E}^\vee}(\mathcal{Z}(T^\flat))_p$ from (10.2.3) and subsequent discussion, we conclude

$$(14.3.7) \quad F(T^\flat) = \widehat{\deg}(\widehat{\mathcal{E}}_{\text{Pet}}^\vee|_{\mathcal{Z}(T^\flat)}) - \frac{1}{2} \deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T^\flat)_{\mathbb{Q}}) \cdot \log(4\pi e^\gamma) = \widehat{\deg}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T^\flat)}),$$

where the last identity follows from (10.2.1). $\square$

Let $T = \text{diag}\{0, T^\flat\} \in \text{Sym}_3(\mathbb{Q})$ be a rank 2 matrix where $T^\flat \in \text{Sym}_2(\mathbb{Q})$ is nondegenerate. Let $y = \text{diag}\{y_1, y^\flat\} \in \text{Sym}_3(\mathbb{R})$ be a positive definite matrix. By (5.2.3) and (4.3.5), we have

$$(14.3.8) \quad \begin{aligned} & (-32\pi^2)^{-1} (C_T^3)'(y, 0) \\ &= -\frac{d}{ds} \Big|_{s=0} \left(y_1^{s/2} W_{T^\flat, \infty}^\star(y^\flat, s+1/2) \Lambda^*(s+1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(s+1/2) \right). \end{aligned}$$

If $T^\flat$ is not positive definite, it follows from [GS19, Proposition 3.3(ii)] that there exists a positive number C such that

$$\frac{d}{dt} \left((C_T^3)'(ty, 0) \right) = O(e^{-Ct}) \quad \text{as } t \rightarrow +\infty.$$

Therefore

$$(14.3.9) \quad \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}^\circ(T, y) = (16\pi^2)^{-1} C_T \int_1^{+\infty} \frac{d}{dt} \left((C_T^3)'(ty, 0) \right) \frac{dt}{t^\rho} = -(16\pi^2)^{-1} (C_T^3)'(y, 0).$$

If $T^\flat$ is positive definite, (14.3.8) implies

$$(14.3.10) \quad (-32\pi^2)^{-1} (C_T^3)'(y, 0) = -\frac{1}{2} \log y_1 \cdot W_{T^\flat, \infty}^\star(y^\flat, 1/2) \Lambda^*(1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(1/2)$$

$$(14.3.11) \quad - \left(W_{T^\flat, \infty}^\star \right)' \left(y^\flat, 1/2 \right) \Lambda^*(1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(1/2)$$

$$(14.3.12) \quad - \frac{d}{ds} \Big|_{s=0} \Lambda^*(s+1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(s+1/2)$$

where we also used the special value formula (4.1.10). Therefore by the functional equation of the product $\Lambda^*(s+1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(s+1/2)$ and (14.3.2),

$$(14.3.13) \quad (-32\pi^2)^{-1} (C_T^3)'(ty, 0) = -\frac{1}{2} \log t \cdot \deg_{\mathcal{E}_{\mathbb{Q}}}(\mathcal{Z}(T^\flat)_{\mathbb{Q}}) + A + F(t) \quad \text{as } t \rightarrow +\infty,$$

where A is independent of t and where the function $F(t)$ is obtained by replacing $y^\flat$ with $ty^\flat$ in (14.3.11). By Lemma 4.1.1, there exists a positive constant C such that

$$(14.3.14) \quad F(t) = O(t^{-1}), \quad F'(t) = O(t^{-1-C}) \quad \text{as } t \rightarrow +\infty.$$

Theorem 14.3.2. *Let $T \in \text{Sym}_3(\mathbb{Q})$ be a matrix of rank 2. For all $y \in \text{Sym}_3(\mathbb{R})_{>0}$,*

$$(14.3.15) \quad \widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = -(32\pi^2)^{-1} (C_T^3)'(y, 0).$$

Proof. Let $\alpha \in \text{GL}_3(\mathbb{Z})$ be a matrix such that (9.6.1) holds, i.e., $\tilde{T} := {}^t\alpha T \alpha = \text{diag}\{\mathbf{0}_1, T^\flat\}$. Let $\tilde{y} = \text{diag}\{y_1, y^\flat\}$ be another positive definite matrix as in (9.6.5).

If $T^\flat$ is not positive definite, we may apply (14.3.9) to find

$$(14.3.16) \quad \begin{aligned} \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y) &= \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) = \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(\tilde{T}, \alpha^{-1} y^t \alpha^{-1}) \\ &= -(32\pi^2)^{-1} (C_{\tilde{T}}^3)'(\alpha^{-1} y^t \alpha^{-1}, 0) \stackrel{(9.6.6)}{=} -(32\pi^2)^{-1} (C_T^3)'(y, 0). \end{aligned}$$

Now assume $T^\flat$ is positive definite. In this case, we have

$$(14.3.17) \quad \begin{aligned} & \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y) \\ &= \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) - \frac{1}{2} \log y_1 \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{Z}(T^\flat)_{\mathbb{Q}}) \end{aligned}$$

$$(14.3.18) \quad = \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(\tilde{T}, \alpha^{-1}y^t\alpha^{-1}) - \frac{1}{2} \log y_1 \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{Z}(T^\flat)_{\mathbb{Q}})$$

$$(14.3.19) \quad = - \left(W_{T^\flat, \infty}^\star \right)' \left(y^\flat, 1/2 \right) \Lambda^*(1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(1/2) - \frac{1}{2} \log y_1 \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{Z}(T^\flat)_{\mathbb{Q}}).$$

$$(14.3.20) \quad = (-32\pi^2)^{-1} \left(C_{\tilde{T}}^3 \right)'(\tilde{y}, 0) + \frac{d}{ds} \Big|_{s=0} \Lambda^*(s+1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(s+1/2)$$

$$(14.3.21) \quad = (-32\pi^2)^{-1} \left(C_T^3 \right)'(y, 0) - \frac{d}{ds} \Big|_{s=-1/2} \Lambda^*(s+1/2, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(s).$$

(14.3.13). For (14.3.17), we use Definition 9.6.2. For (14.3.18), we use the $\mathrm{GL}_3(\mathbb{Z})$ -invariance of the Green currents. For (14.3.19), we combine Lemma 14.1.1 with the estimate of the growth (14.3.14) as in [GS19, Lemma 5.11]. For (14.3.20), we use (14.3.10). For the last identity (14.3.21), we use Lemma 9.6.1 and the functional equation of the product $\Lambda^*(s+1, \eta_{T^\flat}) \prod_p W_{T^\flat, p}^\star(s+1/2)$.

Therefore by Proposition 14.3.1,

$$\widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = \widehat{\deg} \left(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T^\flat)} \right) + \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y) = -(32\pi^2) \left(C_T^3 \right)'(y, 0).$$

□

14.4. The case of corank 2. Recall from §8.1 that $\overline{M}$ (resp. M) denotes the coarse moduli scheme of $\overline{\mathcal{M}}$ (resp. $\mathcal{M}$). In [BGKK07, Theorem 7.61], Burgos Gil, Kramer, and Kühn calculated the height of Hecke correspondences for $\overline{M}$ with respect to the line bundle $\mathcal{E}_{\overline{M}}^\vee$ equipped with the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$, which we denote by $\widehat{\mathcal{E}}_{\overline{M}, \mathrm{Pet}}^\vee$. The difference is recorded given in (10.2.1). A Green current calculation is given in [BK12b], and the results of loc. cit. can be expressed as follows.

Lemma 14.4.1. *For all $T^\flat = (t) \in \mathrm{Sym}_1(\mathbb{Q}) \setminus \{(0)\}$ and $y^\flat \in \mathrm{Sym}_1(\mathbb{R})_{>0}$,*

$$(14.4.1) \quad \begin{aligned} & \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}}^\vee \right)^2|_{\overline{T}_t} \right) + \frac{1}{2} \int_{M(\mathbb{C})} \mathfrak{g}(T^\flat, y^\flat) \wedge \Omega^2 \\ &= - \left((4\pi^2)^{-1} \Lambda'(2) + (48\pi)^{-1} \right) C_{T^\flat}^1(y^\flat, 1) - (48\pi)^{-1} \left(C_{T^\flat}^1 \right)'(y^\flat, 1). \end{aligned}$$

Proof. Berndt and Kühn in [BK12b] use the Hermitian line bundle $\left(\widehat{\mathcal{E}}_{\overline{M}, \mathrm{Pet}}^\vee \right)^{\otimes 12}$ for their height, so their main theorem gives

$$(14.4.2) \quad 144 \cdot \left(\widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}, \mathrm{Pet}}^\vee \right)^2|_{\overline{T}_t} \right) + \frac{1}{2} \int_{M(\mathbb{C})} \mathfrak{g}(T^\flat, y^\flat) \wedge \Omega^2 \right) = -12 \left(\psi(s) a_{T^\flat}(y^\flat, s) \right)' \Big|_{s=1}$$

where $\psi(s)$ is a meromorphic function with the following expansion at $s = 1$,

$$\psi(s) = -1 + 4 \left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \right) (s-1) + O((s-1)^2).$$

By (10.2.1) and e.g. [BGS94, Proposition 3.2.2], the change of height is

$$\widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}, \text{Pet}}^\vee \right)^2 |_{\overline{T}_t} \right) - \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}}^\vee \right)^2 |_{\overline{T}_t} \right) = \log(4\pi e^\gamma) \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee} \left((\overline{T}_t)_{\mathbb{Q}} \right)$$

By (8.5.4), notice that the morphism $i_1 : \mathcal{Z}(T^\flat) \rightarrow \mathcal{T}_t$ generically has degree 2, and compute

$$\deg_{\mathcal{E}_{\mathbb{Q}}^\vee} \left((\overline{T}_t)_{\mathbb{Q}} \right) = \int_{\overline{T}_t(\mathbb{C})} \Omega = \frac{1}{2} \int_{|\mathcal{Z}(T^\flat)(\mathbb{C})|} \Omega = 2 \deg_{\mathcal{E}_{\mathbb{Q}}^\vee} \left(\mathcal{Z}(T^\flat)_{\mathbb{Q}} \right) = (48\pi)^{-1} C_{T^\flat}^1(y^\flat, 1).$$

By (14.1.6), we have

$$-12 \left(\psi(s) a_{T^\flat}(y^\flat, s) \right)' \Big|_{s=1} = 6\pi^{-1} \left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \right) C_{T^\flat}^1(y^\flat, 1) - 3\pi^{-1} (C_{T^\flat}^1)'(y^\flat, 1).$$

The lemma then follows from the functional equation of $\Lambda(s)$. $\square$

Remark 14.4.2. By (9.7.1), formula (14.4.1) can be stated in the following way:

$$\begin{aligned} (14.4.3) \quad & \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^\vee \right)^2 |_{\mathcal{Z}(T^\flat)} \right) + \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}(T^\flat, y^\flat) \wedge \Omega^2 \\ &= -((8\pi^2)^{-1} \Lambda'(2) + (96\pi)^{-1}) C_{T^\flat}^1(y^\flat, 1) - (96\pi)^{-1} (C_{T^\flat}^1)'(y^\flat, 1). \end{aligned}$$

Theorem 14.4.3. *Let $T \in \text{Sym}_3(\mathbb{Q})$ be a matrix of rank 1. For all $y \in \text{Sym}_3(\mathbb{R})_{>0}$,*

$$(14.4.4) \quad \widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = -(32\pi^2)^{-1} (C_T^3)'(y, 0).$$

Proof. Let $\alpha \in \text{GL}_3(\mathbb{Z})$ be a matrix such that (9.6.1) holds, i.e., $\tilde{T} := {}^t\alpha T \alpha = \text{diag}\{\mathbf{0}_2, T^\flat\}$. Let $\tilde{y} = \text{diag}\{y_1, y^\flat\}$ be another positive definite matrix as in (9.6.5).

By Lemma 14.1.1, we have

$$(14.4.5) \quad \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) = \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(\tilde{T}, \alpha^{-1} y^t \alpha^{-1}) = (16\pi^2)^{-1} C_T^1 \int_1^{+\infty} \frac{d}{dt} \left((C_{\tilde{T}}^3)'(t\tilde{y}, 0) \right) \frac{dt}{t^\rho}.$$

By the Definition 9.6.2 of $\mathfrak{g}(T, y)$, combining (5.3.3) with (14.4.5) and (14.1.9), and using Lemma 8.5.1 to evaluate $\deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{Z}(T)_{\mathbb{Q}})$, we find

$$\begin{aligned} (14.4.6) \quad & \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y) - \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}(T^\flat, y^\flat) \wedge \Omega^2 \\ &= \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(T, y) - \int_{\mathcal{M}(\mathbb{C})} \mathfrak{g}(T^\flat, y^\flat) \wedge \Omega^2 - \frac{1}{2} \log \det y_1 \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{Z}(T)_{\mathbb{Q}}) \\ &= -(32\pi^2)^{-1} \frac{\partial f_{\tilde{T}}}{\partial s}(\tilde{y}, 0) - (192\pi)^{-1} \log \det y_1 \cdot C_{T^\flat}^1(y^\flat, 1), \end{aligned}$$

where

$$(14.4.7) \quad -\frac{\partial f_{\tilde{T}}}{\partial s}(\tilde{y}, 0) = -\left(C_{\tilde{T}}^3 \right)'(\tilde{y}, 0) + \left(\frac{\pi}{6} \log \det y_1 + \frac{\pi}{3} + 4\Lambda'(2) \right) C_{T^\flat}^1(y^\flat, 1) + \frac{\pi}{3} (C_{T^\flat}^1)'(y^\flat, 1)$$

as in (5.3.5). By the definition of the arithmetic degree in (9.7.3), we have

$$\widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^\vee \right)^2 \big|_{\mathcal{Z}(T^\flat)} \right) + \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(T, y).$$

Combining (14.4.6) and (14.4.3), we get

$$\widehat{\deg}[\widehat{\mathcal{Z}}(T, y)] = -(32\pi^2)^{-1} \left(C_T^3 \right)'(\tilde{y}, 0) \stackrel{(9.6.6)}{=} -(32\pi^2)^{-1} \left(C_T^3 \right)'(y, 0).$$

□

14.5. The case of corank 3. Let $T = \mathbf{0}_3$. The special cycle is $\mathcal{Z}(\mathbf{0}_3) = \mathcal{M}$. It is proved in [BGKK07, Theorem 7.61] that

$$\widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}, \text{Pet}}^\vee \right)^3 \big|_{\overline{M}} \right) = -\frac{1}{2} \left(\frac{1}{2} \zeta(-1) + \zeta'(-1) \right) = -(4\pi)^{-1} \Lambda'(2) - 48^{-1} + 48^{-1} \log(4\pi e^\gamma).$$

By (10.2.1) and [BGS94, Proposition 3.2.2], we have

$$(14.5.1) \quad \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}}^\vee \right)^3 \big|_{\overline{M}} \right) = \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}, \text{Pet}}^\vee \right)^3 \big|_{\overline{M}} \right) - \frac{3}{2} \log(4\pi e^\gamma) \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\overline{M}_{\mathbb{Q}}) \\ = -(4\pi)^{-1} \Lambda'(2) - 48^{-1}.$$

Theorem 14.5.1. *For all $y \in \text{Sym}_3(\mathbb{R})_{>0}$, we have*

$$(14.5.2) \quad \widehat{\deg}[\widehat{\mathcal{Z}}(\mathbf{0}_3, y)] = -(32\pi^2)^{-1} \left(C_{\mathbf{0}_3}^3 \right)'(y, 0).$$

Proof. By Lemma 14.1.1 and Lemma 5.4.1, particularly (5.4.3) and (5.4.5), we have

$$(14.5.3) \quad \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(\mathbf{0}_3, y) = -(32\pi^2)^{-1} \left(C_{\mathbf{0}_3}^3 \right)'(y, 0) + (16\pi)^{-1} \Lambda'(2) + 192^{-1} + 576^{-1} \log \det y.$$

By the Definition 9.6.2 of $\mathfrak{g}(\mathbf{0}_3, y)$, we have

$$(14.5.4) \quad \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}(\mathbf{0}_3, y) = \frac{1}{2} \int_{\overline{\mathcal{M}}(\mathbb{C})} \mathfrak{g}^\circ(\mathbf{0}_3, y) - \frac{1}{2} \log \det y \cdot \deg_{\mathcal{E}_{\mathbb{Q}}^\vee}(\mathcal{M}_{\mathbb{Q}}) \\ = -(32\pi^2)^{-1} \left(C_{\mathbf{0}_3}^3 \right)'(y, 0) + (16\pi)^{-1} \Lambda'(2) + 192^{-1}.$$

We have

$$(14.5.5) \quad \widehat{\deg} \left(\left(\widehat{\mathcal{E}}^\vee \right)^3 \big|_{\mathcal{M}} \right) \stackrel{(9.7.2)}{=} 4^{-1} \widehat{\deg} \left(\left(\widehat{\mathcal{E}}_{\overline{M}}^\vee \right)^3 \big|_{\overline{M}} \right) \stackrel{(14.5.1)}{=} -(16\pi)^{-1} \Lambda'(2) - 192^{-1}.$$

The theorem now follows from the definition of $\widehat{\deg}[\widehat{\mathcal{Z}}(\mathbf{0}_3, y)]$ in (9.7.3), which is the sum of the two terms (14.5.4) and (14.5.5). □

APPENDIX A. FINITENESS AND BOUNDEDNESS

In this section, we prove the finiteness and boundedness of some sets arising naturally in §9.2 when we define the green currents $\mathbf{g}^\circ(T, \mathbf{y}, \rho)$.

Lemma A.0.1. *Let $U \subset \mathbb{D} \simeq \mathbb{H}_1^2$ be an open set with compact closure. Let $L \subset V$ be a sublattice of rank 4. Let $T \in \text{Sym}_r(\mathbb{Q})$ be a matrix. The following set is finite for all positive integers r :*

$$(A.0.1) \quad S = \{\mathbf{x} \in L^r : T(\mathbf{x}) = T \text{ and } \mathbb{D}_{\mathbf{x}} \cap \overline{U} \neq \emptyset\}.$$

Moreover, there exists a positive number $C > 0$, such that for all elements $\mathbf{x} = (x_1, \dots, x_r) \in \Omega_T(L) \setminus S$, we have

$$(A.0.2) \quad R(\mathbf{x}, z) = \sum_{i=1}^r R(x_i, z) > C, \quad z \in \overline{U}.$$

Proof. We first prove the claim about the finiteness of the set S . Let $\mathbf{x} = (x_1, \dots, x_r)$, the condition $\mathbb{D}_{\mathbf{x}} \cap \overline{U} \neq \emptyset$ is

$$\mathbb{D}_{x_1} \cap \dots \cap \mathbb{D}_{x_r} \cap \overline{U} \neq \emptyset.$$

Therefore we can reduce the lemma to the case that $r = 1$, i.e., we prove that for all numbers $t \in \mathbb{Q}$, the following set is finite

$$S = \{x \in L : \det(x) = t \text{ and } \mathbb{D}_x \cap \overline{U} \neq \emptyset\}.$$

For $t < 0$, the set S is empty. For $t = 0$, the set S consists of the single element $\mathbf{0}_2$, hence it is finite.

Now we consider the case that $t > 0$. By the compactness of $\overline{U}$, we can assume that there exists a number $M > 1$ such that for all points $z = (x_1 + iy_1, x_2 + iy_2) \in \overline{U}$,

$$|x_i| \leq M, \quad M^{-1} \leq |y_i| \leq M, \quad i = 1, 2.$$

Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L$ be an element in S . If $\mathbb{D}_x \cap \overline{U} \neq \emptyset$, we conclude that there exists an element $(z_1, z_2) \in \overline{U}$ such that

$$dz_1 z_2 - cz_1 - bz_2 + a = 0$$

by (7.3.2). Therefore

$$z_2 = \frac{a - cz_1}{b - dz_1}, \quad z_1 = \frac{a - bz_2}{c - dz_2}.$$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we get

$$(A.0.3) \quad y_2 = \frac{t \cdot y_1}{|b - dz_1|^2} \leq \frac{t}{d^2 y_1 + y_1^{-1} (b - dx_1)^2}.$$

Therefore $|d| \leq \sqrt{t(y_1 y_2)^{-1}} \leq M\sqrt{t}$. We can also use the fact that $|b - dz_1|^2 \geq (b - dx_1)^2$ to deduce that

$$|b| \leq \sqrt{t y_1 y_2^{-1}} + |dx_1| \leq M\sqrt{t} + M^2\sqrt{t} \leq 2M^2\sqrt{t}.$$

We can get the same estimate for c by using the expression $z_1 = \frac{a - bz_2}{c - dz_2}$. Finally,

$$|a| = |dz_1 z_2 - cz_1 - bz_2| \leq (4\sqrt{2} + 2) M^3\sqrt{t}.$$

Therefore all the coordinates of x are bounded by some numbers depending only on the set U . Since x belongs to the $\mathbb{Z}$ -lattice L , there are only finitely many x satisfying this condition.

Now we prove the claim about the set $\Omega_T(L) \setminus S$. Notice that the quadratic form $x \in V \mapsto R(x, z) + \det(x)$ is positive definite. By the compactness of $\overline{U}$, there exists a positive number $c > 0$ such that

$$R(x, z) + \det(x) > c \cdot \|x\|^2, \quad \text{for } x \in L \text{ and } z \in \overline{U}.$$

The set $\Omega_T(L) \setminus S$ can be divided into the following parts:

$$S' := \{x \in L^r : \sum_{i=1}^r \|x_i\|^2 \leq c^{-1}(|\text{tr}(T)| + 1)\} \cap \Omega_T(L) \setminus S,$$

$$S'' := \{x \in L^r : \sum_{i=1}^r \|x_i\|^2 > c^{-1}(|\text{tr}(T)| + 1)\} \cap \Omega_T(L) \setminus S.$$

For an element $x \in S''$, we have

$$R(x, z) > |\text{tr}(T)| + 1 - \text{tr}(T) \geq 1.$$

The set S' is finite, therefore the number $C' := \min_{x \in S', z \in \overline{U}} \{R(x, z)\}$ is positive because $\overline{U}$ is compact.

Define

$$C = \min \{C'/2, 1/2\}.$$

Then we have $R(x, z) > C$. □

APPENDIX B. CALCULATIONS OF LOCAL DENSITIES AND THE CASE $p = 2$

B.1. Gross–Keating invariants. We first recall the definition of Gross–Keating invariant from [GK93, §4] and [Bou07, §1]. Let p be a prime number. Let (L, q) be a quadratic $\mathbb{Z}_p$ -lattice of rank $n \geq 1$. We always assume that the quadratic form is nondegenerate. Let $e = (e_1, \dots, e_n)$ be a basis of L such that

$$(B.1.1) \quad q(x_1 e_1 + \dots + x_n e_n) = \sum_{i \leq j} b_{ij}(e) x_i x_j.$$

We write $S(e)$ for the set of tuples $y = (y_1, \dots, y_n) \in \mathbb{Z}^n$ such that

$$(B.1.2) \quad y_1 \leq \dots \leq y_n, \quad \frac{y_i + y_j}{2} \leq \nu_p(b_{ij}(e)) \quad \text{for } 1 \leq i \leq j \leq n.$$

Let $S(L) = \bigcup S(e)$ where the union ranges over all basis e . We order tuples $(y_1, \dots, y_n) \in S$ lexicographically. The Gross–Keating invariants $\text{GK}(L) = (a_1, \dots, a_n)$ are the maximum of $(y_1, \dots, y_n) \in S(L)$. A basis e is called optimal if $(a_1, \dots, a_n) \in S(e)$.

Lemma B.1.1. *Let (L, q) be a nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank $n \geq 1$ with $\text{GK}(L) = (\mu_1, \dots, \mu_n)$,*

(a). *We have*

$$\mu_1 = \min_{x \in L} \{\nu_p(q(x))\}.$$

(b). *If p is odd, the quadratic lattice L is diagonalizable, i.e., there exists a basis $e = (e_1, \dots, e_n)$ such that $(e_i, e_j) = 0$ for $i \neq j$, and $\nu_p(q(e_i)) \leq \nu_p(q(e_j))$ if $1 \leq i \leq j \leq n$. Then we have*

$$\mu_i = \nu_p(q(e_i)), \quad 1 \leq i \leq n.$$

- (c). If $p = n = 2$ and the lattice L is not diagonalizable, then there exists an integer e and a basis $\mathbf{e} = (e_1, e_2)$ such that

$$q(xe_1 + ye_2) = 2^e \cdot \begin{cases} x^2 + xy + y^2, & \text{if } L \text{ is anisotropic;} \\ xy, & \text{if } L \text{ is isotropic.} \end{cases}$$

The Gross-Keating invariant of L is $\text{GK}(L) = (e, e)$.

- (d). If $p = n = 2$ and the lattice L is diagonalizable, then there exists a basis $\mathbf{e} = (e_1, e_2)$ such that

$$q(xe_1 + ye_2) = u_1 2^{a_1} x^2 + u_2 2^{a_2} y^2,$$

where $u_1, u_2 \in \mathbb{Z}_2^\times$, $a_1, a_2 \in \mathbb{Z}$ and $a_1 \leq a_2$. The Gross-Keating invariant of L is given by

$$(B.1.3) \quad \text{GK}(L) = \begin{cases} (a_1, a_2), & \text{if } a_1 \not\equiv a_2 \pmod{2}; \\ (a_1, a_2 + 1), & \text{if } a_1 \equiv a_2 \pmod{2} \text{ and } u_1 + u_2 \equiv 2 \pmod{4}; \\ (a_1, a_2 + 2), & \text{if } a_1 \equiv a_2 \pmod{2} \text{ and } u_1 + u_2 \equiv 0 \pmod{4}. \end{cases}$$

Proof. For part (a), let $\mathbf{f} = (f_1, \dots, f_n)$ be an optimal basis of L . It's obvious that $\min_{i \leq j} \nu_p(b_{ij}(\mathbf{f})) = \min_{x \in L} \{\nu_p(q(x))\}$ because

$$b_{ij}(\mathbf{f}) = \begin{cases} q(f_i + f_j) - q(f_i) - q(f_j), & \text{if } i \neq j; \\ q(f_i), & \text{if } i = j. \end{cases}$$

Hence $\min_{i \leq j} \nu_p(b_{ij}(\mathbf{f})) \geq \min_{x \in L} \{\nu_p(q(x))\}$. The inequality of the other direction $\min_{i \leq j} \nu_p(b_{ij}(\mathbf{f})) \leq \min_{x \in L} \{\nu_p(q(x))\}$ follows from (B.1.1). If $\mu_1 > \min_{x \in L} \{\nu_p(q(x))\}$, let $i \leq j \leq n$ be two integers such that $\nu_p(b_{ij}(\mathbf{f})) = \min_{x \in L} \{\nu_p(q(x))\}$, then

$$\nu_p(b_{ij}(\mathbf{f})) \geq \frac{\mu_i + \mu_j}{2} \geq \mu_1 > \min_{x \in L} \{\nu_p(q(x))\},$$

which is a contradiction, hence $\mu_1 \leq \min_{x \in L} \{\nu_p(q(x))\}$. It's easy to see that the following element belongs to $S(\mathbf{f})$:

$$\left(\min_{x \in L} \{\nu_p(q(x))\}, \dots, \min_{x \in L} \{\nu_p(q(x))\} \right),$$

therefore $\mu_1 \geq \min_{x \in L} \{\nu_p(q(x))\}$. Then we must have $\mu_1 = \min_{x \in L} \{\nu_p(q(x))\}$.

Part (b) is well-known (see for example [Bou07, §2]), we are left to prove (c) and (d). For (c), the existence of the basis can be found in [Bou07, Lemma 3.2]. Now we compute $\text{GK}(L)$. Suppose that $\text{GK}(L) = (\mu_1, \mu_2)$ where $\mu_1 \leq \mu_2$. By (a), we can verify that $\mu_1 = e$. It's easy to see that $(e, e) \in S(L)$, hence $\mu_2 \geq e$. Moreover, we have $\mu_1 + \mu_2 \leq 2e$ by [Bou07, Lemma 1.3(c)], hence $\mu_2 \leq e$. Therefore $\mu_2 = e$.

For (d), part (a) implies that $\mu_1 = a_1$. It's easy to see that $(a_1, a_2) \in S(L)$, hence $\mu_2 \geq a_2$. Let $\mathbf{f} = (f_1, f_2)$ be an optimal basis such that $[f_1, f_2] = [e_1, e_2] \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$q(xf_1 + yf_2) = (a^2 u_1 2^{a_1} + c^2 u_2 2^{a_2}) x^2 + (abu_1 2^{a_1+1} + cdu_2 2^{a_2+1}) xy + (b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) y^2.$$

We first consider the case that $a_1 \not\equiv a_2 \pmod{2}$. Let $a_2 = a_1 + 2\lambda - 1$ where $\lambda \geq 1$. Since $\mu_2 \geq a_2$, we have $\nu_2(b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) \geq \mu_2 \geq a_2$, hence $\nu_2(b^2 u_1 2^{a_1}) \geq a_2$, which implies that $\nu_2(b) \geq \lambda$.

Therefore $d \in \mathbb{Z}_2^\times$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_2)$, and $\nu_2(b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) = a_2$. Since $\mathbf{f}$ is an optimal basis, we have $\mu_2 \leq \nu_2(b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) = a_2$. Therefore $\mu_2 = a_2$.

Next we consider the case that $a_1 \equiv a_2 \pmod{2}$. Let $a_2 = a_1 + 2\lambda$ where $\lambda \geq 0$. By [Bou07, Lemma 1.3(c)],

$$\mu_1 + \mu_2 \leq \nu_2(u_1 u_2 2^{a_1 + a_2 + 2}) = a_1 + a_2 + 2,$$

hence $\mu_2 \leq a_2 + 2$.

- If $u_1 + u_2 \equiv 0 \pmod{4}$. Take $a = 1, b = 2^\lambda, c = 0$ and $d = 1$, we get

$$q(xf_1 + yf_2) = u_1 2^{a_1} x^2 + u_1 2^{a_1 + 1 + \lambda} xy + (u_1 + u_2) 2^{a_2} y^2.$$

It's easy to verify that $(a_1, a_2 + 2) \in S(\mathbf{f})$ since $\nu_2((u_1 + u_2) 2^{a_2}) \geq a_2 + 2$ and $\nu_2(u_1 2^{a_1 + 1 + \lambda}) = a_1 + 1 + \lambda = (a_1 + a_2 + 2)/2$. Therefore $\mu_2 = a_2 + 2$.

- If $u_1 + u_2 \equiv 2 \pmod{4}$. We first prove that $\mu_2 \leq a_2 + 1$. Let's assume the converse that $\mu_2 \geq a_2 + 2$. Since $\nu_2(b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) \geq \mu_2 \geq a_2 + 2$, the element b satisfies that $\nu_2(b) \geq \lambda$ because otherwise $\nu_2(b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) < a_2$, which is a contradiction. If $\nu_2(b) > \lambda$, the element $d \notin \mathbb{Z}_2^\times$ because otherwise $\nu_2(b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2}) = a_2$, which is also a contradiction. Therefore $\nu_2(b) = \lambda$ and $d \in \mathbb{Z}_2^\times$. Let $b = 2^\lambda \cdot b_0$, then

$$b^2 u_1 2^{a_1} + d^2 u_2 2^{a_2} = 2^{a_2} \cdot (u_1 b_0^2 + u_2 d^2) \equiv 2^{a_2} \cdot (u_1 + u_2) \pmod{2^{a_2 + 2}}.$$

Therefore $\mu_2 \leq a_2 + 1$, which is a contradiction, hence $\mu_2 \leq a_2 + 1$. Next we take $a = 1, b = 2^\lambda, c = 0$ and $d = 1$, then

$$q(xf_1 + yf_2) = u_1 2^{a_1} x^2 + u_1 2^{a_1 + 1 + \lambda} xy + (u_1 + u_2) 2^{a_2} y^2.$$

It's easy to verify that $(a_1, a_2 + 1) \in S(\mathbf{f})$ since $\nu_2((u_1 + u_2) 2^{a_2}) = a_2 + 1$ and $\nu_2(u_1 2^{a_1 + 1 + \lambda}) = a_1 + 1 + \lambda > (a_1 + a_2 + 1)/2$. Therefore $\mu_2 = a_2 + 1$.

□

For a nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank n with $\text{GK}(L) = (\mu_1, \dots, \mu_n)$, define

$$(B.1.4) \quad |\text{GK}(L)| = \mu_1 + \dots + \mu_n.$$

Let $\mathbf{e} = (e_1, \dots, e_n)$ be a $\mathbb{Z}_p$ -basis of L , define the inner product matrix of L with respect to the basis $\mathbf{e}$ to be $T(\mathbf{e}) := \left(\frac{1}{2}(e_i, e_j)\right)_{1 \leq i, j \leq n}$. Define $\det L = \det T(\mathbf{e}) \in \mathbb{Q}_p^\times / (\mathbb{Z}_p^\times)^2$, this number is independent of the choice of the basis $\mathbf{e}$.

Lemma B.1.2. *Let L be a nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank n .*

(a). *If $p > 2$, we have*

$$(B.1.5) \quad |\text{GK}(L)| = \nu_p(\det L).$$

(b). *If $p = n = 2$, denote by $\mathbb{Q}_{L,2}$ the degree 2 algebra $\mathbb{Q}_2(\sqrt{\text{disc}(L)})$ over $\mathbb{Q}_2$. Then*

$$(B.1.6) \quad |\text{GK}(L)| = 2 \cdot \left\lceil \frac{\nu_2(\det L)}{2} \right\rceil + \delta_{L,2},$$

where $\delta_{L,2}$ is 1 if $\mathbb{Q}_{L,2}$ is a ramified extension of $\mathbb{Q}_2$, is 2 otherwise.

Proof. Part (a) follows from Lemma B.1.1 (b). Now we focus on the case that $p = n = 2$, if the lattice L is not diagonalizable, we can verify the identity easily by using Lemma B.1.1 (c). Now let's assume that the lattice L is diagonalizable with a diagonalized basis as in Lemma B.1.1 (d). If $a_1 \not\equiv a_2 \pmod{2}$, the algebra $\mathbb{Q}_{L,2}$ is a ramified extension of $\mathbb{Q}_2$, the identity (B.1.6) can be verified by straightforward computation.

If $a_1 \equiv a_2 \pmod{2}$ with $u_1 + u_2 \equiv 2 \pmod{4}$, the mod 8 reduction of (u_1, u_2) have the following possibilities:

$$(u_1, u_2) \equiv (1, 1), (1, 5), (3, 3), (3, 7), (5, 1), (5, 5), (7, 3), (7, 7) \pmod{8},$$

then $\text{disc } L \equiv -1, -5 \pmod{8}$, the corresponding algebra $\mathbb{Q}_{L,2}$ is a ramified extension of $\mathbb{Q}_2$. The identity (B.1.6) is true because both sides are $a_1 + a_2 + 1$.

If $a_1 \equiv a_2 \pmod{2}$ with $u_1 + u_2 \equiv 0 \pmod{4}$, the mod 8 reduction of (u_1, u_2) have the following possibilities:

$$(u_1, u_2) \equiv (1, 3), (1, 7), (3, 1), (3, 5), (5, 3), (5, 7), (7, 1), (7, 5) \pmod{8},$$

then $\text{disc } L \equiv -3, 1 \pmod{8}$, the corresponding algebra $\mathbb{Q}_{L,2}$ is either an unramified quadratic extension of $\mathbb{Q}_2$ ($\text{disc } L \equiv -3 \pmod{8}$), or splits $\mathbb{Q}_2 \times \mathbb{Q}_2$ ($\text{disc } L \equiv 1 \pmod{8}$). The identity (B.1.6) is true because both sides are $a_1 + a_2 + 2$. $\square$

Remark B.1.3. By the proof of Lemma B.1.2, we can see that the $\mathbb{Q}_p$ -algebra $\mathbb{Q}_{L,p} := \mathbb{Q}_p(\sqrt{\text{disc } L})$ is a ramified extension of $\mathbb{Q}_p$ if and only if $a \not\equiv b \pmod{2}$. Otherwise, the algebra $\mathbb{Q}_{L,p}$ is either an unramified quadratic extension of $\mathbb{Q}_p$, or $\mathbb{Q}_p \times \mathbb{Q}_p$.

B.2. Local densities. The goal of this subsection is to compute $\text{Den}^*(X, L)$ defined in (2.3.2), in the case of $F = \mathbb{Q}_p$, by combining the difference formula in [Zhu25], [Zhu24] and explicit calculations in [Wed07].

Lemma B.2.1. *Let L be a nondegenerate integral quadratic $\mathbb{Z}_p$ -lattice of rank 2. For an element $\varepsilon \in \mathbb{Z}_p^\times$ and sufficiently large integers n ,*

$$\begin{aligned} & \text{Den}\left(H^{k+2}, L \oplus \langle \varepsilon p^n \rangle\right) - p^{-2k} \cdot \text{Den}\left(H^{k+2}, L \oplus \langle \varepsilon p^{n-2} \rangle\right) \\ &= \left(1 - p^{-k-1}\right) \left(1 + p^{-k}\right) \cdot \text{Den}\left(H^{k+1}, L\right), \text{ for all } k \geq 0. \end{aligned}$$

Proof. By [Zhu25, Theorem 7.2.1, Remark 7.2.4], for all $k \geq 0$ and $n \geq 2$,

$$\begin{aligned} & \text{Den}\left(H^{k+2}, L \oplus \langle \varepsilon p^n \rangle\right) - p^{-2k} \cdot \text{Den}\left(H^{k+2}, L \oplus \langle \varepsilon p^{n-2} \rangle\right) \\ &= \text{Pden}\left(H^{k+1}, \langle \varepsilon p^n \rangle\right) \cdot \text{Den}\left(H^{k+1} \oplus \langle -\varepsilon p^n \rangle, L\right) \\ &= \left(1 - p^{-k-1}\right) \left(1 + p^{-k}\right) \cdot \text{Den}\left(H^{k+1} \oplus \langle -\varepsilon p^n \rangle, L\right). \end{aligned}$$

By [Zhu24, Lemma 2.2.4], if the integer n is larger than $|\text{GK}(L)|$, we have $\text{Den}\left(H^{k+1} \oplus \langle -\varepsilon p^n \rangle, L\right) = \text{Den}\left(H^{k+1}, L\right)$. Therefore the lemma follows. $\square$

Lemma B.2.2. *Let L be a nondegenerate integral quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$.*

- If $a \not\equiv b \pmod{2}$,

$$(B.2.1) \quad \text{Den}^*(X, L) = X^{-(a+b-1)/2} \cdot \sum_{i=0}^a \sum_{j=0}^{(a+b-1)/2-i} p^{i+j} X^{i+2j}.$$

- If $a \equiv b \pmod{2}$ and L is anisotropic,

$$(B.2.2) \quad \text{Den}^*(X, L) = X^{-(a+b)/2} \cdot \left((1+X) \sum_{i=0}^a \sum_{j=0}^{(a+b)/2-i-1} p^{i+j} X^{i+2j} + p^{(a+b)/2} X^b \cdot \sum_{i=0}^a X^i \right).$$

- If $a \equiv b \pmod{2}$ and L is isotropic,

$$(B.2.3) \quad \text{Den}^*(X, L) = X^{-(a+b)/2} \cdot \left((1-X) \sum_{i=0}^a \sum_{j=0}^{(a+b)/2-i-1} p^{i+j} X^{i+2j} + p^{(a+b)/2} X^b \cdot \sum_{i=0}^a X^i \right).$$

Proof. We only give the detailed proof for the identity (B.2.3), the other two follow from the same idea. For an element $\varepsilon \in \mathbb{Z}_p^\times$ and an integer $n \geq b+4$, Wedhorn [Wed07, (2.8)] gives a precise formula for the polynomial $\text{Den}^*(X, L \oplus \langle \varepsilon p^n \rangle)$. Notice that the symbol $\tilde{\xi} = 1$ by §2.5 ($p > 2$), §2.6 (A2) and (B4) ($p = 2$) of loc. cit.. Then

$$(B.2.4) \quad \frac{\text{Den}(X, L \oplus \langle \varepsilon p^n \rangle)}{(1-p^{-2}X)(1-p^{-2}X^2)} - X^2 \cdot \frac{\text{Den}(X, L \oplus \langle \varepsilon p^{n-2} \rangle)}{(1-p^{-2}X)(1-p^{-2}X^2)} \\ = (1-X^2) \sum_{i=0}^a \sum_{j=0}^{(a+b)/2-i-1} p^{i+j} X^{i+2j} + p^{(a+b)/2} X^b (1+X) \cdot \sum_{i=0}^a X^i$$

The number $\chi_p(L) = 1$ in this case, therefore (B.2.3) follows from (B.2.4) and Lemma B.2.1. $\square$

B.3. Choice of a larger lattice. Let L be a nondegenerate quadratic $\mathbb{Z}_p$ -lattice of rank 2 with $\text{GK}(L) = (a, b)$. We are going to construct a lattice L' such that

- (a). $L \subsetneq L'$ and $L'/L \simeq \mathbb{F}_p$.
- (b). $\text{GK}(L')$ is either $(a, b-2)$, or $(b-2, a)$ if $b-2 < a$.

The construction is nontrivial only when $p = 2$.

Case 1 We first single out a special case that the lattice L is isotropic with $a = b$. Then there exists a basis $\mathbf{e} = (e_1, e_2)$ of L such that $q(xe_1 + ye_2) = p^a xy$. Define L' to be the $\mathbb{Z}_p$ -span of e_1 and $p^{-1}(e_1 + e_2)$, notice that

$$q\left(xe_1 + y\frac{e_1 + e_2}{p}\right) = p^{a-2} \left(\left(\frac{p}{2}x + y\right)^2 - \frac{p^2}{4}x^2 \right),$$

hence $\text{GK}(L') = (a-2, a)$.

Case 2 If $p > 2$ and L is not isotropic with $a = b$, let $\mathbf{e}$ be a basis of L as in Lemma B.1.1 (b). Define L' to be the $\mathbb{Z}_p$ -span of e_1 and $p^{-1}e_2$. Then $\text{GK}(L')$ is either $(a, b-2)$ or $(b-2, a)$ if $b-2 \leq a$.

Case 3 If $p = 2$ and L is not diagonalizable, we have $a = b$ by Lemma B.1.1 (c). The isotropic case has been done before, let's consider the case that L is anisotropic. Let $\mathbf{e}$ be a basis of L as

in Lemma B.1.1 (c), define L' to be the $\mathbb{Z}_2$ -span of e_1 and $2^{-1}e_2$. Then

$$q(xe_1 + y \cdot 2^{-1}e_2) = 2^{a-2} \cdot (4x^2 + 2xy + y^2) = 2^{a-2} \cdot (3x^2 + (x+y)^2).$$

Therefore $\text{GK}(L') = (a-2, a)$ by Lemma B.1.1 (d).

Case 4 If $p = 2$ and L is diagonalizable, let e be a basis of L as in Lemma B.1.1 (d). Then

$$q(xe_1 + y \cdot e_2) = u_1 2^a x^2 + u_2 2^{b-\delta} y^2,$$

where $\delta = 0, 1, 2$ according to (B.1.3). Therefore

4-0 For $\delta = 0$, we have $a \leq b$ and $a \not\equiv b \pmod{2}$. Define L' to be the $\mathbb{Z}_2$ -span of e_1 and $2^{-1}e_2$, then $\text{GK}(L') = (a, b-2)$ if $a \leq b-3$. If $a \geq b-1$, we must have $a = b-1$, then $\text{GK}(L') = (b-2, b-1) = (b-2, a)$.

4-1 For $\delta = 1$, we have $a \leq b-1$ and $a \equiv b-1 \pmod{2}$. If $a \leq b-3$, define L' to be the $\mathbb{Z}_2$ -span of e_1 and $2^{-1}e_2$, then $\text{GK}(L') = (a, b-2)$. If $a > b-3$, we must have $a = b-1$, define L' to be the $\mathbb{Z}_2$ -span of e_1 and $2^{-1}(e_1 + e_2)$, then

$$q\left(xe_1 + y \cdot \frac{e_1 + e_2}{2}\right) = 2^{a-1} \cdot \left(\frac{2u_1 u_2}{u_1 + u_2} \cdot x^2 + \frac{u_1 + u_2}{2} \left(\frac{2u_1}{u_1 + u_2} x + y\right)^2\right).$$

Notice that

$$\frac{2u_1 u_2}{u_1 + u_2} + \frac{u_1 + u_2}{2} \equiv 2u_1 \equiv 2 \pmod{4},$$

hence $\text{GK}(L') = (a-1, a) = (b-2, a)$.

4-2 For $\delta = 2$, we have $a \leq b-2$ and $a \equiv b \pmod{2}$. If $a \leq b-4$, define L' to be the $\mathbb{Z}_2$ -span of e_1 and $2^{-1}e_2$, then $\text{GK}(L') = (a, b-2)$. If $a > b-4$, we must have $a = b-2$, define L' to be the $\mathbb{Z}_2$ -span of e_1 and $2^{-1}(e_1 + e_2)$, then

$$(B.3.1) \quad q\left(xe_1 + y \cdot \frac{e_1 + e_2}{2}\right) = 2^a \cdot \left(u_1 x^2 + u_1 xy + \frac{u_1 + u_2}{4} \cdot y^2\right).$$

Now we prove that the quadratic form (B.3.1) is not diagonalizable. Let's assume the converse that there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_2)$ such that under the transformation $x \mapsto ax + by$, $y \mapsto cx + dy$, the quadratic form (B.3.1) is diagonalized. The coefficient of the term xy after the transformation is $2abu_1 + (ad + bc)u_1 + \frac{u_1 + u_2}{2} \cdot cd = 0$, hence

$$(ad - bc)u_1 = -2abu_1 - 2bcu_1 - \frac{u_1 + u_2}{2} \cdot cd.$$

Notice that the 2-adic valuation of the right hand side is at least 1 since $u_1 + u_2 \equiv 0 \pmod{4}$, while that of the left hand side is 0, we get a contradiction. Therefore the quadratic form (B.3.1) is not diagonalizable, hence its Gross-Keating invariant is $(a, a) = (a, b-2)$ by Lemma B.1.1 (c).

APPENDIX C. THE ABSOLUTE CONVERGENCE OF THE GENUS 3 GREEN CURRENTS

C.1. Statement of the main result. Let $V_\infty = M_2(\mathbb{R})$ and $L = M_2(\mathbb{Z})$. For a matrix $T \in \text{Sym}_3(\mathbb{Q})$ and $\mathbf{y} = a \cdot {}^t a \in \text{Sym}_3(\mathbb{R})_{>0}$, define the scalar kernel $K_T(t, \mathbf{y}, z)$ by the sum

$$(C.1.1) \quad K_T(t, \mathbf{y}, z) \cdot \Omega_z^2 = e^{2\pi t \text{tr}(T\mathbf{y})} \sum_{\substack{\mathbf{x} \in L^3 \\ T(\mathbf{x})=T}} \nu_3(\sqrt{t} \mathbf{x} a)_z, \quad z \in \mathbb{D}, \quad t \geq 1.$$

Recall that the $(1, 1)$ -form Ω is defined by (7.4.5). If T is positive definite, we have $K_T(t, \mathbf{y}, z) = 0$, since V having signature $(2, 2)$ implies that the sum is empty.

For $\text{Re}(\rho) > 0$, recall that we defined a family of locally L^1 forms $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ on $\mathcal{M}(\mathbb{C})$ in (9.2.1), with continuation as a meromorphic family of currents to $\text{Re}(\rho) > -1/2$, as in Lemma 9.2.1. At least for $\text{Re}(\rho) > 2$, the estimate explained after (C.3.3) allows interchange of the $\mathbf{x}$ -sum and the t -integral to give the equivalent formula

$$\mathfrak{g}^\circ(T, \mathbf{y}, \rho) = \left(\int_1^\infty K_T(t, \mathbf{y}, z) t^{-1-\rho} dt \right) \Omega^2.$$

This appendix concerns the extension of $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ to the compactification $\overline{\mathcal{M}}(\mathbb{C})$. The main result of the appendix is the following theorem. We use the notation $z_j = x_j + iy_j$ for $z = (z_1, z_2) \in \mathbb{D}$.

Theorem C.1.1. *For $T \in \text{Sym}_3(\mathbb{Q})$ and $\mathbf{y} \in \text{Sym}_3(\mathbb{R})_{>0}$, the following statements hold.*

- (a). *For $\text{Re}(\rho) > 2$, the top-degree form $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ is L^1 on $\overline{\mathcal{M}}(\mathbb{C})$, and hence defines a current on $\overline{\mathcal{M}}(\mathbb{C})$.*
- (b). *The family $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ has a (unique) meromorphic continuation to $\text{Re}(\rho) > -1/2$, valued in currents on $\overline{\mathcal{M}}(\mathbb{C})$. The only possible pole is a simple pole at 0. The constant term at $\rho = 0$, denoted*

$$(C.1.2) \quad \mathfrak{g}^\circ(T, \mathbf{y}) := \text{CT}_{\rho=0} \mathfrak{g}^\circ(T, \mathbf{y}, \rho),$$

is also a current on $\overline{\mathcal{M}}(\mathbb{C})$, and is represented by an L^1 form on $\overline{\mathcal{M}}(\mathbb{C})$ if T has no positive eigenvalues. Moreover, we have the following.

- (b1). *If T is not positive semi-definite, then the family $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ extends holomorphically to a family of currents on $\overline{\mathcal{M}}(\mathbb{C})$ for all $\rho \in \mathbb{C}$, with $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ given by an L^1 form on $\overline{\mathcal{M}}(\mathbb{C})$ for any fixed ρ .*
- (b2). *If $T = \mathbf{0}_3$, then the residue of $\mathfrak{g}^\circ(T, \mathbf{y}, \rho)$ at $\rho = 0$ is $3[\Omega^2]$, and its constant term $\mathfrak{g}^\circ(T, \mathbf{y})$ is an L^1 form on $\overline{\mathcal{M}}(\mathbb{C})$ satisfying $\mathfrak{g}^\circ(T, \mathbf{y}) = (-3 \log(y_1 y_2) + O_{\mathbf{y}}(1)) \Omega^2$ for $y_1 y_2 \geq 1$ and $|x_1|, |x_2| \leq 1/2$, where $O_{\mathbf{y}}(1)$ means a constant depending on $\mathbf{y}$.*

For $\rho \in \mathbb{C}$, we also need to study absolute convergence of

$$(C.1.3) \quad J(T, \mathbf{y}, \rho) = \int_1^\infty \int_{\mathcal{M}(\mathbb{C})} |K_T(t, \mathbf{y}, z) t^{-1-\rho}| \Omega^2 dt.$$

For this analysis, we estimate $K_T(t, \mathbf{y}, z)$ for $z \in \mathbb{D}_1 := \mathcal{F}_1 \times \mathcal{F}_1 \subseteq \mathbb{D}$ where $\mathcal{F}_1 \subset \mathbb{H}_1$ consists of points $\tau = x + iy$ such that $|x| \leq \frac{1}{2}$ and $y \geq \sqrt{3}/2$.

Theorem C.1.2. For $T \in \text{Sym}_3(\mathbb{Q})$ and $t \geq 1$ and $\mathbf{y} \in \text{Sym}_3(\mathbb{R})_{>0}$ and $z \in \mathbb{D}_1$, we have

$$(C.1.4) \quad |K_T(t, \mathbf{y}, z)| \lesssim_{T, \mathbf{y}} \begin{cases} t^2 & \text{if } T \neq \mathbf{0}_3 \text{ and is positive semi-definite} \\ e^{-c_{T, \mathbf{y}} t} & \text{if } T \text{ has a negative eigenvalue} \\ 1 + e^{-c_{T, \mathbf{y}} t (y_1 y_2)^{-1}} & \text{if } T = \mathbf{0}_3 \end{cases}$$

for some constants $c_{T, \mathbf{y}} > 0$ depending on T and $\mathbf{y}$. Therefore the following statements hold.

- (a). If $T \neq \mathbf{0}_3$ is positive semi-definite, we have $J(T, \mathbf{y}, \rho) < \infty$ if $\text{Re}(\rho) > 2$.
- (b). If $T \neq \mathbf{0}_3$ is not positive semi-definite, we have $J(T, \mathbf{y}, \rho) < \infty$ for all $\rho \in \mathbb{C}$.
- (c). If $T = \mathbf{0}_3$, we have $J(T, \mathbf{y}, \rho) < \infty$ if and only if $\text{Re}(\rho) > 0$.

Proof. For $T \neq \mathbf{0}_3$ positive semi-definite, this follows from the estimates (C.5.2) and (C.6.1). For $T \neq \mathbf{0}_3$ not positive semi-definite, this is the bound (C.5.3), which is also valid for $\text{rank}(T) = 1$ as explained in (C.6). For $T = \mathbf{0}_3$, this follows from the estimates in (C.7.6) and (C.7.7). $\square$

Remark C.1.3. For fixed $t \geq 1$ and $\mathbf{y} \in \text{Sym}_3(\mathbb{R})_{>0}$, the estimate (C.1.4) implies that

$$\sup_{z \in \mathcal{M}(\mathbb{C})} |K_T(t, \mathbf{y}, z)| < \infty.$$

C.2. Some notations. Throughout Appendix C, we write $A \lesssim_\alpha B$ to mean $A \leq C_\alpha B$ for some constant $C_\alpha > 0$ depending on a parameter α . The notation $A \lesssim B$ means that the implied constant is absolute, not depending on any parameters.

For $\mathbf{x} = (x_1, x_2, x_3) \in V_\infty^3$ where $x_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, set $U(\mathbf{x}) = (a_i + d_i)_i$, $V(\mathbf{x}) = (c_i - b_i)_i$, and

$$(C.2.1) \quad P(U, V) = 3 - 2\pi(|U|^2 + |V|^2) + 2\pi^2(|U|^2|V|^2 - (U \cdot V)^2).$$

At $z = (i, i)$, the scalar of ν_3 relative to Ω_z^2 can be obtained by combining the explicit formula (6.1.4) and Lemma 7.6.1

$$(C.2.2) \quad \tilde{\nu}_3(\mathbf{x}) = e^{-\pi \|\mathbf{x}\|^2} P(U(\mathbf{x}), V(\mathbf{x}))$$

where $\|\mathbf{x}\|^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2$.

For a point $z \in \mathbb{D}_1$, let $h_{z_i} = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_i^{1/2} & 0 \\ 0 & y_i^{-1/2} \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, and define $h_z = (h_{z_1}, h_{z_2}) \in H(\mathbb{R})^\circ$. Define

$$\|\mathbf{x}\|_z = \|h_z^{-1} \mathbf{x}\|^2$$

Throughout this appendix, we denote $y_1 y_2$ by Y and y_2/y_1 by k . The change of variables gives

$$(C.2.3) \quad \Omega^2 = -\frac{1}{16\pi^2} \frac{dx_1 \wedge dx_2 \wedge dY \wedge dk}{Y^2 k}.$$

Moreover, the restrictions $y_j \geq \sqrt{3}/2$ on $\mathbb{D}_1$ imply $3/(4Y) \leq k \leq 4Y/3$ on $\mathbb{D}_1$.

Given a tuple $\mathbf{x} = (x_1, x_2, x_3) \in V_\infty^3$, define $d(\mathbf{x}) = (d_1, d_2, d_3) \in \mathbb{R}^3$. We will split the kernel function K_T into the following two parts:

$$(C.2.4) \quad K_T(t, \mathbf{y}, z) = K_T^{d \neq 0}(t, \mathbf{y}, z) + K_T^{d=0}(t, \mathbf{y}, z),$$

where the first part $K_T^{d \neq 0}$ is the summation over all those $\mathbf{x} \in L^3$ with $d(\mathbf{x}) \neq \mathbf{0}$, while the second part $K_T^{d=0}$ is the summation of the remaining terms.

C.3. The Gaussian estimate when $d \neq 0$. For $a \in \text{GL}_3(\mathbb{R})$, denote by λ the smallest eigenvalue of $y = a \cdot {}^t a$.

If $d(\mathbf{x}) \neq \mathbf{0}$, at least one d_i is not 0, then $d_1^2 + d_2^2 + d_3^2 \geq 1$ since all d_i are integers. Notice that the lower-right element of $h_z^{-1} \mathbf{x} \cdot a$ forms a row vector $\sqrt{Y}(d_1, d_2, d_3) \cdot a$,

$$(C.3.1) \quad \|\mathbf{x} \cdot a\|_z^2 \geq \|\sqrt{Y}(d_1, d_2, d_3) \cdot a\|^2 \geq \lambda Y(d_1^2 + d_2^2 + d_3^2) \geq \lambda Y.$$

There is also a bound on all twelve lattice coordinates. Indeed, on the set $\mathbb{D}_1$,

$$\|h_{z_j}\|^2 = y_j + y_j^{-1}(1 + x_j^2) < 3y_j,$$

hence $\|\mathbf{x}\|^2 = \|h_z \cdot h_z^{-1} \mathbf{x}\|^2 < 9Y \|h_z^{-1} \mathbf{x}\|^2 = 9Y \|\mathbf{x}\|_z^2$. On the other hand, we have

$$\|\mathbf{x} \cdot a\|_z^2 = \|h_z^{-1} \mathbf{x} \cdot a\|^2 \geq \lambda \|\mathbf{x}\|_z^2$$

Consequently,

$$(C.3.2) \quad \|\mathbf{x} \cdot a\|_z^2 \geq (9Y)^{-1} \lambda \|\mathbf{x}\|^2.$$

For $\mathbf{x} \in V_\infty^3$ with $T(\mathbf{x}) = T$, we have $R(\mathbf{x} \cdot a, z) = \|\mathbf{x} \cdot a\|_z^2/2 - \text{tr}(Ty)$ by (7.4.3). Since (C.2.1) has degree four, we have

$$(C.3.3) \quad e^{2\pi t \text{tr}(Ty)} |\tilde{\nu}_3(\sqrt{t}(h_z^{-1} \mathbf{x}) \cdot a)| \lesssim (1 + t \|\mathbf{x} \cdot a\|_z^2)^2 e^{-\pi t(\|\mathbf{x} \cdot a\|_z^2 - 2 \text{tr}(Ty))}.$$

Note also that the right-hand side is $\lesssim_{T,y,Y} t^2 e^{-c't \|\mathbf{x}\|^2}$ for some $c' > 0$ depending on y and Y .

C.3.1. The unbounded part. Choose $Y_0 = \max(1, 8 \text{tr}(Ty)/\lambda)$. For $Y \geq Y_0$ and an element $\mathbf{x}$ such that $d(\mathbf{x}) \neq 0$, by (C.3.1)

$$\|\mathbf{x} \cdot a\|_z^2 \geq \lambda Y \geq \lambda Y_0 \geq 8 \text{tr}(Ty) \implies \|\mathbf{x} \cdot a\|_z^2 - 2 \text{tr}(Ty) \geq 3 \|\mathbf{x} \cdot a\|_z^2/4.$$

We have $(1 + x)^2 \lesssim e^{x/4}$ for all $x \geq 0$. Then use (C.3.1) and (C.3.2) on separate $e^{-\pi t \|\mathbf{x} \cdot a\|_z^2/4}$ factors to obtain

$$(1 + t \|\mathbf{x} \cdot a\|_z^2)^2 e^{-\pi t(\|\mathbf{x} \cdot a\|_z^2 - 2 \text{tr}(Ty))} \lesssim e^{\pi t \|\mathbf{x} \cdot a\|_z^2/4} \cdot e^{-3\pi t \|\mathbf{x} \cdot a\|_z^2/4} \lesssim e^{-\pi \lambda t Y/4} e^{-\pi \lambda t \|\mathbf{x}\|^2/(36Y)}.$$

Summing over all $L^3 \cong \mathbb{Z}^{12}$ by using the inequality $\sum_{n \in \mathbb{Z}} e^{-\alpha n^2} \leq 1 + \sqrt{\pi/\alpha}$, we obtain

$$(C.3.4) \quad |K_T^{d \neq 0}(t, y, z)| \lesssim_\lambda e^{-\pi \lambda t Y/4} (1 + (Y/t)^6) \lesssim_\lambda e^{-\pi \lambda t Y/8}.$$

The proof shows that (C.3.4) also holds without the condition $Y \geq Y_0$, in the special case that $T = \mathbf{0}_3$.

C.3.2. The compact part. The region $Y \leq Y_0$ in $\mathbb{D}_1$ is compact because $y_j \leq 2\sqrt{3}^{-1} Y_0$. For $t \geq 1$,

$$(1 + t \|\mathbf{x} \cdot a\|_z^2)^2 e^{-\pi t(\|\mathbf{x} \cdot a\|_z^2 - 2 \text{tr}(Ty))} \leq t^2 (1 + \|\mathbf{x} \cdot a\|_z^2)^2 e^{-\pi(\|\mathbf{x} \cdot a\|_z^2 - 2 \text{tr}(Ty))} \lesssim t^2 e^{2\pi \text{tr}(Ty)} e^{-\pi \|\mathbf{x} \cdot a\|_z^2/2}.$$

The inequality (C.3.2) implies $\|\mathbf{x} \cdot a\|_z^2 \geq (9Y)^{-1} \lambda \|\mathbf{x}\|^2 \geq (9Y_0)^{-1} \lambda \|\mathbf{x}\|^2$ on the region $Y \leq Y_0$. The Gaussian lattice estimate implies

$$(C.3.5) \quad |K_T^{d \neq 0}(t, y, z)| \lesssim_{T,y} t^2$$

for $Y \leq Y_0$. If T is not positive semi-definite, then ${}^t a T a$ has some negative eigenvalue $-\eta$, and we have the improved bound $|K_T^{d \neq 0}(t, y, z)| \lesssim_{T,y} t^2 e^{-2\pi \eta t}$. Note that the set of eigenvalues of

${}^t a T a$ depends only in T and $\mathbf{y}$, and not on the choice of a . If instead $T = \mathbf{0}_3$, we may bound $|\tilde{\nu}_3(\sqrt{t}(h_z^{-1}\mathbf{x}) \cdot a)| \lesssim e^{-\pi\lambda\|\mathbf{x}\|^2/18}$ using (C.3.3) and (C.3.2), since $t \geq 1$, using also $Y \leq Y_0 = 1$ in this case. Summing over $\mathbf{x} \neq 0$ (using the $d \neq 0$ condition) gives $|K_{\mathbf{0}_3}^{d \neq 0}(t, \mathbf{y}, z)| \lesssim_{\mathbf{y}} e^{-\pi\lambda t/(18Y)}$ in the region $Y \leq Y_0$. This bound on $|K_{\mathbf{0}_3}^{d \neq 0}(t, \mathbf{y}, z)|$ also holds without the condition $Y \leq Y_0$, if we instead assume $t \geq Y$.

C.4. The Poisson estimate when $d = 0$. Let $\mathbf{x} \in L^3$ be a tuple such that $d(\mathbf{x}) = \mathbf{0}$. Write $b(\mathbf{x}), c(\mathbf{x}), l(\mathbf{x}) \in \mathbb{Z}^3$ for the upper-right, lower-left, and upper-left rows of the tuple. Then

$$(C.4.1) \quad T = T(\mathbf{x}) = -\frac{1}{2} ({}^t b(\mathbf{x})c(\mathbf{x}) + {}^t c(\mathbf{x})b(\mathbf{x})).$$

Lemma C.4.1. *For fixed $T \in \text{Sym}_3(\mathbb{Q}) \neq \mathbf{0}_3$, there are only finitely many solution pairs (b, c) , where $b, c \in \mathbb{Z}^3$, of the equation (C.4.1).*

Proof. Let (b, c) be a solution pair to the equation (C.4.1). Then the function $v \in \mathbb{Q}^3 \mapsto {}^t v T v$ is $v \mapsto -(b \cdot v)(c \cdot v)$. The unique factorization of the polynomial ring $\mathbb{Q}[v_1, v_2, v_3]$ determines the two linear factors up to scaling and exchange. Choose primitive integral representatives of those factors b_0 and c_0 , then $b = m b_0$, $c = n c_0$ up to swapping b_0 and c_0 , with $m, n \in \mathbb{Z}$ and a fixed nonzero product mn since $T \neq \mathbf{0}_3$. There are finitely many possibilities. $\square$

For each fixed pair (b, c) satisfying (C.4.1), every vector $l \in \mathbb{Z}^3$ can appear as the upper-left element of some tuple $\mathbf{x}$. For $a \in \text{GL}_3(\mathbb{R})$, we introduce the following notations for simplicity:

$$\Lambda_a = \mathbb{Z}^3 \cdot a, \quad B = b \cdot a, \quad C = c \cdot a.$$

For a point $z \in \mathbb{D}$, set $\gamma = -x_1 C - x_2 B$ (with $x_i = \text{Re}(z_i)$ here). Then

$$U(h_z^{-1}\mathbf{x} \cdot a) = \sqrt{Y}^{-1} (l(\mathbf{x}) \cdot a + \gamma), \quad V(h_z^{-1}\mathbf{x} \cdot a) = \sqrt{k}^{-1} C - \sqrt{k} B.$$

For simplicity, let $W = \sqrt{k}^{-1} C + \sqrt{k} B$. Then

$$(C.4.2) \quad |V|^2 = |W|^2 + 4 \text{tr}(T\mathbf{y}).$$

Let $f_{b,c}(U) = e^{-\pi t|U|^2} P(\sqrt{t}U, \sqrt{t}V)$. The contribution of this pair (b, c) to $K_T^{d=0}$ is

$$(C.4.3) \quad K_{b,c}(t, \mathbf{y}, z) := e^{-\pi t|W|^2} \sum_{l \in \Lambda_a} f_{b,c} \left(\frac{l + \gamma}{\sqrt{Y}} \right).$$

The Fourier transform of the function $f(U)$ is:

$$(C.4.4) \quad \hat{f}_{b,c}(\eta) = 2\pi t^{-5/2} e^{-\pi|\eta|^2/t} (|\eta|^2 - \pi t (|V|^2|\eta|^2 - (V \cdot \eta)^2)).$$

Notice that $\hat{f}(\mathbf{0}) = 0$, hence (C.4.3) and Poisson summation implies that

$$(C.4.5) \quad K_{b,c}(t, \mathbf{y}, z) = 2\pi Y^{5/2} t^{-5/2} \det(\mathbf{y})^{-1/2} e^{-\pi t|W|^2} \times \\ \sum_{\substack{\xi \in \Lambda_a^\vee \\ \xi \neq \mathbf{0}}} e^{-\pi t^{-1} Y |\xi|^2} e^{2\pi i \langle \xi, \gamma \rangle} \cdot (|\xi|^2 - \pi t (|V|^2 |\xi|^2 - (V \cdot \xi)^2)).$$

For $q > 0$, define $F_a(q) = q^{5/2} \sum_{\xi \in \Lambda_a^\vee \setminus \mathbf{0}} |\xi|^2 e^{-\pi q |\xi|^2}$. Taking absolute values in (C.4.5), we find

$$(C.4.6) \quad |K_{b,c}(t, \mathbf{y}, z)| \lesssim_{\mathbf{y}} e^{-\pi t|W|^2} (1 + t|V|^2) F_a(Y/t) \lesssim_{\mathbf{y}} e^{-\pi t|W|^2} (1 + t|W|^2 + 4 \text{tr}(T\mathbf{y})t)$$

as $\sup_{q>0} F_a(q) < \infty$. To see $\sup_{q>0} F_a(q) < \infty$, note that comparison with $\int_{\mathbb{R}^3} |\xi|^2 e^{-\pi q |\xi|^2} d\xi$ bounds the sum $\sum_{\xi \in \Lambda_a^\vee \setminus \mathbf{0}} |\xi|^2 e^{-\pi q |\xi|^2}$ by a constant multiple of $q^{-5/2}$. For $q \geq 1$, the sum in fact decays exponentially in q . The second inequality in (C.4.6) follows from (C.4.2). Therefore, we have

$$(C.4.7) \quad |K_{b,c}(t, y, z)| \lesssim_{T,y} 1 + t$$

for all $z \in \mathbb{D}$ and $t \geq 1$.

We now prove (C.1.4) in Theorem C.1.2 for $T \neq \mathbf{0}_3$. The estimate for $K_T^{d \neq 0}$ follows from (C.3.4) and (C.3.5), while that for $K_T^{d=0}$ follows from Lemma C.4.1 and (C.4.7). Combining these two estimates yields the desired bound.

C.5. Proof for matrices of rank ≥ 2 . In this subsection, we give the proof of the remaining statements of Theorem C.1.2 and C.1.1 for the matrices T with $\text{rank}(T) = 2$ or 3 . This case is much easier than the case $\text{rank}(T) \leq 1$ because either $K_T = K_T^{d \neq 0}$ or T has a negative eigenvalue.

We first consider the case that T is positive semi-definite. If $\text{rank } T = 3$, then the kernel function K_T is zero, because there is no tuple $\mathbf{x} \in L^3$ with moment matrix T since the signature of V_∞ is $(2, 2)$. If $\text{rank } T = 2$, every tuple $\mathbf{x}$ contributing to the sum (C.1.3) must have $d(\mathbf{x}) \neq \mathbf{0}$ because the 3-dimensional subspace of matrices with $d = 0$, with quadratic form $-bc$, has positive index one. Hence we have $|K_T(t, y, z)| = |K_T^{d \neq 0}(t, y, z)|$. For the unbounded region $Y \geq Y_0 = \max(1, 8 \text{tr}(Ty)/\lambda)$ of $\mathbb{D}_1$, the estimate function (C.3.4) is integrable jointly in z, t with the factor $t^{-1-\text{Re}(\rho)}$, for all $\rho \in \mathbb{C}$, i.e.,

$$(C.5.1) \quad \begin{aligned} & \int_1^\infty \int_{\mathbb{D}_1 \cap \{Y \geq Y_0\}} |K_T(t, y, z)| \Omega^2 \frac{dt}{t^{1+\text{Re}(\rho)}} = \int_1^\infty \int_{\mathbb{D}_1 \cap \{Y \geq Y_0\}} |K_T^{d \neq 0}(t, y, z)| \Omega^2 \frac{dt}{t^{1+\text{Re}(\rho)}} \\ & \lesssim_\lambda \int_1^\infty \int_{Y_0}^\infty \int_{3/(4Y)}^{4Y/3} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{-\pi \lambda t Y/4} (1 + (Y/t)^6) dx_1 dx_2 \frac{dk dY}{Y^2 k} \frac{dt}{t^{1+\text{Re}(\rho)}} < \infty. \end{aligned}$$

The convergence still holds with powers of $\log t$ in the integrand for derivatives in ρ . Recall that the current $\mathfrak{g}^\circ(T, y, \rho)$ on $\mathcal{M}(\mathbb{C})$ had meromorphic continuation to $\text{Re}(\rho) > -1/2$ as in Lemma 9.2.1, with a possible simple pole at $\rho = 0$ by [GS19, Proposition 2.12, (2.91)]. The estimate (C.5.1) shows that the restriction of $\mathfrak{g}^\circ(T, y, \rho)$ to a sufficiently small neighborhood in $\overline{\mathcal{M}}(\mathbb{C})$ of the boundary $\overline{\mathcal{M}}(\mathbb{C}) \setminus \mathcal{M}(\mathbb{C})$ admits a holomorphic continuation to all $\rho \in \mathbb{C}$ and can be represented by an L^1 form in that neighborhood.

For the integral $J(T, y, \rho)$, the estimates (C.3.4) and (C.3.5) give

$$(C.5.2) \quad |K_T(t, y, z)| \lesssim_{T,y} t^2.$$

Since $\mathcal{M}(\mathbb{C})$ has finite volume, we find $J(T, y, \rho) \lesssim_{T,y} \text{vol}(\mathcal{M}(\mathbb{C})) \int_1^\infty t^{1-\text{Re}(\rho)} dt < \infty$ for $\text{Re}(\rho) > 2$.

Next we consider the case that T has a negative eigenvalue. Following the notations of §C.4, for any solution pair (b, c) of the equation (C.4.1), we have ${}^t a T a = -({}^t B C + {}^t C B)/2$. The matrix ${}^t a T a$ preserves $\text{span}\{B, C\} \subseteq \mathbb{R}^3$ and acts as 0 on $\text{span}\{B, C\}^\perp$. Note that $\text{span}\{B, C\}$ has rank 2 because ${}^t a T a$ has rank ≥ 2 . Consider an orthonormal basis $\{u, v\}$ of $\text{span}\{B, C\}$ with $u = |B|^{-1} B$. In the basis $\{u, v\}$, the matrix ${}^t a T a$ acts on $\text{span}\{B, C\}$ by the matrix

$$\begin{pmatrix} -B \cdot C & -\frac{1}{2}|B|(C \cdot v) \\ -\frac{1}{2}|B|(C \cdot v) & 0 \end{pmatrix}.$$

In this case, the unique negative eigenvalue of ${}^t a T a$ is $-\eta = -\frac{1}{2}(B \cdot C + |B||C|) < 0$. Consequently,

$$|W|^2 = k|B|^2 + k^{-1}|C|^2 + 2B \cdot C \geq 2(|B||C| + B \cdot C) = 4\eta.$$

In combination with (C.4.6), we thus find $|K_{b,c}(t, y, z)| \lesssim_{T,y} (1+t)e^{-2\pi\eta t}$ and hence $|K_T^{d=0}(t, y, z)| \lesssim_{T,y} (1+t)e^{-2\pi\eta t}$ by Lemma C.4.1. Not assuming existence of a solution pair (b, c) to (C.4.1), we have $\sup_{Y \geq Y_0} |K_T^{d \neq 0}(t, y, z)| \lesssim_{\lambda} e^{-\pi\lambda t/8}$. Combining with the estimate on the compact subset $Y \leq Y_0$ in §C.3.2, we get

$$(C.5.3) \quad |K_T(t, y, z)| \lesssim_{T,y} (1+t^2)e^{-c't},$$

for some $c' > 0$ depending on T and y . Therefore, for all $\rho \in \mathbb{C}$, we have

$$(C.5.4) \quad \int_1^\infty \int_{\mathcal{M}(\mathbb{C})} |K_T(t, y, z)| \Omega_z^2 \cdot t^{-1-\operatorname{Re}(\rho)} dt < \infty.$$

Therefore $J(T, y, \rho) < \infty$ for all $\rho \in \mathbb{C}$. Moreover, the convergence (C.5.4) is still true with powers of $\log t$ in the integrand for derivatives in ρ . Thus, for the case that T has a negative eigenvalue, the family of currents $\mathbf{g}^\circ(T, y, \rho)$ extends holomorphically to all $\rho \in \mathbb{C}$ and is valued in L^1 forms on $\overline{\mathcal{M}}(\mathbb{C})$.

C.6. Proof for matrices of rank 1. In this subsection, we give the proof of Theorem C.1.2 and C.1.1 for the matrices T with $\operatorname{rank}(T) = 1$. If the only nonzero eigenvalue of T is negative, an argument similar to the case of T with negative eigenvalues in §C.5 still works and we have a similar bounding result (C.5.3), hence (C.5.4). Therefore we only focus on the case that the only nonzero eigenvalue of T is positive.

We first prove the absolute convergence in Theorem C.1.2. Finiteness of pairs (b, c) by Lemma C.4.1, the bound on $K_{b,c}$ from (C.4.7), and the bound for $K_T^{d \neq 0}$ from (C.3.4) and (C.3.5) imply

$$(C.6.1) \quad |K_T(t, y, z)| \lesssim_{T,y} (1+t^2).$$

As $\mathcal{M}(\mathbb{C})$ has finite volume, we find $J(T, y, \rho) \lesssim_{T,y} \int_1^\infty (1+t^2)t^{-1-\operatorname{Re}(\rho)} dt < \infty$ for $\operatorname{Re}(\rho) > 2$.

Next we consider the more difficult problem of extending the current to the compactification, as stated in Theorem C.1.1. For the term $K_T^{d \neq 0}$ in (C.2.4), the estimate (C.3.4) still works, so the integral $\int_1^\infty K_T^{d \neq 0}(t, y, z) \Omega_z^2 \cdot t^{-1-\rho} dt$ converges for all $\rho \in \mathbb{C}$ in the unbounded region $Y \geq Y_0$, and hence defines family of L^1 forms on $\mathbb{D}_1 \cap \{Y \geq Y_0\}$ which is holomorphic in ρ . Therefore we are left with the term $K_T^{d=0}$.

By Lemma C.4.1, equation (C.4.1) has finitely many solution pairs (b, c) . If there are no such pairs there is nothing further to prove. For each pair (b, c) , the two vectors b and c are linearly dependent since otherwise the rank of T would be 2. Then there exists a primitive vector (meaning the gcd of the three coordinates are 1) $p \in \mathbb{Z}^3$ such that $b = b_0 p$ and $c = c_0 p$, where $b_0, c_0 \in \mathbb{Z} \setminus \{0\}$. The positive semi-definiteness of the matrix T implies $b_0 c_0 < 0$.

Let $\ell_p = |p \cdot a|^2 > 0$ and $k_0 = -c_0/b_0 > 0$. Then $\operatorname{tr}(Ty) = -b_0 c_0 \ell_p$. In the notation of §C.4, the vector W , as a function in k ,

$$(C.6.2) \quad W = W(k) = \sqrt{k}^{-1} c_0 p \cdot a + \sqrt{k} b_0 p \cdot a = (\sqrt{k}^{-1} c_0 + \sqrt{k} b_0) p \cdot a.$$

Hence $|W(k)|^2 = -\ell_p b_0 c_0 (k_0 k^{-1} + k k_0^{-1} - 2)$. Consequently, for all $r > 0$, we get the following estimate for $t \geq 1$,

$$(C.6.3) \quad \int_0^\infty e^{-rt|W(k)|^2} \frac{dk}{k} \lesssim_{r,T,y} t^{-1/2}.$$

Now we are going to split the integral $\int_1^\infty K_T^{d=0}(t, y, z) \Omega_z^2 \cdot t^{-1-\rho} dt$ at $t = Y$, and we study the two parts $1 \leq t \leq Y$ and $t \geq Y$ separately. For the range $1 \leq t \leq Y$, let $q = Y/t \geq 1$. Then the exponential decay of $F_a(q)$ in §C.4, together with (C.4.6), gives

$$(C.6.4) \quad |K_{b,c}(t, y, z)| \lesssim_{T,y} (1+t) q^{5/2} e^{-c'q} e^{-c't|W(k)|^2}$$

where c' depends on y . The factor $1 + t|W|^2 + 4\text{tr}(Ty)t$ has been absorbed into the Gaussian $e^{-c't|W(k)|^2}$. The formulas (C.2.3) and (C.6.3) give

$$(C.6.5) \quad \int_{|x_1| < 1/2} \int_{|x_2| < 1/2} \int_{k \in \mathbb{R}_{>0}} \int_{Y \geq t} |K_{b,c}(t, y, z)| \Omega_z^2 \lesssim_{T,y} (1+t) t^{-1/2} \int_t^\infty (Y/t)^{5/2} e^{-c'Y/t} \frac{dY}{Y^2} \lesssim_y t^{-1/2}.$$

It follows that, for $Y \geq 1$, the integral $\int_1^Y K_T^{d=0}(t, y, z) \Omega_z^2 \cdot t^{-1-\rho} dt$ converges for $\text{Re}(\rho) > -1/2$, where it defines a holomorphic family of L^1 -forms on $\mathbb{D}_1$.

The range $t \geq Y$ is considerably more delicate. In the sum on the right-hand side of (C.4.3), there are two possibilities for the vector $l \in \Lambda_a$: it either lies in the subspace spanned by $p \cdot a$ or it does not. Set $v := p \cdot a$ and $\ell_p := |v|^2$. Let $\pi_\perp: \mathbb{R}^3 \rightarrow \{v\}^\perp \subseteq \mathbb{R}^3$ be the orthogonal projection, and set $\Lambda_\perp := \pi_\perp(\Lambda_a)$.

For each $\lambda \in \Lambda_\perp$, choose a lift $\tilde{\lambda} \in \Lambda_a$ with $\pi_\perp(\tilde{\lambda}) = \lambda$, and such that $\tilde{0} = 0$. Define $\theta_\lambda \in \mathbb{R}$ by the formula $\tilde{\lambda} = \theta_\lambda v + \lambda$. Then every $l \in \Lambda_a$ has a unique expression

$$(C.6.6) \quad l = (m + \theta_\lambda)v + \lambda \quad m \in \mathbb{Z}, \lambda \in \Lambda_\perp.$$

Then the function $K_{b,c}$ from (C.4.3), as a sum over such l , splits into the following two parts:

$$K_{b,c}(t, y, z) = e^{-\pi t|W|^2} \sum_{m \in \mathbb{Z}} f_{b,c} \left(\frac{m \cdot (p \cdot a) + \gamma}{\sqrt{Y}} \right) + e^{-\pi t|W|^2} \sum_{\substack{m \in \mathbb{Z} \\ \mathbf{0} \neq \lambda \in \Lambda_\perp}} f_{b,c} \left(\frac{(m + \theta_\lambda) \cdot (p \cdot a) + \lambda + \gamma}{\sqrt{Y}} \right).$$

We call the first term the *parallel part* and denote it by $K_{b,c}^\parallel$, since the vector l is parallel to $p \cdot a$. We call the second term the *nonparallel part* and denote it by $K_{b,c}^\nparallel$.

C.6.1. Nonparallel part. We first consider those vectors $l \in \Lambda_a$ not parallel to $p \cdot a$, i.e., with $\lambda \neq 0$ in the decomposition of l from (C.6.6). Let $\delta := \min_{\mathbf{0} \neq \lambda \in \Lambda_\perp} \{|\lambda|\} > 0$.

By the explicit formula (C.2.1) in combination with (C.4.2),

$$(C.6.7) \quad |P(\sqrt{t}U, \sqrt{t}V)| \lesssim_{T,y} (1+t)(1+t|U|^2)(1+t|W|^2)$$

for $t \geq 1$. The individual summand in $K_{b,c}(t, y, z)$ corresponding to l is bounded by a multiple (depending on T and y) of the function

$$(C.6.8) \quad (1+t)e^{-rt(|W|^2+|U|^2)} = (1+t)e^{-rt|W|^2-rt|l+\gamma|^2/Y},$$

for some $r > 0$, where we used $\sup_{x \in \mathbb{R}} ((1 + |x|)e^{-r|x|}) < \infty$ for all $r > 0$, to absorb $(1 + t|U|^2)(1 + t|W|^2)$ into the Gaussian $e^{-\pi t(|U|^2 + |W|^2)}$, up to changing the constant π .

We next estimate

$$\sum_{\substack{m \in \mathbb{Z} \\ \mathbf{0} \neq \lambda \in \Lambda_{\perp}}} e^{-rt|l+\gamma|^2/Y}.$$

By the decomposition (C.6.6), we have $l + \gamma = (m + \theta_{\lambda}) \cdot (p \cdot a) + \lambda - x_1 c_0 p \cdot a - x_2 b_0 p \cdot a$. We then have

$$(C.6.9) \quad |l + \gamma|^2 = \ell_p |m + \theta_{\lambda} - x_1 c_0 - x_2 b_0|^2 + |\lambda|^2.$$

After summing over m , the function $x_1 \mapsto \sum_{m \in \mathbb{Z}} e^{-rt\ell_p |m - x_1 c_0 - x_2 b_0|^2/Y}$ has period $|c_0|^{-1}$, hence

$$(C.6.10) \quad \begin{aligned} & \int_{-1/2}^{1/2} \sum_{m \in \mathbb{Z}} e^{-rt\ell_p |m + \theta_{\lambda} - x_1 c_0 - x_2 b_0|^2/Y} dx_1 = |c_0|^{-1} \int_{-|c_0|/2}^{|c_0|/2} \sum_{m \in \mathbb{Z}} e^{-rt\ell_p |m + \theta_{\lambda} - x - x_2 b_0|^2/Y} dx \\ &= \int_{-1/2}^{1/2} \sum_{m \in \mathbb{Z}} e^{-rt\ell_p |m + \theta_{\lambda} - x - x_2 b_0|^2/Y} dx = \int_{\mathbb{R}} e^{-rt\ell_p |x - x_2 b_0|^2/Y} dx = \sqrt{\pi \ell_p^{-1}} \sqrt{Y/(rt)}. \end{aligned}$$

For the sum over nonzero $\lambda \in \Lambda_{\perp}$, we have $\sum_{\mathbf{0} \neq \lambda \in \Lambda_{\perp}} e^{-rt|\lambda|^2/Y} < e^{-(r\delta^2/2Y)t} \sum_{\mathbf{0} \neq \lambda \in \Lambda_{\perp}} e^{-r|\lambda|^2/2}$ since $t \geq Y$.

Taking the product of these two estimates and setting $M' := \sqrt{\pi r^{-1} \ell_p^{-1}} \cdot \sum_{\mathbf{0} \neq \lambda \in \Lambda_{\perp}} e^{-r|\lambda|^2/2}$ gives

$$(C.6.11) \quad \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |K_{b,c}^{\#}(t, y, z)| dx_1 dx_2 \lesssim_{T,Y} M' (1+t) e^{-rt|W|^2} e^{-(r\delta^2/2Y)t} \cdot \sqrt{Y/t}.$$

By (C.6.3), integration against $k^{-1} dk$ gives

$$(C.6.12) \quad \int_0^{\infty} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |K_{b,c}^{\#}(t, y, z)| dx_1 dx_2 \frac{dk}{k} \lesssim_{T,Y} M' (1+t) \sqrt{Y/t} t^{-1/2} e^{-(r\delta^2/2Y)t} \lesssim_{T,Y} \sqrt{Y} \cdot e^{-(r\delta^2/2Y)t}.$$

Using the volume form (C.2.3), we find that

$$\int_{\mathbb{D}_1} \int_{\max\{Y, 1\}}^{\infty} |K_{b,c}^{\#}(t, y, z)| \Omega^2 t^{-1-\text{Re}(\rho)} dt$$

is bounded by a constant multiple of the integral

$$(C.6.13) \quad \int_{3/4}^{\infty} Y^{-3/2} \int_Y^{\infty} e^{-(r\delta^2/2Y)t} t^{-1-\text{Re}(\rho)} dt dY = \left(\int_1^{\infty} e^{-(r\delta^2/2)t} t^{-1-\text{Re}(\rho)} dt \right) \int_{3/4}^{\infty} Y^{-3/2-\text{Re}(\rho)} dY.$$

For $\rho \in \mathbb{C}$ such that $\text{Re}(\rho) > -1$, the integral (C.6.13) converges. Moreover, the convergence of (C.6.13) still holds with powers of $\log t$ in the integrand for derivatives in ρ .

C.6.2. Parallel part. It remains to consider the parallel part. For this, it is more convenient to change from z_1, z_2 to the following two coordinates:

$$(C.6.14) \quad w_m = m - c_0 z_1 - b_0 z_2 \quad \text{for } (m \in \mathbb{Z}), \quad \zeta = z_2.$$

In these coordinates, we have $|U|^2 + |W|^2 = Y^{-1}\ell_p|w_m|^2$. For fixed m , the corresponding term in $K_{b,c}^\parallel$ is

$$(C.6.15) \quad e^{-\pi t|W|^2} f_{b,c} \left(\frac{m \cdot (p \cdot a) + \gamma}{\sqrt{Y}} \right) = (3 - 2\pi t Y^{-1} \ell_p |w_m|^2 - 8\pi t \operatorname{tr}(Ty)) e^{-\pi t Y^{-1} \ell_p |w_m|^2}.$$

Since x_1 and x_2 are bounded, for any $\varepsilon > 0$, there are only finitely many integers m such that $\{|w_m| < \varepsilon\} \cap \mathbb{D}_1 \neq \emptyset$.

Let $h \in C^\infty(\overline{\mathcal{M}}(\mathbb{C}))$ be a compactly supported function. We still use h to denote its pullback to $\mathbb{D}_1$. We are going to analyze the (absolute) convergence of the following integral:

$$(C.6.16) \quad \int_{\mathbb{D}_1} \int_Y^\infty K_{b,c}^\parallel(t, y, z) h(z) t^{-1-\rho} dt \cdot \Omega_z^2.$$

To do this, we split the integral on $\mathbb{D}_1$ into two parts: the first part is those $z \in \mathbb{D}_1$ such that $|w_m(z)| \geq \varepsilon$ for all the integers m , while the second part is the finite union $\bigcup_m \{z \in \mathbb{D}_1 : |w_m(z)| < \varepsilon\}$.

On the first part where $|w_m(z)| \geq \varepsilon$ for all m , similar estimation techniques as the nonparallel case imply that

$$(C.6.17) \quad |(C.6.15)| \lesssim_{T,y} (1+t) e^{-rt/Y} e^{-rt|w_m|^2/Y} \lesssim_{T,y} (1+t) e^{-rt/Y} e^{-rt((m-c_0x_1-b_0x_2)^2/Y + \ell_p^{-1}|W(k)|^2)}$$

for some r (possibly depending on T, y, ε), where the $|w_m(z)| \geq \varepsilon$ condition is used to produce the term $e^{-rt/Y}$. After summing over all the integers m , the periodization and k -integration used in the nonparallel case give the analogues of (C.6.11), (C.6.12) and (C.6.13). Thus the parallel part $K_{b,c}^\parallel$ defines an L^1 -form on $\mathbb{D}_1$ over the first part holomorphically for $\operatorname{Re}(\rho) > -1$.

On the second part, we consider the integral (C.6.16) within a tube $|w_m| < \varepsilon$. For all $m' \neq m$ and for ε sufficiently small (assume this), we have $|w_{m'}| > \varepsilon$ on the tube $|w_m| < \varepsilon$, therefore their summation defines an L^1 -form over the first part holomorphically for $\operatorname{Re}(\rho) > -1/2$. Hence we are left to consider the integral of (C.6.15) against h over the tube. Using the substitution $t = Y\tau$,

$$(C.6.18) \quad \int_Y^\infty (C.6.15) t^{-1-\rho} dt = Y^{-\rho} (A_\rho(w_m) - 8\pi \operatorname{tr}(Ty) Y H_\rho(w_m)),$$

where

$$(C.6.19) \quad A_\rho(w_m) = \int_1^\infty (3 - 2\pi \ell_p \tau |w_m|^2) e^{-\pi \ell_p \tau |w_m|^2} \tau^{-1-\rho} d\tau, \quad H_\rho(w_m) = \int_1^\infty e^{-\pi \ell_p \tau |w_m|^2} \tau^{-\rho} d\tau.$$

Under the new coordinates w_m and ζ , we have

$$\Omega^2 = -\frac{c_0^2}{32\pi^2} \frac{dw_m \wedge d\bar{w}_m \wedge d\zeta \wedge d\bar{\zeta}}{Y^2}.$$

Recalling that $Y = y_1 y_2$ and $y_1 = -c_0^{-1}(b_0 y_2 + \operatorname{Im}(w_m))$ by (C.6.14), we get $y_2^2 \lesssim_T Y \lesssim_T y_2^2$ on the tube $|w_m| < \varepsilon$, for sufficiently small ε (relative to T).

We first consider the A_ρ -part. Notice that for $\rho \in \mathbb{C}$ with $\operatorname{Re}(\rho) > -1$,

$$(C.6.20) \quad \begin{aligned} \int_{|w_m| < \varepsilon} |A_\rho(w_m)| dw_m d\bar{w}_m &< \int_1^\infty \int_{\mathbb{C}} |(3 - 2\pi \ell_p \tau |w_m|^2) e^{-\pi \ell_p \tau |w_m|^2} \tau^{-1-\rho}| dw_m d\bar{w}_m d\tau \\ &\lesssim_{T,y} \int_1^\infty \tau^{-2-\operatorname{Re}(\rho)} d\tau = (1 + \operatorname{Re}(\rho))^{-1}, \end{aligned}$$

where the second inequality follows from integrating the Gaussian. For ρ in any compact subset $K \subseteq \{\operatorname{Re}(\rho) > -1\}$, we have $Y^{-2-\operatorname{Re}(\rho)} \lesssim_{T,K} y_2^{-4-2\operatorname{Re}(\rho)}$. Then for $\rho \in K$, we have

$$(C.6.21) \quad \int_{\{|w_m| < \varepsilon\} \times \mathcal{F}_1} |Y^{-\rho} A_\rho(w_m) h(w_m, \zeta)| \Omega^2 = \frac{c_0^2}{32\pi^2} \int_{\mathcal{F}_1} \int_{|w_m| < \varepsilon} |Y^{-\rho} A_\rho(w_m) h(w_m, \zeta) Y^{-2} dw_m d\bar{w}_m d\zeta d\bar{\zeta}| \\ \lesssim_{T,Y,K} \|h\|_\infty (1 + \operatorname{Re}(\rho))^{-1} \int_1^\infty y_2^{-4-2\operatorname{Re}(\rho)} dy_2 < \infty.$$

Moreover, the convergence (C.6.21) still holds with powers of $\log Y$ in the integrand for derivatives in ρ .

Next we consider the H_ρ -part. For $\zeta \in \mathcal{F}_1$ and $\rho \in \mathbb{C}$, define $\phi_{\zeta,\rho}(w_m) = h(w_m, \zeta) Y^{-\rho-1}$. Let $\xi = \operatorname{Re}(w_m)$ and $\eta = \operatorname{Im}(w_m)$, so $w_m = \xi + i\eta$. We first want to get an estimate for

$$(C.6.22) \quad \|\phi_{\zeta,\rho}\|_{C^2} := \max_{j+k \leq 2} \sup_{|w_m| \leq \varepsilon} |\partial_\xi^j \partial_\eta^k \phi_{\zeta,\rho}(w_m)|.$$

In the standard “cusp coordinates” associated to (∞, ∞) on $\overline{\mathcal{M}}(\mathbb{C})$, we have

$$q_1 = e^{2\pi i z_1} = e^{2\pi i c_0^{-1}(m - b_0 \zeta - w_m)} \quad \text{and} \quad q_2 = e^{2\pi i z_2} = e^{2\pi i \zeta}.$$

The function h is smooth in the coordinates q_1, q_2 . Since $\frac{d}{d\xi} q_1 = -2\pi i c_0^{-1} q_1$ and $\frac{d}{d\eta} q_1 = 2\pi c_0^{-1} q_1$, and since h is compactly supported on $\overline{\mathcal{M}}(\mathbb{C})$, we have $\|h\|_{C^2,w} := \max_{j+k \leq 2} \sup_{|w_m| \leq \varepsilon} |\partial_\xi^j \partial_\eta^k h(w_m, \zeta)| < \infty$.

Notice also that $\partial_\xi Y = 0$ and $\partial_\eta Y = -c_0^{-1} y_2$. Therefore, in any compact subset $K \subseteq \{\operatorname{Re}(\rho) > -1\}$, we have

$$(C.6.23) \quad \|\phi_{\zeta,\rho}\|_{C^2} \lesssim_{T,K} \|h\|_{C^2,w} \cdot y_2^{-2-2\operatorname{Re}(\rho)}.$$

Taylor expansion of $\phi_{\zeta,\rho}$ at $w_m = 0$ gives

$$(C.6.24) \quad \phi_{\zeta,\rho}(w_m) = \phi_{\zeta,\rho}(0) + \partial_\xi \phi_{\zeta,\rho}(0) \xi + \partial_\eta \phi_{\zeta,\rho}(0) \eta + O(\|\phi_{\zeta,\rho}\|_{C^2} |w_m|^2), \quad |w_m| < \varepsilon.$$

Then

$$(C.6.25) \quad \int_{|w_m| < \varepsilon} e^{-\pi \ell_p \tau |w_m|^2} \phi_{\zeta,\rho}(w_m) \frac{i}{2} dw_m d\bar{w}_m = \frac{\phi_{\zeta,\rho}(0)(1 - e^{-\pi \ell_p \tau \varepsilon^2})}{\ell_p \tau} + O(\tau^{-2} \|\phi_{\zeta,\rho}\|_{C^2}).$$

Therefore

$$(C.6.26) \quad \int_{\{|w_m| < \varepsilon\}} Y^{-1-\rho} H_\rho(w_m) h(w_m, \zeta) \frac{i}{2} dw_m d\bar{w}_m = \frac{\phi_{\zeta,\rho}(0)}{\ell_p \rho} - \phi_{\zeta,\rho}(0) \ell_p^{-1} \int_1^\infty e^{-\pi \ell_p \varepsilon^2 \tau} \tau^{-1-\rho} d\tau \\ + (\text{remainder term}),$$

for $\operatorname{Re}(\rho) > 0$, where the (remainder term) is the integration of the term $O(\tau^{-2} \|\phi_{\zeta,\rho}\|_{C^2})$ in (C.6.25) against τ . This (remainder term) extends holomorphically to $\operatorname{Re}(\rho) > -1$ by the estimate (C.6.23). The second term in (C.6.26) is also holomorphic for all $\rho \in \mathbb{C}$ due to the Gaussian term $e^{-\pi \ell_p \varepsilon^2 \tau}$ in the integrand. Moreover, we have $\phi_{\zeta,\rho}(0) = (-c_0 b_0^{-1})^{\rho+1} y_2^{-2-2\rho} h(0, \zeta)$, so $|\phi_{\zeta,\rho}(0)| \lesssim_{T,K,h} y_2^{-2-2\operatorname{Re}(\rho)}$ for $\rho \in K$. Therefore we can abbreviate (C.6.26) as

$$(C.6.27) \quad \int_{\{|w_m| < \varepsilon\}} Y^{1-\rho} H_\rho(w_m) h(w_m, \zeta) \frac{i}{2} dw_m d\bar{w}_m = -\frac{(c_0 b_0^{-1})^{\rho+1} y_2^{-2-2\rho} h(0, \zeta)}{\ell_p \rho} + t_m(\rho, \zeta; h),$$

where in a compact subset $K \subseteq \{\operatorname{Re}(\rho) > -1\}$, the function $t_m(\rho, \zeta; h)$ is holomorphic in ρ and we have $|t_m(\rho, \zeta; h)| \lesssim_{T,Y,K,\varepsilon} \|h\|_{C^2,w} y_2^{-2-2\operatorname{Re}(\rho)}$. Therefore, (C.6.27) is integrable against $d\zeta \wedge d\bar{\zeta}$ over $\zeta \in \mathcal{F}_1$ when $\operatorname{Re}(\rho) > -1/2$. Since $Y = Y(w_m, \zeta)$ and $H_\rho(w_m)$ are positive, the integral

$$(C.6.28) \quad \int_{\{|w_m| < \varepsilon\} \times \mathcal{F}_1} \int_Y K_{b,c}^\parallel(t, y, z) h(z) t^{-1-\rho} dt \cdot \Omega_z^2$$

converges absolutely when $\operatorname{Re}(\rho) > -1/2$, and defines a meromorphic function in ρ with a possible simple pole at $\rho = 0$. From the bounds above in terms of $\|h\|_{C^2,w}$, the constant term of (C.6.28) at $\rho = 0$ is continuous as a function of h , i.e. the constant term defines a $(2, 2)$ -current.

C.7. Proof for the zero matrix. In this subsection, we give the proof of Theorem C.1.2 and Theorem C.1.1 for the matrix $T = \mathbf{0}_3$. We focus on the situation with $d = 0$ for the moment. In that case, equation (C.4.1) forces $b = \mathbf{0}$ or $c = \mathbf{0}$ because if some $b_i \neq 0$, the diagonal condition from T gives $c_i = 0$, and the off-diagonal conditions from T then give $c_j = 0$ for every $j \neq i$. There are now infinitely many such pairs (b, c) , since if $b = \mathbf{0}$, then c can be any vector in $\mathbb{Z}^3$, and conversely. Then

$$(C.7.1) \quad K_{\mathbf{0}_3}^{d=0}(t, y, z) = \sum_{c \in \mathbb{Z}^3} K_{\mathbf{0},c}(t, y, z) + \sum_{b \in \mathbb{Z}^3} K_{b,\mathbf{0}}(t, y, z) - K_{\mathbf{0},\mathbf{0}}(t, y, z).$$

For $q > 0$ and $\tau = x + iy \in \mathbb{H}_1$, define

$$(C.7.2) \quad H(q, \tau) = \sum_{l, \mu \in \Lambda_a} e^{-\pi(l-x\mu|^2+y^2|\mu|^2)/q} P\left(\frac{l-x\mu}{\sqrt{q}}, \frac{y\mu}{\sqrt{q}}\right),$$

here we follow the notation of §C.4 with $\Lambda_a = \mathbb{Z}^3 \cdot a$. For $z = (z_1, z_2) \in \mathbb{H}_1^2 = \mathbb{D}$, a calculation gives

$$(C.7.3) \quad \sum_{c \in \mathbb{Z}^3} K_{\mathbf{0},c}(t, y, z) = H(Y/t, z_1), \quad \sum_{b \in \mathbb{Z}^3} K_{b,\mathbf{0}}(t, y, z) = H(Y/t, z_2), \quad K_{\mathbf{0},\mathbf{0}}(t, y, z) = \sum_{l \in \Lambda_a} e^{-\pi|l|^2/q} P(lq^{-1/2}, 0).$$

For the range $q = Y/t \leq 1$, for $\tau \in \mathcal{F}_1$, we have

$$(C.7.4) \quad |l - x\mu|^2 + y^2|\mu|^2 \geq |l|^2 + \frac{3}{4}|\mu|^2 - |l \cdot \mu| \geq \frac{1}{4}(|l|^2 + |\mu|^2).$$

Since P is a polynomial in its variables, we can absorb P into the square-root of the Gaussian term and sum over Λ_a^2 . This gives $H(q, \tau) \lesssim_y 1$ for $q \leq 1$ and $\tau \in \mathcal{F}_1$.

For $q \geq 1$, we use Poisson summation in l and the Fourier transform from (C.4.4) to give

$$H(q, \tau) = 2\pi q^{3/2} |\det a|^{-1} \sum_{\substack{\mu \in \Lambda_a \\ \mathbf{0} \neq \xi \in \Lambda_a^\vee}} e^{-\pi y^2 |\mu|^2 / q - \pi q |\xi|^2} e^{-2\pi i x \langle \xi, \mu \rangle} (q |\xi|^2 - \pi y^2 (|\mu|^2 |\xi|^2 - (\mu \cdot \xi)^2)).$$

For the polynomial in the bracket, a calculation gives $|q |\xi|^2 - \pi y^2 (|\mu|^2 |\xi|^2 - (\mu \cdot \xi)^2)| \leq |\xi|^2 q (1 + 2\pi y^2 |\mu|^2 / q) \leq 2\pi |\xi|^2 q (1 + y^2 |\mu|^2 / q)$. Hence

$$|H(q, \tau)| \leq 4\pi^2 q^{5/2} \left(\sum_{\mu \in \Lambda_a} (1 + y^2 |\mu|^2 / q) e^{-\pi y^2 |\mu|^2 / q} \right) \left(\sum_{\mathbf{0} \neq \xi \in \Lambda_a^\vee} |\xi|^2 e^{-\pi q |\xi|^2} \right).$$

We find $\sum_{\mu \in \Lambda_a} (1 + y^2 |\mu|^2 / q) e^{-\pi y^2 |\mu|^2 / q} \lesssim_y 1 + q^{3/2}$ for $y \geq \sqrt{3}/2$, using the Gaussian integral comparison technique used to bound $F_a(q)$ in §C.4. For the ξ sum, let $\delta_a = \min_{\mathbf{0} \neq \xi \in \Lambda_a^\vee} |\xi| > 0$,

$$\sum_{\mathbf{0} \neq \xi \in \Lambda_a^\vee} |\xi|^2 e^{-\pi q |\xi|^2} \leq e^{-\pi(q-1)\delta_a^2} \sum_{\mathbf{0} \neq \xi \in \Lambda_a^\vee} |\xi|^2 e^{-\pi |\xi|^2} = e^{-\pi \delta_a^2 q} \cdot \left(e^{\pi \delta_a^2} \sum_{\mathbf{0} \neq \xi \in \Lambda_a^\vee} |\xi|^2 e^{-\pi |\xi|^2} \right).$$

Hence for $q \geq 1$, we have

$$(C.7.5) \quad |H(q, \tau)| \lesssim_y q^4 e^{-\pi \delta_a^2 q}$$

For all $q > 0$, we thus have $\sup_{\tau \in \mathcal{F}_1} |H(q, \tau)| \lesssim_y 1$ using exponential decay of the ξ sum for $q \geq 1$. For all $q > 0$, Poisson summation again gives

$$K_{\mathbf{0}, \mathbf{0}}(t, y, z) = 2\pi q^{5/2} |\det a|^{-1} \sum_{\mathbf{0} \neq \xi \in \Lambda_a^\vee} |\xi|^2 e^{-\pi q |\xi|^2} = 2\pi |\det a|^{-1} F_a(q)$$

with $F_a(q)$ as used in Section C.4. By the estimates in loc. cit., we have $|K_{\mathbf{0}, \mathbf{0}}(t, y, z)| \lesssim_y 1$ (uniformly in z). The decomposition in (C.7.1) now gives $\sup_{z \in \mathbb{D}_1} |K_{\mathbf{0}_3}^{d=0}(t, y, z)| \lesssim_y 1$.

Recall that $\sup_{z \in \mathbb{D}_1} |K_{\mathbf{0}_3}^{d \neq 0}(t, y, z)| \lesssim_y 1$ by (C.3.4) for $Y \geq Y_0$ and Section C.3.2 for $Y \leq Y_0$. We thus have $\sup_{z \in \mathbb{D}_1} |K_{\mathbf{0}_3}(t, y, z)| \lesssim_y 1$, which proves one direction of Theorem C.1.2 (c) for $T = \mathbf{0}_3$, i.e. that the integral defining $J(\mathbf{0}_3, y, \rho)$ converges absolutely for $\text{Re}(\rho) > 0$, using $\int_{\mathcal{M}(\mathbb{C})} \Omega^2 < \infty$. In particular, $\mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)$ is a holomorphic family of L^1 forms on $\overline{\mathcal{M}}(\mathbb{C})$ for $\text{Re}(\rho) > 0$.

Now we explain meromorphic continuation of the L^1 form $\mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)$ for $\text{Re}(\rho) > -1$. The $l = \mu = 0$ term in the summation (C.7.2) is 3, while all other terms are $\lesssim e^{-\pi(|l|^2 + |\mu|^2)/(8q)}$ by (C.7.4). Combining with the estimate on $|K_{\mathbf{0}_3}^{d \neq 0}(t, y, z)|$ from Section C.3.2, we get that for $t \geq \max(1, Y)$, we have

$$(C.7.6) \quad |K_{\mathbf{0}_3}(t, y, z) - 3| \leq |K_{\mathbf{0}_3}^{d \neq 0}(t, y, z)| + |K_{\mathbf{0}_3}^{d=0}(t, y, z) - 3| \lesssim_y e^{-c't/Y},$$

for some $c' > 0$ depending only on y . The estimate in (C.7.6) also shows the divergence for $\text{Re}(\rho) \leq 0$ claimed in Theorem C.1.2 (c).

When $1 \leq t \leq Y$, we have the exponential estimate $|K_{\mathbf{0}, \mathbf{0}}(t, y, z)| \lesssim_y e^{-c''q}$ for some $c'' > 0$ depending only on y , as follows from the exponential estimate on the Gaussian sum from Section C.4. Combining this with the estimates (C.7.5) and (C.3.4), we have

$$(C.7.7) \quad |K_{\mathbf{0}_3}(t, y, z)| \leq |K_{\mathbf{0}_3}^{d=0}(t, y, z)| + |K_{\mathbf{0}_3}^{d \neq 0}(t, y, z)| \lesssim_y q^4 e^{-c'''q} + e^{-c'''tY} (1 + q^6) \lesssim_y 1$$

for $1 \leq t \leq Y$, and some constant $c''' > 0$ which depends only on y . For any $c_0 > 0$ and $Y \geq 1$,

$$(C.7.8) \quad \int_1^Y e^{-c_0 t Y} (1 + q^6) \frac{dt}{t} \stackrel{s=Y/t}{\leq} \int_Y^{Y^2} e^{-c_0 s} (1 + s^6) \frac{ds}{s} < \int_1^\infty e^{-c_0 s} (1 + s^6) \frac{ds}{s},$$

$$\int_1^Y q^4 e^{-c_0 q} \frac{dq}{q} \stackrel{q=Y/t}{\leq} \int_1^Y q^4 e^{-c_0 q} \frac{dq}{q} < \int_1^\infty q^4 e^{-c_0 q} \frac{dq}{q}, \quad \int_Y^\infty e^{-c_0 t/Y} \frac{dt}{t} \leq \int_1^\infty e^{-c_0 t} \frac{dt}{t}.$$

It follows that for $Y \geq 1$, we have

$$(C.7.9) \quad \int_1^Y |K_{\mathbf{0}_3}(t, y, z)| \frac{dt}{t} \lesssim_y 1, \quad \int_Y^\infty |K_{\mathbf{0}_3}(t, y, z) - 3| \frac{dt}{t} \lesssim_y 1,$$

uniformly in z .

For $\operatorname{Re}(\rho) > 0$, we have

$$(C.7.10) \quad \int_1^\infty K_{\mathbf{0}_3}(t, y, z) \frac{dt}{t^{1+\rho}} = \frac{3}{\rho} + \int_1^\infty (K_{\mathbf{0}_3}(t, y, z) - 3) \frac{dt}{t^{1+\rho}}$$

where the integral converges absolutely as already explained in the paragraph before (C.7.6), because $|K_{\mathbf{0}_3}(t, y, z)| \lesssim_y 1$.

The second integral splits into the following two parts:

$$(C.7.11) \quad \int_1^Y (K_{\mathbf{0}_3}(t, y, z) - 3) \frac{dt}{t^{1+\rho}}, \quad \int_Y^\infty (K_{\mathbf{0}_3}(t, y, z) - 3) \frac{dt}{t^{1+\rho}}.$$

The first one converges absolutely for all $\rho \in \mathbb{C}$, again using $|K_{\mathbf{0}_3}(t, y, z)| \lesssim_y 1$. The second one also converges absolutely for all $\rho \in \mathbb{C}$ by the estimate (C.7.6). Therefore, for any fixed $z \in \mathbb{D}_1$, the formula (C.7.10) gives a meromorphic continuation of $\mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)_z$ to $\rho \in \mathbb{C}$, holomorphic away from $\rho = 0$ where there is a simple pole, i.e.

$$(C.7.12) \quad \mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)_z = \frac{3\Omega_z^2}{\rho} + \left[\left(\int_1^\infty (K_{\mathbf{0}_3}(t, y, z) - 3) \frac{dt}{t^{1+\rho}} \right) \Omega_z^2 \right].$$

For any fixed $\rho \neq 0$, the formula (C.7.12) defines a measurable $(2, 2)$ -form $\mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)$ on $\overline{\mathcal{M}}(\mathbb{C})$. Next we consider the absolute integrability of $\mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)$ and its constant term at $\rho = 0$. We first estimate the integral of $(K_{\mathbf{0}_3}(t, y, z) - 3)\Omega^2$ over $\overline{\mathcal{M}}(\mathbb{C})$, before integrating over t . By (C.2.3), we have

$$(C.7.13) \quad \int_{\overline{\mathcal{M}}(\mathbb{C})} |K_{\mathbf{0}_3}(t, y, z) - 3| \Omega^2 \lesssim \int_{3/4}^\infty \int_{3/(4Y)}^{4Y/3} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |K_{\mathbf{0}_3}(t, y, z) - 3| dx_1 dx_2 \frac{dk dY}{Y^2 k} \\ \lesssim_y \int_{3/4}^t e^{-c't/Y} \frac{\log(4Y/3)}{Y^2} dY + \int_t^\infty \frac{\log(4Y/3)}{Y^2} dY \lesssim_y \frac{1 + \log t}{t},$$

where we used (C.7.6) for the range $Y \leq t$ and (C.7.7) for the range $t \leq Y$. Therefore, the difference $\mathfrak{g}^\circ(\mathbf{0}_3, y, \rho) - 3\rho^{-1}[\Omega^2]$ is an L^1 form on $\overline{\mathcal{M}}(\mathbb{C})$ for $\operatorname{Re}(\rho) > -1$, and admits a holomorphic extension to $\rho = 0$ where its value is also an L^1 form $\overline{\mathcal{M}}(\mathbb{C})$.

For the constant term at $\rho = 0$, and for $Y \geq 1$, we have

$$(C.7.14) \quad \operatorname{CT}_{\rho=0} \mathfrak{g}^\circ(\mathbf{0}_3, y, \rho) = \left(\int_1^\infty (K_{\mathbf{0}_3}(t, y, z) - 3) \frac{dt}{t} \right) \Omega^2 \\ = \left(-3 \log Y + \int_1^Y K_{\mathbf{0}_3}(t, y, z) \frac{dt}{t} + \int_Y^\infty (K_{\mathbf{0}_3}(t, y, z) - 3) \frac{dt}{t} \right) \Omega^2 = (-3 \log Y + O_y(1)) \Omega^2,$$

by (C.7.9), where $O_y(1)$ means a finite number independent of the point z , where the constant could depend on y . The absolute integrability of $\mathfrak{g}^\circ(\mathbf{0}_3, y) := \operatorname{CT}_{\rho=0} \mathfrak{g}^\circ(\mathbf{0}_3, y, \rho)$ over $\overline{\mathcal{M}}(\mathbb{C})$ can also be seen from integrability of $(1 + \log Y)\Omega^2$ over $\mathbb{D}_1$.

REFERENCES

- [BF10] Jan Hendrik Bruinier and Jens Funke. On the injectivity of the Kudla–Millson lift and surjectivity of the Borcherds lift. In *Moonshine: the first quarter century and beyond*, volume 372 of *London Math. Soc. Lecture Note Ser.*, pages 12–39. Cambridge Univ. Press, Cambridge, 2010. 30
- [BGKK07] José Ignacio Burgos Gil, Jürg Kramer, and Ulf Kühn. Cohomological arithmetic chow rings. *Journal of the Institute of Mathematics of Jussieu*, 6(1):1–172, 2007. 4, 47, 48, 86, 88
- [BGS94] Jean-Benoît Bost, Henri Gillet, and Christophe Soulé. Heights of projective varieties and positive green forms. *Journal of the American Mathematical Society*, 7:903–1027, 1994. 87, 88
- [BH26] Jan Hendrik Bruinier and Benjamin Howard. *Arithmetic Volumes of Unitary Shimura Varieties*, volume 318 of *Memoirs of the American Mathematical Society*, no. 1619. American Mathematical Society, Providence, RI, 2026. 2
- [BK12a] Rolf Berndt and Ulf Kühn. On Kudla’s Green function for signature (2,2), Part I. *arXiv preprint*, 2012. Preprint available at arXiv. 25, 29, 30, 31, 44, 47, 81
- [BK12b] Rolf Berndt and Ulf Kühn. On Kudla’s Green function for signature (2,2), Part II. *arXiv preprint*, 2012. Preprint available at arXiv. 43, 80, 81, 82, 86
- [Bou07] Irene I. Bouw. 12. Invariants of Ternary Quadratic Forms. In *ARGOS Seminar on Intersections of Modular Correspondences*, pages 113–137, 2007. 90, 91, 92
- [BY20] Jan Bruinier and Tonghai Yang. Arithmetic degrees of special cycles and derivatives of siegel eisenstein series. *Journal of European Math. Soc.*, page pp60, 2020. 2, 82
- [Che24] Ryan C. Chen. Co-rank 1 Arithmetic Siegel–Weil: combined version. *Online*, May 2, 2024. 2, 6, 9, 13, 15, 39, 40, 46, 47, 48, 51, 53, 54, 60, 68, 69, 70
- [CLZ] Ryan Chen, Weixiao Lu, and Wei Zhang. Faltings heights and the subleading terms of adjoint l -functions. In preparation. 57, 71, 72
- [CY20] Sungmun Cho and Takuya Yamauchi. A reformulation of the Siegel series and intersection numbers. *Mathematische Annalen*, 377(3):1757–1826, 2020. 39
- [Del85] Pierre Deligne. Preuve des conjectures de Tate et de Shafarevitch. In *Séminaire Bourbaki*, volume 1983/84, number 121–122 in *Astérisque*, pages 25–41. Société mathématique de France, 1985. Exposé no. 616. 5
- [DR73] Pierre Deligne and Michael Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 349, pages 143–316. Springer, Berlin-New York, 1973. 32
- [DY19] Tuoping Du and Tonghai Yang. Arithmetic Siegel–Weil formula on $X_0(N)$. *Adv. Math.*, 345:702–755, 2019. 2
- [Fal83] Gerd Faltings. Endlichkeitssätze für abelsche varietäten über zahlkörpern. *Inventiones Mathematicae*, 73(3):349–366, 1983. German; “Finiteness theorems for abelian varieties over number fields.”. 49
- [Gil09] Henri Gillet. Arithmetic intersection theory on Deligne–Mumford stacks, 2009. 47
- [GK93] Benedict Gross and Kevin Keating. On the intersection of modular correspondences. *Inventiones Mathematicae*, 112(2):225–246, 1993. 1, 4, 8, 39, 48, 55, 57, 82, 90
- [Gro86] Benedict Gross. On canonical and quasi-canonical liftings. *Inventiones Mathematicae*, 84(2):321–326, 1986. 58
- [GS19] Luis E. García and Siddarth Sankaran. Green forms and the arithmetic Siegel–Weil formula. *Inventiones Mathematicae*, 215(3):863–975, 2019. 2, 3, 6, 13, 14, 15, 16, 18, 19, 21, 22, 23, 25, 29, 30, 31, 33, 40, 41, 42, 43, 44, 45, 46, 47, 81, 82, 83, 84, 86, 100
- [Gö07a] Ulrich Görtz. Arithmetic intersection numbers. In *Argos seminar on intersections of modular correspondences*, number 312 in *Astérisque*, pages 15–24. Société Mathématique de France, Paris, 2007. 82
- [Gö07b] Ulrich Görtz. A sum of representation numbers. In *Argos seminar on intersections of modular correspondences*, number 312 in *Astérisque*, pages 9–14. Société Mathématique de France, Paris, 2007. 39
- [Hai14] Richard Hain. Lectures on moduli spaces of elliptic curves, 2014. 43
- [HK91] Michael Harris and Stephen S. Kudla. The central critical value of the triple product l -function. *Annals of Mathematics*, 133(3):605–672, 1991. 6, 24

- [HM22] Ben Howard and Keerthi Madapusi. Kudla’s modularity conjecture on integral models of orthogonal shimura varieties. *arXiv preprint arXiv:2211.05108*, 2022. [32](#), [33](#)
- [How89] Roger Howe. Remarks on classical invariant theory. *Transactions of the American Mathematical Society*, 313(2):539–570, 1989. [24](#)
- [How12] Benjamin Howard. Complex multiplication cycles and Kudla-Rapoport divisors. *Ann. of Math. (2)*, 176(2):1097–1171, 2012. [34](#)
- [HZ76] Friedrich Hirzebruch and Don Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–113, 1976. [28](#)
- [IK18] Tamotsu Ikeda and Hidenori Katsurada. On the Gross-Keating invariant of a quadratic form over a non-archimedean local field. *Amer. J. Math.*, 140(6):1521–1565, 2018. [8](#), [9](#)
- [Ike17] Tamotsu Ikeda. On the functional equation of the Siegel series. *J. Number Theory*, 172:44–62, 2017. [9](#)
- [KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic Moduli of Elliptic Curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985. [33](#), [58](#)
- [KM86] Stephen S. Kudla and John J. Millson. The Theta Correspondence and Harmonic Forms. I. *Mathematische Annalen*, 274(3):353–378, 1986. [23](#), [25](#), [29](#)
- [KM87] Stephen S. Kudla and John J. Millson. The Theta Correspondence and Harmonic Forms. II. *Mathematische Annalen*, 277(2):267–314, 1987. [23](#), [25](#), [29](#)
- [KM90] Stephen S. Kudla and John J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, 71:121–172, 1990. [25](#)
- [KR88] Stephen S. Kudla and Stephen Rallis. On the Weil–Siegel formula. *Journal für die reine und angewandte Mathematik*, 387:1–68, 1988. [18](#)
- [KR90] Stephen S. Kudla and Stephen Rallis. Degenerate principal series and invariant distributions. *Israel Journal of Mathematics*, 69:25–45, 1990. [11](#)
- [KR94] Stephen S. Kudla and Stephen Rallis. A regularized Siegel–Weil formula: The first term identity. *Annals of Mathematics*, 140(1):1–80, 1994. [6](#), [11](#), [18](#), [23](#), [24](#)
- [KR11] Stephen Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties I. Unramified local theory. *Invent. Math.*, 184(3):629–682, 2011. [2](#)
- [KR14] Stephen Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties II: Global theory. *J. Reine Angew. Math.*, 697:91–157, 2014. [2](#)
- [KRS92] Stephen S. Kudla, Stephen Rallis, and David Soudry. On the degree 5 L-function for $\mathrm{Sp}(2)$. *Inventiones Mathematicae*, 107(3):483–542, 1992. [23](#)
- [KRY04] Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. Derivatives of Eisenstein series and Faltings heights. *Compos. Math.*, 140(4):887–951, 2004. [49](#), [51](#)
- [KRY06] Stephen Kudla, Michael Rapoport, and Tonghai Yang. *Modular Forms and Special Cycles on Shimura Curves (AM-161)*, volume 161. Princeton university press, 2006. [2](#)
- [Kud94] Stephen Kudla. Splitting metaplectic covers of dual reductive pairs. *Israel Journal of Mathematics*, 87(1-3):361–401, 1994. [13](#), [14](#)
- [Kud97] Stephen S. Kudla. Central derivatives of Eisenstein series and height pairings. *Annals of Mathematics*, 146(3):545–646, 1997. [1](#), [13](#), [14](#), [34](#)
- [Kud03] Stephen S. Kudla. Integrals of borchers forms. *Compositio Mathematica*, 137(3):293–349, 2003. [25](#), [31](#), [38](#), [80](#)
- [Kud04] Stephen S. Kudla. Special cycles and derivatives of eisenstein series. In Henri Darmon and Shou-Wu Zhang, editors, *Heegner Points and Rankin L-Series*, volume 49 of *Mathematical Sciences Research Institute Publications*, pages 243–270. Cambridge University Press, Cambridge, 2004. [1](#)
- [Liu11] Yifeng Liu. Arithmetic theta lifting and L -derivatives for unitary groups, I. *Algebra Number Theory*, 5(7):849–921, 2011. [2](#)
- [LZ22a] Chao Li and Wei Zhang. Kudla–Rapoport cycles and derivatives of local densities. *Journal of the American Mathematical Society*, 35(3):705–797, 2022. [2](#)

- [LZ22b] Chao Li and Wei Zhang. On the arithmetic Siegel–Weil formula for GSpin Shimura varieties. *Inventiones mathematicae*, 228(3):1353–1460, 2022. [2](#)
- [Meu07] Volker Meusers. 9. Canonical and quasi-canonical liftings in the split case. In *ARGOS Seminar on Intersections of Modular Correspondences*, pages 87–98, 2007. [58](#)
- [Mum85] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics. Oxford University Press, 1985. With appendices by C. P. Ramanujam and Yuri Manin, Reprint of the second (1974) edition. [51](#)
- [Nag92] Shoyu Nagaoka. A note on the siegel-eisenstein series of weight 2 on $Sp_2(\mathbb{Z})$. *Manuscripta Mathematica*, 77(1):71–88, 1992. [39](#)
- [NT91] Yukiyoshi Nakkajima and Yuichiro Taguchi. A generalization of the Chowla–Selberg formula. *Journal für die reine und angewandte Mathematik*, 419:119–124, 1991. [5](#), [49](#), [51](#)
- [Rap07] Michael Rapoport. 13. Deformations of isogenies of formal groups. In *ARGOS Seminar on Intersections of Modular Correspondences*, pages 139–170, 2007. [56](#), [57](#)
- [Roh84] David E. Rohrlich. A modular version of Jensen’s formula. *Math. Proc. Cambridge Philos. Soc.*, 95(1):15–20, 1984. [4](#)
- [RR93] Ramaswamy Ranga Rao. On some explicit formulas in the theory of Weil representation. *Pacific Journal of Mathematics*, 157(2):335–371, 1993. [13](#)
- [RZ96] Michael Rapoport and Thomas Zink. *Period spaces for p -divisible groups*, volume 141 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996. [54](#), [66](#), [68](#), [69](#)
- [Shi82] Goro Shimura. Confluent hypergeometric functions on tube domains. *Math. Ann.*, 260(3):269–302, 1982. [15](#)
- [SSY23] Siddarth Sankaran, Yousheng Shi, and Tonghai Yang. A genus two arithmetic Siegel–Weil formula on $X_0(N)$. *Transactions of the American Mathematical Society*, 376(06):3995–4041, 2023. [2](#)
- [Ste21] Oliver Stein. Analytic properties of Eisenstein series and standard L -functions. *Nagoya Mathematical Journal*, 244:168–203, 2021. [18](#)
- [Tat67] John T Tate. p -divisible groups. In *Proceedings of a conference on Local Fields*, pages 158–183. Springer, 1967. [72](#)
- [Voi21] John Voight. *Quaternion algebras*, volume 288 of *Graduate Texts in Mathematics*. Springer, Cham, [2021] ©2021. [66](#)
- [Wed07] Torsten Wedhorn. 15. Calculation of representation densities. In *ARGOS Seminar on Intersections of Modular Correspondences*, pages 179–190, 2007. [93](#), [94](#)
- [Wei65] André Weil. Sur la formule de Siegel dans la théorie des groupes classiques. *Acta Mathematica*, 113(1):1–87, 1965. [6](#)
- [Wew07a] Stefan Wewers. 14. An alternative approach using ideal bases. In *ARGOS Seminar on Intersections of Modular Correspondences*, pages 171–177, 2007. [59](#)
- [Wew07b] Stefan Wewers. 8. Canonical and quasi-canonical liftings. In *ARGOS Seminar on Intersections of Modular Correspondences*, pages 67–86, 2007. [58](#)
- [Zhu24] Baiqing Zhu. Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$: singular terms, 2024. [2](#), [93](#)
- [Zhu25] Baiqing Zhu. Arithmetic Siegel–Weil formula on $\mathcal{X}_0(N)$. *Algebra & Number Theory*, 19(9):1771–1822, 2025. [2](#), [93](#)

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