

CO-RANK 1 ARITHMETIC SIEGEL–WEIL III: GEOMETRIC LOCAL-TO-GLOBAL

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ABSTRACT. This is the third in a sequence of four papers, where we prove the arithmetic Siegel–Weil formula in co-rank 1 for Kudla–Rapoport special cycles on exotic smooth integral models of unitary Shimura varieties of arbitrarily large even arithmetic dimension. Our arithmetic Siegel–Weil formula implies that degrees of Kudla–Rapoport arithmetic special 1-cycles are encoded in the first derivatives of unitary Eisenstein series Fourier coefficients.

In this paper, we finish the reduction process from global arithmetic intersection numbers for special cycles to the local geometric quantities in our companion papers.

Building on our previous companion papers, we also propose a construction for arithmetic special cycle classes associated to possibly singular matrices of arbitrary co-rank.

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1. INTRODUCTION

This paper is a continuation of our companion papers [Che24a; Che24b], and will be followed by [Che24c]. We refer the reader to the introductions of [Che24a; Che24c] for further motivation, overview, and strategy for our four-part series of papers. The discussion in Section 1.1 is an abridged version of loc. cit..

In Section 1.2 below, we survey the remainder of the reduction process from global arithmetic Siegel–Weil to our local main theorems (which were stated in terms of local special cycles and local Whittaker functions). The reduction process will require work on both the geometric and analytic sides of the arithmetic Siegel–Weil formula, and uses (geometric) “local height decomposition” inputs from [Che24a, Sections 10 and 11]. The present paper focuses on geometric aspects. Analytic aspects will be treated in our companion paper [Che24c], where we combine our preceding results to complete the proof of our co-rank 1 arithmetic Siegel–Weil formula.

In Section 1.3, we outline the structure of this paper and its relation with our companion papers [Che24a; Che24b; Che24c].

1.1. Arithmetic Siegel–Weil. For the introduction, fix an imaginary quadratic field F/\mathbb{Q} with ring of integers \mathcal{O}_F and odd discriminant Δ . Let V be a non-degenerate F/\mathbb{Q} Hermitian space of signature $(n-1, 1)$ with pairing $(-, -)$. Set $G = U(V)$. For the introduction, we assume V contains a full-rank self-dual¹ \mathcal{O}_F -lattice.

Since the work of Kudla–Rapoport [KR14] (also Rapoport–Smithling–Zhang [RSZ21]), it has been customary to define special cycles $\mathcal{Z}(T) \rightarrow \mathcal{M}$ over (stacky) integral models $\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_F$ for Shimura varieties associated to $G' := \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \times G$. In this paper, we mainly take $\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_F$ to be the “exotic smooth” Rapoport–Smithling–Zhang (RSZ) integral model of odd relative dimension $n-1$ [RSZ21, §6] (empty if $n \equiv 0 \pmod{4}$). The stack \mathcal{M} admits a moduli description: it parameterizes tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ where A_0 and A are abelian schemes (dimensions 1 and n respectively) with \mathcal{O}_F -actions ι_0 and ι , and with compatible quasi-polarizations λ_0 and λ . The datum $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ satisfies a few additional conditions, which we suppress in the introduction (see [Che24a, Section 3.1] and Section 2.1). We are able to prove versions of our main global results for more general \mathcal{M} (including odd arithmetic dimension n) at the price of discarding finitely many primes (particularly ramified primes for odd n); see [Che24c, Remark 9.1.5].

The moduli stack \mathcal{M} carries a natural family of Hermitian \mathcal{O}_F -lattices

$$\mathcal{L}at \rightarrow \mathcal{M} \quad \mathcal{L}at := \underline{\mathrm{Hom}}_{\mathcal{O}_F}(A_0, A). \tag{1.1.1}$$

Given² any $T \in \mathrm{Herm}_m(\mathbb{Q})$, the associated *Kudla–Rapoport* special cycle $\mathcal{Z}(T) \rightarrow \mathcal{M}$ is defined as the substack

$$\mathcal{Z}(T) := \{\underline{x} \in \mathcal{L}at^m : (\underline{x}, \underline{x}) = T\} \subseteq \mathcal{L}at^m \tag{1.1.2}$$

¹Our convention is that *self-duality* for Hermitian lattices is always with respect to the trace pairing, here $\mathrm{tr}_{F/\mathbb{Q}}(-, -)$.

²The notation Herm_m denotes a scheme over $\mathrm{Spec} \mathbb{Z}$, e.g. $\mathrm{Herm}_m(\mathbb{Q})$ denotes $m \times m$ Hermitian matrices with entries in F .

consisting of m -tuples with Gram matrix T . More precisely, see Section 2.1. The morphism $\mathcal{Z}(T) \rightarrow \text{Spec } \mathcal{O}_F$ is smooth of relative dimension $n - 1 - \text{rank}(T)$ in the generic fiber over $\text{Spec } F$. If T is not positive semi-definite, then $\mathcal{Z}(T)$ is empty.

We propose a new candidate definition of arithmetic cycle classes

$$[\widehat{\mathcal{Z}}(T)] := [\widehat{\mathcal{Z}}(T)_{\mathcal{H}}] + \sum_{p \text{ prime}} [\mathbb{L}\mathcal{Z}(T)_{\mathcal{V},p}] \in \widehat{\text{Ch}}^m(\mathcal{M})_{\mathbb{Q}} \quad (1.1.3)$$

associated to arbitrary (possibly singular) T , where $\widehat{\text{Ch}}^m(\mathcal{M})_{\mathbb{Q}}$ is an arithmetic Chow group associated to \mathcal{M} (Section 3.1). Here, $[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}]$ should describe “horizontal” contributions and $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V},p}$ should describe “vertical” contributions. The vertical (positive characteristic) contributions $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V},p}$ were defined in our companion paper [Che24a, Section 3.6]. As we explain in Section 3.2 of this paper, the class $[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}]$ may be constructed using currents $g_{T,y}$ (associated to T and allowed to vary with a parameter $y \in \text{Herm}_m(\mathbb{R})_{>0}$) satisfying a *modified current equation*, i.e. that

$$-\frac{1}{2\pi i} \partial \bar{\partial} g_{T,y} + \delta_{\mathcal{Z}(T)_c} \wedge [c_1(\widehat{\mathcal{E}}_c^{\vee})^{m-\text{rank}(T)}] \quad (1.1.4)$$

is represented by a smooth (m, m) -form. For such currents, we apply the proposal of [GS19, §5.4] to the flat part³ $\mathcal{Z}(T)_{\mathcal{H}}$ of $\mathcal{Z}(T)$.

For general T , there is no precise definition of $[\widehat{\mathcal{Z}}(T)]$ which has been proposed in the prior literature [Li24, Remark 4.4.2]. Our candidate definition may need modification on a compactification, but we expect it to apply in already-compact situations (e.g. the Rapoport–Smithling–Zhang [RSZ21] setup for CM extensions of totally real fields $\neq \mathbb{Q}$). In general, it may also be necessary to modify the Green currents differently than in [GS19, Definition 4.7]; see discussion below.

Currents satisfying (1.1.4) were constructed by Garcia and Sankaran [GS19], using the Mathai–Quillen theory of superconnections [GS19, (4.38)]. For their arithmetic Siegel–Weil results, however, they need a non “linearly invariant” modification of their current [GS19, Definition 4.7] (see discussion below).

We choose to instead use the star-product approach of Kudla [Kud97a] (as formulated by Liu for unitary groups [Liu11]) to define the currents $g_{T,y}$ for our arithmetic Siegel–Weil results. Traditionally, the star product approach was used for nonsingular T (or at least block diagonal T , with diagonal entries 0 or nonsingular). In [Che24b, Section 2.4], we gave a (new) *linearly invariant* modification in the case of singular $T \in \text{Herm}_n(\mathbb{Q})$ with rank $n - 1$, which will appear in our arithmetic Siegel–Weil result for singular T . The discussion in [Che24b, Section 2.4] focused on the local version (on the Hermitian symmetric domain); we descend it to the complex Shimura variety in Section 5.4 of this paper.

As part of the expected automorphic behavior of $[\widehat{\mathcal{Z}}(T)]$, it is expected that these classes should satisfy a certain “linear invariance” property for the action⁴ of $\text{GL}_m(\mathcal{O}_F)$ on Hermitian matrices

³Given an algebraic stack \mathcal{X} over a Dedekind domain R , its *flat part* or *horizontal part* of $\mathcal{X}_{\mathcal{H}}$ is the largest closed substack $\mathcal{X}_{\mathcal{H}} \subseteq \mathcal{X}$ which is flat over $\text{Spec } R$. The stack $\mathcal{X}_{\mathcal{H}}$ is also the scheme-theoretic image of the generic fiber of \mathcal{X} . Given a formal algebraic stack \mathcal{X} over $\text{Spf } R$ for a complete discrete valuation ring R , its *flat part* or *horizontal part* $\mathcal{X}_{\mathcal{H}}$ is the largest closed substack $\mathcal{X}_{\mathcal{H}} \subseteq \mathcal{X}$ which is flat over $\text{Spf } R$ (in the sense discussed in Section 4.7).

⁴For any $\gamma \in \text{GL}_m(\mathcal{O}_F)$ and any Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$, we say e.g. that T and ${}^t \bar{\gamma} T \gamma$ are $\text{GL}_m(\mathcal{O}_F)$ -*equivalent*, and that they lie in the same $\text{GL}_m(\mathcal{O}_F)$ -*equivalence class*.

T . We verify this for the classes we define: for any $g_{T,y}$ satisfying

$$g_{T,y} = g^{t\bar{\gamma}T\gamma, \gamma^{-1}y^t\bar{\gamma}^{-1}}, \quad (1.1.5)$$

we show

$$[\widehat{\mathcal{Z}}(T)] = [\widehat{\mathcal{Z}}(t\bar{\gamma}T\gamma)] \quad (1.1.6)$$

where $[\widehat{\mathcal{Z}}(T)]$ is formed with respect to y and $[\widehat{\mathcal{Z}}(t\bar{\gamma}T\gamma)]$ is formed with respect to $\gamma^{-1}y^t\bar{\gamma}^{-1}$. In fact, we prove refined results: we showed that the vertical part at each prime p is linearly invariant on the level of Grothendieck groups [Che24a, (3.6.11)], and we will see that the horizontal part is linear invariant on its own Section 3.2. The currents $g_{T,y}$ appearing in our main arithmetic Siegel–Weil results satisfy the linear invariance property in (1.1.5); see Section 5.4 (also [Che24b, Section 2.4]). Note that the Garcia–Sankaran Green currents in [GS19, (4.38)] also satisfy the same linear invariance property (but the modified currents in [GS19, Definition 4.7] do not).

Due to non-properness of $\mathcal{M} \rightarrow \text{Spec } \mathcal{O}_F$ in general, one should likely modify $[\widehat{\mathcal{Z}}(T)]$ on a suitable compactification of \mathcal{M} . If $\mathcal{Z}(T) \rightarrow \text{Spec } \mathcal{O}_F$ is proper, however, we consider certain “arithmetic degrees without boundary contributions” (a real number) when $m \leq n$, given by

$$\begin{aligned} \widehat{\text{deg}}([\widehat{\mathcal{Z}}(T)] \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{n-m}) &:= \left(\int_{\mathcal{M}_{\mathbb{C}}} g_{T,y} \wedge c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^\vee)^{n-m} \right) \\ &+ \widehat{\text{deg}}((\widehat{\mathcal{E}}^\vee)^{n-\text{rank}(T)}|_{\mathcal{Z}(T), \mathscr{X}}) \\ &+ \sum_{p \text{ prime}} \text{deg}_{\mathbb{F}_p}(\mathbb{L} \mathcal{Z}(T)_{\mathscr{Y}, p} \cdot (\mathcal{E}^\vee)^{n-m}) \log p \end{aligned} \quad (1.1.7)$$

conditional on convergence of the integral, for a certain metrized tautological bundle $\widehat{\mathcal{E}}$ on \mathcal{M} (Section 2.1); we do check convergence of the integral in the settings of our arithmetic Siegel–Weil results. Here we set $\mathcal{M}_{\mathbb{C}} := \mathcal{M} \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathbb{C}$ for either embedding $F \rightarrow \mathbb{C}$. The middle term is mixed characteristic in nature: for $\text{rank}(T) = n - 1$, it is (essentially) a weighted sum of Faltings heights of abelian varieties ([Che24c, Remark 9.1.4]). For proper $\mathcal{Z}(T) \rightarrow \mathcal{O}_F$, the quantity in (1.1.7) should coincide with the arithmetic degree (without boundary contributions) of a version of $[\widehat{\mathcal{Z}}(T)]$ on any reasonable compactification of \mathcal{M} .

We consider the (normalized) $U(m, m)$ Siegel Eisenstein series

$$E^*(z, s)_n^\circ := \Lambda_m(s)_n^\circ \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P_1(\mathbb{Z}) \backslash SU(m, m)(\mathbb{Z})} \frac{\det(y)^{s-s_0}}{|\det(cz + d)^n| |\det(cz + d)|^{2(s-s_0)}}. \quad (1.1.8)$$

Here $\Lambda_m(s)_n^\circ$ is a certain normalizing factor [Che24c, (6.1.2)], consisting of various L -functions and Γ functions, etc.. We wrote $P_1 := P \cap SU(m, m)$ for $P \subseteq U(m, m)$ denoting the Siegel parabolic ($m \times m$ block upper triangular), the element $z = x + iy$ lies in Hermitian upper-half space (i.e. $x \in \text{Herm}_m(\mathbb{R})$ and $y \in \text{Herm}_m(\mathbb{R})_{>0}$, meaning y is positive definite), and $s_0 = (n - m)/2$. Given $T \in \text{Herm}_m(\mathbb{Q})$, the Eisenstein series $E^*(z, s)_n^\circ$ has T -th Fourier coefficient

$$E_T^*(y, s)_n^\circ := 2^{m(m-1)/2} |\Delta|^{-m(m-1)/4} \int_{\text{Herm}_m(\mathbb{Z}) \backslash \text{Herm}_m(\mathbb{R})} E^*(z, s)_n^\circ e^{-2\pi i \text{tr}(Tz)} dx \quad (1.1.9)$$

for $z = x + iy$ in Hermitian upper-half space, where this integral is taken with respect to the Euclidean measure on $\text{Herm}_m(\mathbb{R})$.

In our four-part sequence of papers, our main result is an *arithmetic Siegel–Weil formula* in co-rank 1, i.e. we show that the formula

$$\frac{h_F}{w_F} \frac{d}{ds} \Big|_{s=s_0} \frac{2\Lambda_n(s-s_0)_n^\circ}{\kappa\Lambda_m(s)_n^\circ} E_T^*(y, s)_n^\circ \stackrel{?}{=} \widehat{\text{vol}}_{\mathcal{E}^\vee}([\widehat{\mathcal{Z}}(T)]). \quad (1.1.10)$$

holds when $m = n$ and T is singular of co-rank 1. Here, we write h_F (resp. w_F) for the class number of (resp. number of roots of unity in) \mathcal{O}_F , and $\kappa = 1$ (resp. $\kappa = 2$) if $m < n$ (resp. $m = n$). We also prove (the closely related version of) (1.1.10) when T is nonsingular of rank $n - 1$; in that case, the special value at $s = s_0$ simultaneously has geometric meaning (“geometric Siegel–Weil”). We refer to the introduction of our companion paper [Che24c] for further discussion on this.

As formulated in (1.1.10), the arithmetic Siegel–Weil formula was possibly considered essentially previously known (at least up to a volume constant) for nonsingular $T \in \text{Herm}_n(\mathbb{Q})$ by the local theorems [Liu11; LZ22; LL22] (see discussion following [Che24a, (1.3.8)]). Our four-part series resolves the case where T is singular of co-rank 1. Non-Archimedean aspects of the arithmetic Siegel–Weil formula (along with combined non-Archimedean and Archimedean results) were open for corank ≥ 1 , prior to our work. As formulated in (1.1.10), the arithmetic Siegel–Weil formula is essentially open in corank ≥ 2 . We refer to the introductions of our companion papers [Che24a; Che24c] for a more comprehensive review.

1.2. Local-to-global. Our proof of arithmetic Siegel–Weil in co-rank 1 is local in nature. In our previous companion papers [Che24a; Che24b], we formulated and proved the key “local arithmetic Siegel–Weil” formulas at all places, which are local analogues of (the co-rank 1 case of) (1.1.10). At the Archimedean place (resp. non-Archimedean places), the geometric side of our local theorem takes place on an appropriate Hermitian symmetric domain (resp. Rapoport–Zink space).

The main objective of this paper is to explain how to patch these local geometric quantities to the (global) arithmetic intersection number considered in (1.1.7).

For motivation, we first illustrate (a special case of) the local-to-global reduction process on the analytic side of the arithmetic Siegel–Weil formula, involving Eisenstein series and local Whittaker functions. In the special case of (1.1.10) when $T = \text{diag}(0, T^b)$ for T^b nonsingular of rank $n - 1$ and $y = \text{diag}(1, y^b)$, we have a local decomposition

$$\frac{1}{2} \frac{d}{ds} \Big|_{s=0} E_T^*(y, s)_n^\circ = \frac{d}{ds} \Big|_{s=0} \left(\frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ} E_{T^b}^*(y^b, s+1/2)_n^\circ \right) \quad (1.2.1)$$

$$= \left(\frac{d}{ds} \Big|_{s=0} \frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ} \right) E_{T^b}^*(y^b, 1/2)_n^\circ \quad (1.2.2)$$

$$+ \left(\frac{d}{ds} \Big|_{s=1/2} W_{T^b, \infty}^*(y^b, 1/2)_n^\circ \right) \prod_p W_{T^b, p}^*(1/2)_n^\circ \quad (1.2.3)$$

$$+ \sum_p \left(\frac{d}{ds} \Big|_{s=1/2} W_{T, p}^*(s)_n^\circ \right) W_{T^b, \infty}^*(y, 1/2)_n^\circ \prod_{\ell \neq p} W_{T^b, \ell}^*(1/2)_n^\circ \quad (1.2.4)$$

by the Leibniz rule and the Euler product

$$E_{T^b}^*(y^b, s)_n^\circ = W_{T^b, \infty}^*(y^b, s)_n^\circ \prod_p W_{T^b, p}^*(s)_n^\circ \quad (1.2.5)$$

of local (normalized) Whittaker functions over all places ([Che24c, Section 6.1]). If T is not in block diagonal form but ${}^t\bar{\gamma}T\gamma$ is block diagonal for some $\gamma \in \mathrm{GL}_n(\mathcal{O}_F)$, a certain “linear invariance” property for Eisenstein series ([Che24c, Section 2.3]) gives a similar local decomposition. If no such γ exists, we instead take $\gamma \in \mathrm{GL}_n(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ (“local diagonalizability”); this introduces a discrepancy of $\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell$, but vary p removes this discrepancy by \mathbb{R} -linear independence of $\log p$ over all primes p (see [Che24c, Section 9.1]). This “ p -local diagonalization” argument is a new feature on the analytic side for our local-to-global reduction of arithmetic Siegel–Weil.

To reduce (1.1.10) (for T singular of co-rank 1) to our local main theorems from [Che24a, Section 9] and [Che24b, Section 4], we exhibit a local decomposition on the geometric side analogous to (1.2.2), (1.2.3), and (1.2.4).

On the geometric side of the arithmetic Siegel–Weil formula, we will use uniformization (Archimedean and non-Archimedean) to pass from arithmetic intersection numbers for global special cycles to geometric quantities defined for local special cycles. (This local-to-global reduction is treated more carefully in Section 5.4.) As mentioned in [Che24a, Section 1.7], a key difficulty is that our arithmetic intersection numbers involve a mixed characteristic contribution (e.g. Faltings heights of abelian varieties, coming from components of $\mathcal{Z}(T)$ flat over $\mathrm{Spec} \mathcal{O}_F$) which do not admit an obvious *canonical* local decomposition. This is the main difference between our local-to-global arguments and previous work such as [KR14; LZ22], which only involve intersection numbers for purely vertical special cycles (empty in the generic fiber).

As we sketched in [Che24a, Section 1.7], we consider the local decomposition of the *difference* of Faltings heights between isogenous abelian varieties. The somewhat delicate passage from Faltings heights (and “tautological heights”) to local quantities calculate-able on Rapoport–Zink spaces was treated in [Che24a, Sections 10 and 11], but the precise relation with global special cycles was not completed there. In Section 4.9 of this paper, we explain how to use the results from [Che24a, Sections 10 and 11] to calculate arithmetic intersection numbers for global special cycles via uniformization.

For suitable open compact subgroups $K'_f = K_{0,f} \times K_f \subseteq (\mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m)(\mathbb{A}_f) \times U(V)(\mathbb{A}_f) = G'(\mathbb{A}_f)$ (with \mathbb{A}_f the adèle ring of \mathbb{Q}), there is a version $\mathcal{M}_{K'_f}$ of \mathcal{M} with level K'_f structure (essentially as considered by Kottwitz [Kot92]); we postpone the definition to Section 2.2. Similarly write $\mathcal{Z}(T)_{K'_f}$ for the pullback of $\mathcal{Z}(T) \rightarrow \mathcal{M}$ to $\mathcal{M}_{K'_f}$.

We first consider Archimedean intersection numbers, involving Green currents (in the terminology above, this is still “purely vertical” from the perspective of this paper). Fix an embedding $F \rightarrow \mathbb{C}$, and set $\mathcal{M}_{K'_f, \mathbb{C}} := \mathcal{M}_{K'_f} \times_{\mathrm{Spec} \mathcal{O}_F} \mathbb{C}$. For sufficiently small K'_f and any $T \in \mathrm{Herm}_m(\mathbb{Q})$ (for any m), there are *complex uniformization isomorphisms* (Section 5.4)

$$\mathcal{M}_{K'_f, \mathbb{C}}^{\mathrm{an}} \cong G'(\mathbb{Q}) \backslash (\mathcal{D} \times G'(\mathbb{A}_f) / K'_f) \tag{1.2.6}$$

$$\mathcal{Z}(T)_{K'_f} \cong G'(\mathbb{Q}) \backslash \left(\prod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_{\infty}) \times \mathcal{D}'(\underline{x}_f) \right). \tag{1.2.7}$$

Here \mathcal{D} is a Hermitian symmetric domain (parameterizing maximal negative definite subspaces of the complex Hermitian space $V \otimes_{\mathbb{Q}} \mathbb{R}$), the notation $\mathcal{D}(\underline{x}) \subseteq \mathcal{D}$ denotes an Archimedean *local special*

cycle (the complex submanifold corresponding to subspaces perpendicular to every element of the m tuple \underline{x}), and the set $\mathcal{D}'(\underline{x}_f) = ((\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m)(\mathbb{A}_f)/K_{0,f} \times \mathcal{D}(\underline{x}_f)) \subseteq G'(\mathbb{A}_f)/K'_f$ is what we call an “away-from- ∞ ” local special cycle (Section 5.1).

Consider any $y \in \text{Herm}_m(\mathbb{R})_{>0}$ and $\underline{x} \in V^m$ with $(\underline{x}, \underline{x}) = T$. For the moment, assume $m \geq n-1$ if T is positive definite. If T is singular, we also require $m = n$ and $\text{rank}(T) = n-1$. In these cases, we constructed currents $[\xi(\underline{x}, y)]$ on the Hermitian symmetric domain \mathcal{D} [Che24b, Section 2.4]. These will descend to currents $g_{T,y}$ on the (analytification of the) complex Shimura variety $\mathcal{M}_{K'_f, \mathbb{C}}^{\text{an}}$. We have the local and global Archimedean intersection numbers

$$\text{Int}_\infty(T, y) := \int_{\mathcal{D}} [\xi(\underline{x}, y)] \wedge c_1(\widehat{\mathcal{E}}^\vee)^{n-m} \quad \text{Int}_{\infty, \text{global}}(T, y) := \int_{\mathcal{M}_{\mathbb{C}}^{\text{an}}} g_{T,y} \wedge c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^\vee)^{n-m} \quad (1.2.8)$$

respectively. Via uniformization, these are related by the formula (“local to global”, Section 5.4)

$$\text{Int}_{\infty, \text{global}}(T, y) = \text{Int}_\infty(T, y) \frac{[\widehat{\mathcal{O}}_F : K_{0,f}]}{|\mathcal{O}_F^\times/h_F|} \cdot \text{deg} \left[U(V)(\mathbb{Q}) \backslash \prod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_f) \right] \quad (1.2.9)$$

where $\widehat{\mathcal{O}}_F := \mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, and where $\text{deg}[-]$ denotes groupoid cardinality. For simplicity, consider the case when $T^\flat := T$ is moreover nonsingular (the body of this paper treats the more general case). Our main Archimedean local theorem [Che24b, Theorem 4.1.1] gives

$$\text{Int}_\infty(T^\flat, y) = - \frac{d}{ds} \Big|_{s=1/2} W_{T^\flat}^*(y, s)_n^\circ. \quad (1.2.10)$$

By “local Siegel–Weil” (see [Che24c, Section 7.4]), we have

$$\text{deg} \left[U(V)(\mathbb{Q}) \backslash \prod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T^\flat}} \mathcal{D}(\underline{x}_f) \right] = \frac{2h_F}{w_F} \prod_p W_{T^\flat, p}^*(1/2)_n^\circ \quad (1.2.11)$$

and the comparison with (1.2.3) now emerges.

We next consider “non-Archimedean” intersection numbers, which will involve positive characteristic contributions (e.g. from components $\mathcal{Z}(T)$ in positive characteristic) and parts of mixed characteristic contributions (from components of $\mathcal{Z}(T)$ flat over \mathcal{O}_F , giving Faltings heights of abelian varieties). We allow possibly stacky level K'_f , but assume that K'_f is standard at p (Section 2.2). For the moment, we assume p is a prime which is nonsplit in \mathcal{O}_F . Set $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and let \check{F}_p be the completion of the maximal unramified extension of F_p . We then have (stacky) *Rapoport–Zink uniformization isomorphisms* (Section 4.6)

$$\check{\mathcal{M}}_{K'_f} \cong [I'(\mathbb{Q}) \backslash (\mathcal{N}' \times G'(\mathbb{A}_f^p)/K_f'^p)] \quad (1.2.12)$$

$$\check{\mathcal{Z}}(T)_{K'_f} \cong \left[I'(\mathbb{Q}) \backslash \left(\prod_{\substack{\underline{x} \in \mathbf{V}^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{Z}'(\underline{x}_p) \times \mathcal{Z}'(\underline{x}^p) \right) \right]. \quad (1.2.13)$$

The notations $\check{\mathcal{Z}}(T)_{K'_f}$ and $\check{\mathcal{Z}}(T)_{K'_f}$ denote the respective formal completions along supersingular loci (after base-change to $\text{Spec } \mathcal{O}_{\check{F}_p}$). Here $I' \cong \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \times U(\mathbf{V})$, where \mathbf{V} is a certain positive definite F/\mathbb{Q} Hermitian space which is isomorphic to V at all places other than ∞ and p . Here

$\mathcal{N}' = \mathcal{N}(1, 0)' \times \mathcal{N}$ is a Rapoport–Zink space, which is a locally Noetherian formal scheme and a certain moduli space of p -divisible groups. The notation $\mathcal{Z}'(\underline{\mathbf{x}}_p) = \mathcal{N}(1, 0)' \times \mathcal{Z}(\underline{\mathbf{x}}_p) \subseteq \mathcal{N}'$ denotes a certain local Kudla–Rapoport special cycle (closed formal subscheme) at p , and the set $\mathcal{Z}'(\underline{\mathbf{x}}^p) = (\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m)(\mathbb{A}_f^p)/K_{0,f}^p \times \mathcal{Z}(\underline{\mathbf{x}}^p) \subseteq G'(\mathbb{A}_f^p)/K_f'^p$ is what we call an “away-from- p ” local special cycle.

For illustration purposes, consider a nonsingular $T^\flat \in \text{Herm}_{n-1}(\mathbb{Q})$ (again, the body of this paper treats a more general setup). Suppose $\underline{\mathbf{x}}^\flat \in \mathbf{V}^{n-1}$ is any tuple with Gram matrix T^\flat . We then have local and global non-Archimedean vertical intersection numbers at p

$$\text{Int}_{\mathcal{V},p}(T^\flat) := 2[\check{F}_p : \check{\mathbb{Q}}_p]^{-1} \deg_{\mathbb{F}_p}(\mathbb{L}\mathcal{Z}(\underline{\mathbf{x}}^\flat)_{\mathcal{V}} \cdot \mathcal{E}^\vee) \log p \quad \text{Int}_{\mathcal{V},p,\text{global}}(T^\flat) := \deg_{\mathbb{F}_p}(\mathbb{L}\mathcal{Z}(T)_{\mathcal{V},p} \cdot \mathcal{E}^\vee) \log p \quad (1.2.14)$$

corresponding to the “vertical parts” $\mathbb{L}\mathcal{Z}(\underline{\mathbf{x}}_p)$ and $\mathbb{L}\mathcal{Z}(T^\flat)_{\mathcal{V},p}$ of local and global special cycles respectively, intersected against an appropriate dual tautological bundle \mathcal{E}^\vee .

At least if $p \neq 2$, we also have local ($\text{Int}_{\mathcal{H},p}(T)$) and global ($\text{Int}_{\mathcal{H},\ell,\text{global}}(T)$) non-Archimedean horizontal intersection numbers at p (elements of $\mathbb{Q} \cdot \log p$) given by

$$\text{Int}_{\mathcal{H},p}(T^\flat) \sim \text{local change of Faltings heights along “minimal isogenies”} \quad (1.2.15)$$

$$\widehat{\deg}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T^\flat)_{\mathcal{H}}}) - (\deg_{\mathbb{Z}} \mathcal{Z}(T^\flat)_{\mathcal{H}}) \cdot h_{\widehat{\mathcal{E}}^\vee}^{\text{CM}} = \sum_{\ell} \text{Int}_{\mathcal{H},\ell,\text{global}}(T^\flat) \quad (1.2.16)$$

where the sum ranges over all primes ℓ , with all but finitely many terms equal to 0. Here $h_{\widehat{\mathcal{E}}^\vee}^{\text{CM}}$ is a certain height constant, see (2.1.13). The precise definition of $\text{Int}_{\mathcal{H},p}(T^\flat)$ that we use is more complicated; we defer to Section 4.9 (and the motivation in [Che24a, Section 1.6]). The symbol $\widehat{\deg}$ denotes an arithmetic degree of a certain metrized line bundle [Che24a, Section 3.5], and $\deg_{\mathbb{Z}} \mathcal{Z}(T^\flat)_{\mathcal{H}}$ is the “degree” of $\mathcal{Z}(T^\flat)_{\mathcal{H}} \rightarrow \text{Spec } \mathbb{Z}$ (more precisely, see Section 4.9).

We consider *total “intersection numbers”*

$$\text{Int}_p(T^\flat) := \text{Int}_{\mathcal{H},p}(T^\flat) + \text{Int}_{\mathcal{V},p}(T^\flat) \quad \text{Int}_{p,\text{global}}(T^\flat) := \text{Int}_{\mathcal{H},p,\text{global}}(T^\flat) + \text{Int}_{\mathcal{V},p,\text{global}}(T^\flat) \quad (1.2.17)$$

(local and global) at p , and have the following local-to-global relation, whose precise formulation and proof is the main objective of this paper (see Sections 4.8 and 4.9).

Theorem. *We have*

$$\text{Int}_{p,\text{global}}(T) = \text{Int}_p(T^\flat) \frac{[\widehat{\mathcal{O}}_F : K_{0,f}]}{|\mathcal{O}_F^\times|/h_F} \cdot \deg \left[U(\mathbf{V})(\mathbb{Q}) \setminus \prod_{\substack{\underline{\mathbf{x}} \in V^m \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T^\flat}} \mathcal{Z}(\underline{\mathbf{x}}^p) \right]. \quad (1.2.18)$$

The main ingredients are Rapoport–Zink uniformization (Sections 4.8 and 4.9) and the “local decomposition of heights” input from our companion paper [Che24a, Sections 10 and 11]. Our local main non-Archimedean theorem from [Che24a, Section 9] gives

$$\text{Int}_p(T^\flat) = - \frac{d}{ds} \Big|_{s=1/2} W_{T^\flat}^*(s)_n^\circ. \quad (1.2.19)$$

By “local Siegel–Weil” (see [Che24c, Section 7.4]), we have

$$\deg \left[U(\mathbf{V})(\mathbb{Q}) \backslash \prod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T^b}} \mathcal{Z}(\underline{\mathbf{x}}^p) \right] = \frac{2h_F}{w_F} W_{T^b, \infty}^*(y^b, 1/2)_n^\circ \prod_{\ell \neq p} W_{T^b, \ell}^*(1/2)_n^\circ \quad (1.2.20)$$

for any $y^b \in \text{Herm}_{n-1}(\mathbb{R})_{>0}$, and the comparison with (1.2.4) now emerges.

The height constant $h_{\mathcal{E}_V}^{\text{CM}}$ is essentially the derivative appearing in (1.2.2) (see [Che24c, (9.1.2)]), and the degree $\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}}$ is essentially the special value $E_{T^b}^*(y^b, s)_n^\circ$ appearing in loc. cit. (by geometric Siegel–Weil as in [Che24c, Remark 8.1.2] or [Che24c, (1.2.10)]). The comparison with (1.2.2) now emerges.

A similar local-to-global reduction is possible (and carried out in Section 4) if p is split (allowing $p = 2$), though we now need Rapoport–Zink uniformization involving ordinary abelian varieties. In the split case, we only uniformize special cycles $\mathcal{Z}(T)$ with T of rank $\geq n - 1$.

1.3. Outline. We briefly summarize the remaining content in this paper, and discuss the relation with our companion papers [Che24a; Che24b; Che24c]. Further explanations may be found at the beginning of some sections.

The remaining sections into Parts 1 and 2 and appendices.

In Part 1 “Global special cycles”, we set up the global moduli stacks (RSZ) and special cycles (KR) appearing in our main global theorems. The exotic smooth case was discussed in our companion paper [Che24a, Section 3.1]. In the present paper, we also allow a more general setup at the price of possibly discarding finitely many primes (particularly the ramified primes in the case of odd arithmetic dimension). Our main theorems will also apply in these situations [Che24c, Remark 9.1.5]. In Section 3, we define associated arithmetic special cycle classes and discuss arithmetic degrees.

In Part 2 “Uniformization” we discuss complex and Rapoport–Zink uniformization of special cycles in our setup, and finish the reduction process from global heights/intersection numbers to the geometric quantities appearing in the “local arithmetic Siegel–Weil” theorems of our companion papers [Che24a; Che24b]. Strictly speaking, the Rapoport–Zink uniformization we need at split places does not seem covered by the literature (not supersingular locus). We treat inert/ramified/split in parallel. For the most part, we disallow $p = 2$ only in the ramified case. Using the analogous local construction from [Che24b, Section 2.4], we explain a modified Green current for singular T (of rank $n - 1$ and size $n \times n$) in Section 5.4.

The analytic side of the local-to-global reduction process will appear in our final companion paper [Che24c], where we complete the proof of our (global) co-rank 1 arithmetic Siegel–Weil formula using essentially all preceding results.

Appendix A contains some notation on abelian schemes, and records a proof for quasi-compactness of special cycles (which does not seem explicitly available in the literature). Appendix B concerns p -divisible groups, where we fix some notation and record some (presumably standard) facts.

Our algebro-geometric conventions follow the Stacks project [SProject] unless stated otherwise.

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Part 1. Global special cycles

2. MODULI STACKS OF ABELIAN VARIETIES

Fix an imaginary quadratic field extension F/\mathbb{Q} with ring of integers \mathcal{O}_F and write $a \mapsto a^\sigma$ for the nontrivial automorphism σ of F . We write $\Delta \in \mathbb{Z}_{<0}$ and $\sqrt{\Delta} \in \mathcal{O}_F$ (pick a square root) for (generators of the) discriminant and different, respectively.

In [Che24a, Section 3] of our companion paper, we stated our main global theorems in terms of certain “exotic smooth” moduli stacks $\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_F$ of RSZ (Rapoport–Smithling–Zhang) [RSZ21]. Since our proofs are eventually local in nature, our global theorems also apply for other RSZ integral models after inverting finitely many primes. We explain these more general integral models (associated to non-degenerate Hermitian \mathcal{O}_F -lattices L) in Section 2.1. Sections 2.2 and 2.3 and Appendix A contain some technical facts (level structure, generic smoothness of special cycles, and quasi-compactness of special cycles respectively) which were skipped in [Che24a, Section 3].

2.1. Integral models. For the reader’s convenience, we recall the notion of a Hermitian abelian scheme in the terminology of [Che24a, Definition 3.1.1].

Definition 2.1.1. Let S be a scheme over $\mathrm{Spec} \mathcal{O}_F$. By a *Hermitian abelian scheme* over S , we mean a tuple (A, ι, λ) where

- A is an abelian scheme over S of constant relative dimension n
- $\iota: \mathcal{O}_F \rightarrow \mathrm{End}(A)$ is a ring homomorphism
- $\lambda: A \rightarrow A^\vee$ is a quasi-polarization satisfying:
 - (Action compatibility) The Rosati involution \dagger on $\mathrm{End}^0(A)$ satisfies $\iota(a)^\dagger = \iota(a^\sigma)$ for all $a \in \mathcal{O}_F$.

For any fixed $n \geq 1$ (even or odd), the *moduli stack of Hermitian abelian schemes* \mathcal{M} is the stack⁵ in groupoids over $\mathrm{Spec} \mathcal{O}_F$ with

$$\mathcal{M}(S) := \{\text{groupoid of relative } n\text{-dimensional Hermitian abelian schemes over } S\} \quad (2.1.1)$$

for \mathcal{O}_F -schemes S .

For an integer r with $0 \leq r \leq n$, we also recall

(Kottwitz $(n - r, r)$ signature condition) For all $a \in \mathcal{O}_F$, the characteristic polynomial of $\iota(a)$ acting on $\mathrm{Lie} A$ is $(x - a)^{n-r}(x - a^\sigma)^r \in \mathcal{O}_S[x]$

for pairs (A, ι) , where $A \rightarrow S$ is a relative n -dimensional abelian scheme with \mathcal{O}_F -action ι , and S is an \mathcal{O}_F -scheme. Here we view \mathcal{O}_S as an \mathcal{O}_F -algebra via the structure map $S \rightarrow \mathrm{Spec} \mathcal{O}_F$. This defines a substack⁶

$$\mathcal{M}(n - r, r) \subseteq \mathcal{M} \quad (2.1.2)$$

⁵By a *stack in groupoids* over some base scheme S , we always mean a (not necessarily algebraic) stack in groupoids as in [SProject, Definition 02ZI] over the fppf site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

⁶A *substack* will always mean a strictly full substack.

consisting of Hermitian abelian schemes of signature $(n - r, r)$. The inclusion $\mathcal{M}(n - r, r) \rightarrow \mathcal{M}$ is representable by schemes (in the sense of [SProject, Section 04ST]) and is a closed immersion. There is an isomorphism⁷ $\mathcal{M}(n - r, r) \rightarrow \mathcal{M}(r, n - r)$ given by $(A, \iota, \lambda) \mapsto (A, \iota \circ \sigma, \lambda)$.

For any integer $d \geq 1$, there is a substack $\mathcal{M}^{(d)} \subseteq \mathcal{M}$ consisting of Hermitian abelian schemes (A, ι, λ) where λ is polarization of constant degree $\deg \lambda := \deg \ker \lambda = d$. If $\mathcal{A}_{n,d}$ (over $\text{Spec } \mathcal{O}_F$) denotes the moduli stack of (relative) n -dimensional abelian schemes equipped with a polarization of degree d , the forgetful map $\mathcal{M}^{(d)} \rightarrow \mathcal{A}_{n,d}$ is representable by schemes, finite, and unramified (e.g. via Lemma A.2.3).

Hence $\mathcal{M}^{(d)}$ is a Noetherian Deligne–Mumford stack which is separated and finite type over $\text{Spec } \mathcal{O}_F$ (because this is true of $\mathcal{A}_{n,d}$ as proved with level structure in the classical [MFK94, §7.2 Theorem 7.9]; one can deduce the stacky version upon inverting primes dividing the level, taking stack quotients, and patching over $\text{Spec } \mathcal{O}_F$).

We set

$$\mathcal{M}(n - r, r)^{(d)} := \mathcal{M}(n - r, r) \cap \mathcal{M}^{(d)} \quad (2.1.3)$$

where the right-hand side is an intersection of substacks of \mathcal{M} . There is an open and closed disjoint union decomposition⁸

$$\mathcal{M}^{(d)}[1/\Delta] = \coprod_{(n-r,r)} \mathcal{M}(n - r, r)^{(d)}[1/\Delta] \quad (2.1.4)$$

over $\text{Spec } \mathcal{O}_F[1/\Delta]$, where the disjoint union runs over all possible signatures $(n - r, r)$.

The structure morphism $\mathcal{M}(n - r, r)^{(d)}[1/(d\Delta)] \rightarrow \text{Spec } \mathcal{O}_F[1/\Delta]$ is smooth of relative dimension $(n - r)r$ (e.g. recall that being smooth of some relative dimension may be checked fppf locally on the target for morphisms of algebraic stacks; then apply Remark 2.3.6 below). We set $\mathcal{M}_0 := \mathcal{M}(1, 0)^{(1)}$. The structure morphism $\mathcal{M}_0 \rightarrow \text{Spec } \mathcal{O}_F$ is proper, quasi-finite, and étale by [How12, Proposition 3.1.2] or [How15, Proposition 2.1.2].

Given any non-degenerate Hermitian \mathcal{O}_F -lattice L of rank n and signature $(n - r, r)$, we define an associated substack

$$\mathcal{M} \subseteq \mathcal{M}_0 \times_{\text{Spec } \mathcal{O}_F} \mathcal{M}(n - r, r) \quad (2.1.5)$$

as follows (cf. [KR14, Proposition 2.12], there in a principally polarized situation). Write $(-, -)$ for the pairing on L , let b_L be the smallest positive integer such that $b_L \cdot (-, -)$ is \mathcal{O}_F -valued.

If L is self-dual⁹ of signature $(n - 1, 1)$ and $2 \nmid \Delta$, we take $\mathcal{M} \rightarrow \mathcal{O}_F$ to be the exotic smooth moduli stack described in [Che24a, Section 3.1]. We refer to this as the *even exotic smooth case*.

Otherwise, let L' be the Hermitian \mathcal{O}_F lattice which is the \mathcal{O}_F -module L but with Hermitian pairing $b_L \cdot (-, -)$. Form the dual lattice L'^\vee of L' with respect to the Hermitian pairing, and set $d'_L := |L'^\vee/L'|$. Let $d_L \in \mathbb{Z}_{>0}$ be the product of ramified primes and the primes p for which $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is not self-dual.

⁷As in the Stacks project (e.g. [SProject, Section 04XA]), we often abuse terminology and say “isomorphism” of stacks instead of “equivalence”.

⁸Here, the notation $\mathcal{M}^{(d)}[1/\Delta]$ means $\mathcal{M}^{(d)} \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathcal{O}_F[1/\Delta]$. We often use such shorthand, along with subscripts for base change, e.g. $\mathcal{M}_S^{(d)} := \mathcal{M}^{(d)} \times_{\text{Spec } \mathcal{O}_F} S$ over an understood base.

⁹As in [Che24a, Section 2.2], our convention is that *self-duality* (without additional specification) is understood with respect to the trace pairing; here that means $\text{tr}_{\mathcal{O}_F/\mathbb{Z}}(-, -)$.

Definition 2.1.2. In the preceding situation, we let $\mathcal{M} \subseteq \mathcal{M}_0 \times_{\text{Spec } \mathcal{O}_F} \mathcal{M}(n-r, r)[1/d_L]$ be the substack

$$\mathcal{M}(S) := \left\{ (A_0, \iota_0, \lambda_0, A, \iota, \lambda) : \begin{array}{l} \text{Hom}_{\mathcal{O}_F \otimes \hat{\mathbb{Z}}^p}(T^p(A_{0, \bar{s}}), T^p(A_{\bar{s}})) \cong L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \\ \text{for every geometric point } \bar{s} \text{ of } S, \text{ with } p = \text{char}(\bar{s}), \\ \text{and } b_L \cdot \lambda \text{ is a polarization of degree } d'_L \end{array} \right\} \quad (2.1.6)$$

for schemes S over $\text{Spec } \mathcal{O}_F[1/d_L]$, where

$$(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0(S) \quad (A, \iota, \lambda) \in \mathcal{M}(n-r, r)(S). \quad (2.1.7)$$

If L is self-dual of signature $(n-1, 1)$ and $2 \nmid \Delta$, then applying Definition 2.1.2 over $\text{Spec } \mathcal{O}_F[1/\Delta]$ would give $\mathcal{M}[1/\Delta]$ (for the exotic smooth moduli stack \mathcal{M}).

In Definition 2.1.2, the notation $\text{Hom}_{\mathcal{O}_F \otimes \hat{\mathbb{Z}}^p}(T^p(A_{0, \bar{s}}), T^p(A_{\bar{s}})) \cong L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$ asserts the existence of isomorphisms of Hermitian lattices, and the elements of $\text{Hom}_{\mathcal{O}_F \otimes \hat{\mathbb{Z}}^p}(T^p(A_{0, \bar{s}}), T^p(A_{\bar{s}}))$ are not required to respect Hermitian pairings. As usual, $T^p(-)$ is the away-from- p adèlic Tate module (if $p = 0$, this is over the full finite adèles) and $\hat{\mathbb{Z}}^p = \prod_{\ell \neq p} \mathbb{Z}_\ell$.

In all cases, note that \mathcal{M} depends only on the *adèlic isomorphism class*¹⁰ of L . The stack \mathcal{M} and its special cycles are the global moduli stacks of main interest in this work. We generally suppress L from notation, but sometimes write \mathcal{M}^L instead of \mathcal{M} to emphasize L dependence.

We claim that \mathcal{M} is a Noetherian Deligne–Mumford stack which is separated and smooth of relative dimension $(n-r)r$ over $\text{Spec } \mathcal{O}_F[1/d_L]$. The even exotic smooth case was discussed in [Che24a, Section 3.1]. The general case is similar: there is an open and closed disjoint union decomposition

$$\mathcal{M}_0 \times_{\text{Spec } \mathcal{O}_F} \mathcal{M}(n-r, r)^{(d)}[1/(d\Delta)] = \coprod_{L''} \mathcal{M}^{L''} \quad (2.1.8)$$

running over representatives L'' , one for each adèlic isomorphism class of non-degenerate Hermitian \mathcal{O}_F -lattices of signature $(n-r, r)$ satisfying $L'' \subseteq L''^\vee$ and $|L''^\vee/L''| = d$. We have used flatness of $\mathcal{M}(n-r, r)^{(d)}[1/(d\Delta)] \rightarrow \text{Spec } \mathcal{O}_F[1/(d\Delta)]$ in the open and closed decomposition (to lift to characteristic 0; cf. [KR14, Proposition 2.12] [RSZ18, Remark 4.2]). With notation as above, the map

$$\mathcal{M}^L \longrightarrow \mathcal{M}^{L'} \quad (2.1.9)$$

$$(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \longmapsto (A_0, \iota_0, \lambda_0, A, \iota, b_L \lambda)$$

is an isomorphism for any L , after restricting to $\text{Spec } \mathcal{O}_F[1/(d_L\Delta)]$.

Remark 2.1.3. If L has rank $n = 1$, we can construct \mathcal{M} without discarding any primes. Then $\mathcal{M} \rightarrow \text{Spec } \mathcal{O}_F$ is smooth, by smoothness of $\mathcal{M}(1, 0)^{(d)} \rightarrow \text{Spec } \mathcal{O}_F$ for any $d \in \mathbb{Z}_{>0}$.

Given any L with associated moduli stack \mathcal{M} , and given any $T \in \text{Herm}_m(\mathbb{Q})$, there is a *Kudla–Rapoport special cycle* $\mathcal{Z}(T) \rightarrow \mathcal{M}$ as in [KR14, Definition 2.8] (there in a principally polarized situation). For the reader’s convenience, we recall the definition (which is also [Che24a, Definition 3.2.1]).

¹⁰We say that non-degenerate Hermitian \mathcal{O}_F lattices L and L' are *adèlically isomorphic* (or are in the same *adèlic isomorphism class*) if there exist isomorphisms of $\mathcal{O}_F \otimes \mathbb{Z}_p$ -Hermitian lattices $L \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for every prime p , as well as isomorphisms of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{R}$ -Hermitian spaces $L \otimes_{\mathbb{Z}} \mathbb{R} \cong L' \otimes_{\mathbb{Z}} \mathbb{R}$ (classical terminology: *genus*).

Definition 2.1.4 (Kudla–Rapoport special cycles). Given an integer $m \geq 0$, let $T \in \text{Herm}_m(\mathbb{Q})$ be a $m \times m$ Hermitian matrix (with coefficients in F). The *Kudla–Rapoport (KR) special cycle* $\mathcal{Z}(T)$ is the stack in groupoids over $\text{Spec } \mathcal{O}_F$ defined as follows: for schemes S over $\text{Spec } \mathcal{O}_F$, we take $\mathcal{Z}(T)(S)$ to be the groupoid

$$\mathcal{Z}(T)(S) := \left\{ \begin{array}{l} (A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(S) \\ (A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x}) : \underline{x} = [x_1, \dots, x_m] \in \text{Hom}_{\mathcal{O}_F}(A_0, A)^m \\ (\underline{x}, \underline{x}) = T \end{array} \right\} \quad (2.1.10)$$

where $(\underline{x}, \underline{x})$ is the matrix with i, j -th entry given by $x_i^\dagger x_j \in \text{End}_{\mathcal{O}_F}^0(A_0)$ (a quasi-endomorphism), with \dagger denoting the Rosati involution. We sometimes refer to elements $x \in \text{Hom}_{\mathcal{O}_F}(A_0, A)$ as *special homomorphisms*.

By Lemma A.2.3, the forgetful map $\mathcal{Z}(T) \rightarrow \mathcal{M}$ is representable by schemes, finite, and unramified (and of finite presentation). Hence $\mathcal{Z}(T)$ is a separated Deligne–Mumford stack of finite type over $\text{Spec } \mathcal{O}_F$.

In the situation of Definition 2.1.4, recall $\text{End}_{\mathcal{O}_F}(A_0) = \mathcal{O}_F$ (if the right-hand side is abuse of notation for global sections of the constant sheaf \mathcal{O}_F on S). If the Hermitian pairing on L is \mathcal{O}_F -valued, we thus have $\mathcal{Z}(T) = \emptyset$ unless T has coefficients in \mathcal{O}_F . If L is self-dual and $2 \nmid \Delta$, we have $\mathcal{Z}(T) = \emptyset$ unless $\sqrt{\Delta} \cdot T$ has coefficients in \mathcal{O}_F . Positivity of the Rosati involution also implies that the special cycle $\mathcal{Z}(T)$ is empty unless T is positive semi-definite of rank $\leq n$.

In the even exotic smooth case, recall that we considered a certain metrized dual tautological bundle $\widehat{\mathcal{E}}^\vee$ on \mathcal{M} [Che24a, Sections 3.4 and 3.5]. A similar setup applies on \mathcal{M}^L for any L ; we now briefly recall the definitions for the reader’s convenience.

We write Ω_0^\vee for the line bundle on \mathcal{M}_0 given by the functorial association $(A_0, \iota_0, \lambda_0) \mapsto \text{Lie } A_0$ for objects $(A_0, \iota_0, \lambda_0) \in \mathcal{M}_0(S)$. Given any object $(A, \iota, \lambda) \in \mathcal{M}(n-r, r)[1/\Delta](S)$, there is a unique direct sum decomposition

$$\text{Lie } A = (\text{Lie } A)^+ \oplus (\text{Lie } A)^- \quad (2.1.11)$$

where the ι action on $(\text{Lie } A)^+$ (resp. $(\text{Lie } A)^-$) is \mathcal{O}_F -linear (resp. σ -linear).

Definition 2.1.5. By the *tautological bundle* \mathcal{E} on $\mathcal{M}(n-r, r)$, we mean the rank r locally free sheaf \mathcal{E} (for the fppf topology) whose dual is $\mathcal{E}^\vee := (\text{Lie } A)^-$ for $(A, \iota, \lambda) \in \mathcal{M}(n-1, 1)[1/\Delta]$.

By the *tautological bundle* \mathcal{E} on \mathcal{M} , we mean the locally free sheaf whose dual is $\Omega_0^\vee \otimes \mathcal{E}^\vee$ (pullbacks suppressed from notation).

The bundles \mathcal{E}^\vee , Ω_0^\vee , and \mathcal{E}^\vee may each be equipped with certain Hermitian metrics; we write $\widehat{\mathcal{E}}^\vee$, $\widehat{\Omega}_0^\vee$, and $\widehat{\mathcal{E}}^\vee$ for the resulting metrized bundles. We normalize these metrics as in [Che24a, Section 3.5] (in particular, the metric on $\widehat{\mathcal{E}}^\vee$ involves the factor of $4\pi e^\gamma$ which appears in [Che24a, (3.5.3)]). For readers interested in Faltings height, we also consider the metrized Hodge (determinant) bundle $\widehat{\omega}$ on \mathcal{M} , which is pulled back from the Hodge determinant bundle ω on $\mathcal{M}(n-r, r)$ and with metric normalized as in [Che24a, (3.5.1)].

The Faltings height of any elliptic curve with CM by \mathcal{O}_F is (with our normalizations)

$$h_{\text{Fal}}^{\text{CM}} := (\widehat{\omega}_{A_0}) = \frac{1}{2} \frac{L'(1, \eta)}{L(1, \eta)} + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)} + \frac{1}{4} \log |\Delta| - \frac{1}{2} \log(2\pi) \quad (2.1.12)$$

where η is the quadratic character associated to F/\mathbb{Q} , and Γ is the usual gamma function. This comes from the classical Chowla–Selberg formula (the statement above is as in [KRY04, Proposition 10.10]). It will also be convenient to consider the height constants

$$h_{\text{tau}}^{\text{CM}} := -h_{\text{Fal}}^{\text{CM}} + \frac{1}{4} \log |\Delta| - \frac{1}{2} \log(4\pi e^\gamma) \quad h_{\widehat{\mathcal{E}}_v}^{\text{CM}} := -h_{\text{Fal}}^{\text{CM}} - \frac{1}{4} \log |\Delta| + h_{\text{tau}}^{\text{CM}}. \quad (2.1.13)$$

These will re-appear in Section 4.9.

The next lemma will provide a base point for non-Archimedean uniformization (Section 4.3). The notation A_0^σ denotes the abelian scheme A_0 but with \mathcal{O}_F -action $\iota_0 \circ \sigma$.

Lemma 2.1.6. *Let L be any non-degenerate Hermitian \mathcal{O}_F -lattice of rank n and signature $(n-r, r)$, with associated moduli stack \mathcal{M} . There exists a finite degree field extension E/F and $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(\mathcal{O}_E[1/d_L])$ such that A is \mathcal{O}_F -linearly isogenous to $A_0^{n-r} \times (A_0^\sigma)^r$. In particular, \mathcal{M} is nonempty.*

Proof. First consider $\kappa = \mathbb{C}$ (equipped with a morphism $\mathcal{O}_F \rightarrow \mathbb{C}$). Fix the trivializations of roots of unity $\mathbb{Z}/d\mathbb{Z} \xrightarrow{\sim} \mu_d(\mathbb{C})$ sending $1 \mapsto e^{-2\pi i/d}$.

Choose $\sqrt{\Delta}$ to be the square-root whose image under $F \rightarrow \mathbb{C}$ has positive imaginary part. We pass between Hermitian and alternating forms using the generator $\sqrt{\Delta}$ of the different ideal (as in [Che24a, Section 2.1]). Express L as a triple (L, ι, λ) where $\iota: \mathcal{O}_F \rightarrow \text{End}_{\mathbb{Z}}(L)$ is an action and λ is a \mathcal{O}_F -compatible alternating pairing on L .

Take $(A_0, \iota_0, \lambda_0)$ to be the complex elliptic curve \mathbb{C}/\mathcal{O}_F . If $L_0 := \mathcal{O}_F$ is the rank one Hermitian \mathcal{O}_F -lattice with Hermitian pairing $(x, y) = x^\sigma y$, we have $H_1(A_0, \mathbb{Z}) \cong L_0$ as Hermitian lattices.

Take any orthogonal decomposition $L_F = W \oplus W^\perp$ where W is positive definite of rank $n-r$ and W^\perp is negative definite of rank r . Define the $\mathbb{C} = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{R}$ -action on $L \otimes_{\mathbb{Z}} \mathbb{R}$ to agree with ι on $W \otimes_F \mathbb{C}$ and to agree with $\iota \circ \sigma$ on $W^\perp \otimes_F \mathbb{C}$. This complex structure gives a tuple (A, ι, λ) , where $A := (L \otimes_{\mathbb{Z}} \mathbb{R})/L$ is an abelian variety with \mathcal{O}_F -action ι and action compatible quasi-polarization λ . We have $H_1(A, \mathbb{Z}) \cong L$ as Hermitian lattices. By the usual comparison of $H_1(-, \mathbb{Z})$ with p -adic Tate modules [Mum85, §24 Theorem 1], we conclude $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(\mathbb{C})$.

We claim that A is \mathcal{O}_F -linearly isogenous to $A_0^{n-r} \times (A_0^\sigma)^r$. Indeed, any \mathcal{O}_F -linear inclusion $\mathcal{O}_F^{n-r} \hookrightarrow L \cap W$ and any σ -linear inclusion $\mathcal{O}_F^r \hookrightarrow L \cap W^\perp$ will define an \mathcal{O}_F -linear isogeny $A_0^{n-r} \times (A_0^\sigma)^r \rightarrow A$.

Since A_0 is defined over some number field $\overline{\mathbb{Q}}$, it follows that A and any isogeny $A_0^{n-r} \times (A_0^\sigma)^r \rightarrow A$ may also be defined over $\overline{\mathbb{Q}}$ (here using characteristic zero, so the kernel of the isogeny is étale). Descend these objects to some number field E .

Over a number field, it is a classical fact that any elliptic curve with \mathcal{O}_F -action has everywhere potentially good reduction [Deu41]. After extending E if necessary, we thus obtain $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(\mathcal{O}_E[1/d_L])$ (the \mathcal{O}_F -actions extend by the Néron mapping property, and the polarizations extend to polarizations as in the proof of [FC90, Theorem 1.9]). The Néron mapping property extends the isogeny $A_0^{n-r} \times (A_0^\sigma)^r \rightarrow A$ over $\text{Spec } \mathcal{O}_E[1/d_L]$. \square

2.2. Level structure. Let L be any non-degenerate Hermitian \mathcal{O}_F -lattice of rank n , and form the associated moduli stack \mathcal{M} . We discuss level structure for \mathcal{M} .

Set $V = L \otimes_{\mathcal{O}_F} F$, and form the unitary group $U(V)$ (over $\text{Spec } \mathbb{Q}$). Let $L_0 = \mathcal{O}_F$ any self-dual Hermitian \mathcal{O}_F -lattice of rank 1. We set

$$\begin{aligned} K_{L_0,p} &:= \text{Stab}_{U(V_0)(\mathbb{Q}_p)}(L_0 \otimes \mathbb{Z}_p) & K_{L,p} &:= \text{Stab}_{U(V)(\mathbb{Q}_p)}(L \otimes \mathbb{Z}_p) \\ K_{L_0,f} &= \prod_p K_{L_0,p} & K_{L,f} &= \prod_p K_{L,p} \end{aligned}$$

for all p , where $\text{Stab}_{U(V)(\mathbb{Q}_p)}(L \otimes \mathbb{Z}_p)$ denotes the stabilizer of $L \otimes \mathbb{Z}_p$ in $U(V)(\mathbb{Q}_p)$, etc.. We say that $K_{L,f} \subseteq U(V)(\mathbb{A}_f)$ is the *adèlic stabilizer* of L . Set $K'_{L,f} = K_{L_0,f} \times K_{L,f}$. Note that there is no dependence (up to functorial isomorphism) on the choice of L_0 , or the choice of L within its adèlic isomorphism class. We use the usual notation where $K_{L,f}^p$ means to omit the p -th factor in the product, etc..

For integers $N \geq 1$, we define the “principal congruence subgroups”

$$K_p(N) := \ker(K_{L,p} \rightarrow \text{GL}(L \otimes \mathbb{Z}_p / N\mathbb{Z}_p)) \quad K_f(N) = \prod_p K_{L,p}(N)$$

(suppressing L dependence from notation) and similarly define $K_{0,p}(N_0)$ and $K_f(N_0)$ for $N_0 \geq 1$. Given a pair $N' = (N_0, N)$ of integers $N_0, N \geq 1$, we set $K'_f(N') := K_{0,f}(N_0) \times K_f(N)$ and $K'_p(N') := K_{0,p}(N_0) \times K_p(N)$, etc.. Given $N' = (N_0, N)$, we sometimes abuse notation, e.g. $N' \geq a$ means $N_0, N \geq a$, and the notation $\mathcal{X}[1/N']$ for an algebraic stack \mathcal{X} will mean inverting all primes dividing N' . If $N_0 = N$, we write $K'_f(N) := K'_f(N')$.

Let $K'_f = K_{0,f} \times K_f \subseteq K'_{L,f}$ be any open compact subgroup which admits product factorizations $K_{0,f} = \prod_p K_{0,p}$ and $K_f = \prod_p K_p$. We set $K'_p := K_{0,p} \times K_p$, etc.. Let $N_{K'_f}$ be the product of primes p for which $K_{0,p} \neq K_{L_0,p}$ or $K_p \neq K_{L,p}$. We say that K'_f is *standard* at p if $p \nmid N_{K'_f}$.

Notation 2.2.1. For $K'_f \subseteq K'_{L,f}$ as above, we reserve the term *small* or *small level* to mean that $K'_f \subseteq K'_f(N')$ for some $N' \geq 3$.

Consider $\alpha = (A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(S)$ for some scheme S . Suppose p is a prime which is invertible on S . For any integer $e \geq 0$, we consider the fppf sheaf

$$\underline{\text{Level}}(p^e) \subseteq \underline{\text{Isom}}(A_0[p^e], L_0 \otimes_{\mathbb{Z}} \mathbb{Z}/p^e\mathbb{Z}) \times \underline{\text{Isom}}(A[p^e], (L_0 \otimes_{\mathcal{O}_F} L) \otimes_{\mathbb{Z}} \mathbb{Z}/p^e\mathbb{Z}) \quad (2.2.1)$$

on S , where $\underline{\text{Level}}(p^e)$ is the open and closed subfunctor corresponding to isomorphisms $A_0[p^e] \rightarrow L_0/p^e L_0$ and $A[p^e] \rightarrow L/p^e L$ which lift to \mathcal{O}_F -linear isomorphisms

$$\begin{aligned} T_p(A_{0,\bar{s}}) &\xrightarrow{\sim} L_0 && \text{unitary up to scalar} \\ \text{Hom}_{\mathcal{O}_F}(T_p(A_{0,\bar{s}}), T_p(A_{\bar{s}})) &\xrightarrow{\sim} L && \text{unitary} \end{aligned}$$

over every geometric point \bar{s} of S . Since L_0 is rank one, the “unitary up to scalar” condition is automatic.

Consider any open compact subgroup $K'_p \subseteq K'_{L,p}$. For each $e \geq 0$, we write $K'_p \bmod p^e$ (temporary notation) for the image of K'_p in $\text{GL}_1(L_0 \otimes_{\mathbb{Z}} \mathbb{Z}/p^e\mathbb{Z}) \times \text{GL}_n(L \otimes_{\mathbb{Z}} \mathbb{Z}/p^e\mathbb{Z})$. There is a canonical action of $K'_p \bmod p^e$ on $\underline{\text{Level}}(p^e)$.

Definition 2.2.2 (Level structure). Let $K'_f \subseteq K'_{L,f}$ be any factorizable open compact subgroup, as above. Consider an object $\alpha = (A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(S)$ for some scheme $S \rightarrow \text{Spec } \mathcal{O}_F[1/N_{K'_f}]$.

If p is prime which is invertible on S , the *sheaf of level K'_p structures for α* is the quotient (coequalizer)¹¹

$$\underline{\text{Level}}_{K'_p} := (K'_p \bmod p^e)_{e \geq 0} \backslash (\underline{\text{Level}}(p^e))_{e \geq 0} \quad (2.2.2)$$

in the category of pro-objects for the category of fppf sheaves on S . If p is not invertible on S , we let $\underline{\text{Level}}_{K'_p}$ be the constant sheaf valued in a singleton set.

The *sheaf of level K'_f structures for α* is the product

$$\underline{\text{Level}}_{K'_f} := \prod_p \underline{\text{Level}}_{K'_p}. \quad (2.2.3)$$

over all primes p . This is an fppf sheaf on S , and is locally constant in the étale topology.

A K'_p *level structure* (rep. K'_f *level structure*) for α is a global section of $\underline{\text{Level}}_{K'_p}$ (resp. $\underline{\text{Level}}_{K'_f}$).

Remark 2.2.3. Let K'_f , α , and S be as in Definition 2.2.2. Fix a geometric point \bar{s} of S , with $\text{char}(\bar{s}) = p \geq 0$. Assume moreover that S is connected. In this case, giving a K'_f -level structure for α is (canonically) same as giving a pair $(\tilde{\eta}_0, \tilde{\eta})$ where $\tilde{\eta}_0$ (resp. $\tilde{\eta}$) is a $\pi_{1,\text{ét}}(S, \bar{s})$ -stable $K_{0,f}^p$ -orbit (resp. K_f^p -orbit) of isomorphisms

$$\begin{aligned} \eta_0: T^p(A_{0,\bar{s}}) &\xrightarrow{\sim} L_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p && \text{unitary up to scalar} \\ \eta: \text{Hom}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p}(T^p(A_{0,\bar{s}}, A_{\bar{s}})) &\xrightarrow{\sim} L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p && \text{unitary.} \end{aligned}$$

In the notation of Definition 2.2.2, note that $K'_p = K'_p(p^e)$ implies $\underline{\text{Level}}_{K'_p} = (\underline{\text{Level}}(p^e))$. In Remark 2.2.3, note that the “unitary up to scalar” condition is automatic because L_0 has rank 1. Even when S is not connected, we abuse notation and write $(\tilde{\eta}_0, \tilde{\eta})$ for level K'_f structure in the sense of Section 2.2

Given an open compact K'_f as in Definition 2.2.2, we now define a stack in groupoids $\mathcal{M}_{K'_f}$ over $\text{Spec } \mathcal{O}_F[1/(d_L N_{K'_f})]$ with

$$\mathcal{M}_{K'_f}(S) := \{(\alpha, \tilde{\eta}_0, \tilde{\eta}) : \alpha \in \mathcal{M}(s) \text{ and } (\tilde{\eta}_0, \tilde{\eta}) \text{ a level } K'_f \text{ structure for } \alpha\} \quad (2.2.4)$$

for schemes S over $\text{Spec } \mathcal{O}_F[1/(d_L N_{K'_f})]$. Given $T \in \text{Herm}_m(\mathbb{Q})$, we write $\mathcal{Z}(T)_{K'_f} := \mathcal{Z}(T) \times_{\mathcal{M}} \mathcal{M}_{K'_f}$ (“level K'_f special cycle”).

Write $\mathcal{A}_{n,d,N}$ for the moduli stack over $\text{Spec } \mathcal{O}_F[1/N]$ of (relative) n -dimensional abelian schemes A with degree d polarization and a chosen isomorphism $A[N] \rightarrow L \otimes \mathbb{Z}/N\mathbb{Z}$ of group schemes (not necessarily compatible with symplectic pairings). We similarly form $\mathcal{A}_{1,1,N_0}$ using the lattice L_0 (and pick a basis of L_0 for convenience). Recall that $\mathcal{A}_{n,d,N}$ is representable by a separated Deligne–Mumford stack of finite type over $\text{Spec } \mathcal{O}_F[1/N]$, and that $\mathcal{A}_{n,d,N}$ is a scheme quasi-projective over $\text{Spec } \mathcal{O}_F[1/N]$ if $N \geq 3$ (see [MFK94, §7.2 Theorem 7.9]).

Let $b_L, d_L \in \mathbb{Z}_{>0}$ be associated to L , as discussed before Definition 2.1.2. If $K'_f(N')$ is the principal congruence subgroup of some level $N' = (N_0, N)$, consider the forgetful morphism

$$\mathcal{M}_{K'_f(N')} \longrightarrow \mathcal{A}_{1,1,N_0} \times \mathcal{A}_{n,d_L,N} \quad (2.2.5)$$

¹¹Note that this quotient $\underline{\text{Level}}_{K'_p}$ is (isomorphic to) a sheaf (not just a pro-sheaf), so it is sensible to refer to $\underline{\text{Level}}_{K'_p}$ an fppf sheaf on S (rather than a pro-object).

which forgets the \mathcal{O}_F -actions and sends $\lambda \mapsto b_L \lambda$. For level structure, see the description in (2.2.1). The induced map

$$\mathcal{M}_{K'_f(N')} \rightarrow \mathcal{M}[1/N'] \times_{(\mathcal{A}_{1,1} \times \mathcal{A}_{n,d_L})} (\mathcal{A}_{1,1,N_0} \times \mathcal{A}_{n,d_L,N})[1/d_L] \quad (2.2.6)$$

is representable by schemes and is an open and closed immersion. Hence $\mathcal{M}_{K'_f(N')} \rightarrow \mathcal{A}_{1,1,N_0} \times \mathcal{A}_{n,d_L,N}[1/d_L]$ is finite (and representable by schemes).

Lemma 2.2.4.

- (1) For any open compact subgroup K'_f as in Definition 2.2.2, the stack $\mathcal{M}_{K'_f}$ is a separated Deligne–Mumford stack of finite type over $\mathrm{Spec} \mathcal{O}_F$. If K'_f is small, then $\mathcal{M}_{K'_f}$ is a quasi-projective scheme over $\mathrm{Spec} \mathcal{O}_F$.
- (2) For any inclusion $K'_f \subseteq K''_f$, the forgetful morphism $\mathcal{M}_{K'_f} \rightarrow \mathcal{M}_{K''_f}[1/N_{K''_f}]$ (i.e. expand a K'_f -orbit to a K''_f orbit) is finite étale of degree $|K''_f/K'_f|$. If $K'_f \subseteq K''_f$ is a normal subgroup, then $\mathcal{M}_{K'_f} \rightarrow \mathcal{M}_{K''_f}[1/N_{K''_f}]$ is a torsor for the finite discrete group K''_f/K'_f .

Proof. The second sentence in part (2) is clear from construction (and makes sense before we know these stacks are algebraic). When $K'_f = K'_f(N')$ for some N' , the claims in part (1) follow from (2.2.6).

For general K'_f , select $N' = (N_0, N)$ such that $\mathcal{M}_{K'_f(N')}$ is a scheme and $K'_f(N') \subseteq K'_f$. Then $\mathcal{M}_{K'_f(N')} \rightarrow \mathcal{M}_{K'_f}[1/N']$ is a torsor for the finite discrete group $K'_f/K'_f(N')$ (in particular, finite étale), and hence admits the stack quotient presentation $\mathcal{M}_{K'_f}[1/N'] \cong [\mathcal{M}_{K'_f(N')}/(K'_f/K'_f(N'))]$, which shows that $\mathcal{M}_{K'_f}[1/N']$ is Deligne–Mumford. Picking another $M' = (M_0, M)$ such that $\mathrm{gcd}(N_0N, M_0M)$ is only divisible by primes dividing $N_{K'_f}$, we find that $\mathcal{M}_{K'_f}[1/M']$ is Deligne–Mumford as well. These two charts show that $\mathcal{M}_{K'_f}$ is Deligne–Mumford, as well as separated and finite type over $\mathrm{Spec} \mathcal{O}_F$.

If $K'_f \subseteq K''_f$, then for any scheme S with a morphism $S \rightarrow \mathcal{M}_{K''_f}$, the 2-fiber product $\mathcal{M}_{K'_f} \times_{\mathcal{M}_{K''_f}} S$ is fibered in setoids, hence equivalent to a sheaf (of sets). But since $\mathcal{M}_{K'_f(N')} \rightarrow \mathcal{M}_{K'_f}[1/N']$ is a $K'_f/K'_f(N')$ -torsor and affine morphisms satisfy fpqc descent [SProject, Section 0244], we conclude that $\mathcal{M}_{K'_f}[1/N'] \times_{\mathcal{M}_{K''_f}} S$ is represented by a scheme. As above, we may pick some other M' to patch and show that the morphism $\mathcal{M}_{K'_f} \rightarrow \mathcal{M}_{K''_f}[1/N_{K''_f}]$ is representable by schemes. Since $\mathcal{M}_{K'_f(N')} \rightarrow \mathcal{M}_{K'_f}[1/N']$ and $\mathcal{M}_{K'_f(N')} \rightarrow \mathcal{M}_{K''_f}[1/N']$ are both finite étale surjections, we conclude that $\mathcal{M}_{K'_f} \rightarrow \mathcal{M}_{K''_f}[1/N_{K''_f}]$ is finite étale by varying N' again (using standard facts like [SProject, Lemma 02K6, Lemma 01KV, Lemma 0AH6, Lemma 02LS]). The remaining claims follow from this. \square

Lemma 2.2.5. Fix any prime p and a matrix $T \in \mathrm{Herm}_m(\mathbb{Q})$ with $m \geq 0$. The morphism $\mathcal{Z}(T)_{K'_f(p^e)} \rightarrow \mathcal{M}_{K'_f(p^e)}$ is a disjoint union of closed immersions for all $e \gg 0$.

Proof. For $e \in \mathbb{Z}_{\geq 0}$, we define a stack (used only in this proof) $\mathcal{M}(p^e)$ over $\mathrm{Spec} \mathcal{O}_F[1/(d_L p)]$ as follows. For schemes S over $\mathrm{Spec} \mathcal{O}_F[1/(d_L p)]$, we take $\mathcal{M}(p^e)(S)$ to be the groupoid

$$\mathcal{M}(p^e)(S) := \left\{ (A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x}) : \begin{array}{l} (A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{M}(S) \\ \underline{x} = [x_1, \dots, x_m] \in \mathrm{Hom}_{\mathcal{O}_F}(A_0[p^e], A[p^e])^m \end{array} \right\}. \quad (2.2.7)$$

We have a commutative diagram

$$\begin{array}{ccc}
 & & \mathcal{M}(p^e) \\
 & \nearrow & \downarrow \\
 \mathcal{Z}(T)[1/p] & \longrightarrow & \mathcal{M}.
 \end{array} \tag{2.2.8}$$

The forgetful morphism $\mathcal{M}(p^e) \rightarrow \mathcal{M}[1/p]$ is representable by schemes and a finite étale surjection. Thus, $\mathcal{M}(p^e)$ is representable by a separated Deligne–Mumford stack of finite type over $\mathrm{Spec} \mathcal{O}_F$.

We claim that $\mathcal{Z}(T) \rightarrow \mathcal{M}(p^e)$ is a closed immersion for $e \gg 0$. This may be checked fppf locally on the target. Suppose $S \rightarrow \mathcal{M}[1/p]$ is an fppf cover by a Noetherian scheme S (possible since \mathcal{M} is locally Noetherian and quasi-compact). It is enough to check that $\mathcal{Z}(T) \times_{\mathcal{M}} S \rightarrow \mathcal{M}(p^e) \times_{\mathcal{M}} S$ is a closed immersion of schemes. Since the morphism $\mathcal{Z}(T) \rightarrow \mathcal{M}$ (resp. $\mathcal{M}(p^e) \rightarrow \mathcal{M}[1/p]$) is finite and unramified (resp. finite), we conclude that $\mathcal{Z}(T)[1/p] \rightarrow \mathcal{M}(p^e)$ is also finite and unramified.

To prove the claim, it remains only to check that the morphism of schemes $\mathcal{Z}(T) \times_{\mathcal{M}} S \rightarrow \mathcal{M}(p^e) \times_{\mathcal{M}} S$ is universally injective for $e \gg 0$ (for morphisms of schemes, being a closed immersion is the same as being proper, unramified, and universally injective [SProject, Lemma 04XV]).

We first show that universal injectivity holds fiber-wise over every point $s \in S$ for e sufficiently large (with e possibly depending on s). For any point s on S with residue field $k(s)$, we know that $\mathcal{Z}(T)_{k(s)} \rightarrow \mathcal{M}(p^e)_{k(s)}$ is universally injective for $e \gg 0$ (possibly depending on s) because the map $\mathrm{Hom}(A_1, A_2) \rightarrow \mathrm{Hom}(T_p(A_1), T_p(A_2))$ is injective for abelian varieties A_1, A_2 over any field of characteristic $\neq p$ (apply this over a geometric point mapping to s and use finiteness of $\mathcal{Z}(T)$).

Being universally injective may be checked fiber-wise over S , so we need to show that there is a value of e which works for all points $s \in S$ simultaneously. We can select $e \gg 0$ so that $\mathcal{Z}(T) \times_{\mathcal{M}} S \rightarrow \mathcal{M}(p^e) \times_{\mathcal{M}} S$ is universally injective (hence a closed immersion) over the generic point of each irreducible component of S . For such e , a limiting argument (“spreading out”) implies that $\mathcal{Z}(T) \times_{\mathcal{M}} S \rightarrow \mathcal{M}(p^e) \times_{\mathcal{M}} S$ is a closed immersion over an open dense subset of S . Applying Noetherian induction on S proves the claim.

To finish the proof of the lemma, we observe that $\mathcal{M}(p^e) \times_{\mathcal{M}} \mathcal{M}_{K'_f(p^e)} \rightarrow \mathcal{M}_{K'_f(p^e)}$ is a finite disjoint union of isomorphisms, corresponding to the constant sheaf valued in $\mathrm{Hom}_{\mathcal{O}_F}(L_0 \otimes_{\mathbb{Z}} \mathbb{Z}/p^e \mathbb{Z}, L \otimes_{\mathbb{Z}} \mathbb{Z}/p^e \mathbb{Z})^m$. Hence $\mathcal{Z}(T)_{K'_f(p^e)} \rightarrow \mathcal{M}_{K'_f(p^e)}$ is a disjoint union of closed immersions. \square

Recall that we defined certain derived vertical special cycle classes ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{Y}, p} \in \mathrm{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}}$ in our companion paper [Che24a, Section 3.6]. There we restricted to the even exotic smooth case (to avoid discussing the more general setup of Section 2.1), but the same constructions apply in the situation of Section 2.1. We also obtain classes ${}^{\mathbb{L}}\mathcal{Z}(T)_{(p), K'_f}$ and ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{Y}, p, K'_f}$ (defined as in [Che24a, Section 3.6], but now pulled-back to $\mathcal{M}_{K'_f}$ for some level K'_f away from p , i.e. $p \nmid N_{K'_f}$) which are compatible with varying level K'_f .

2.3. Generic smoothness. We explain a generic smoothness result for special cycles (Lemma 2.3.5). The other lemmas are auxiliary. The proof proceeds by reducing to p -divisible groups over a base where p is locally nilpotent, and then checking formal smoothness using Serre–Tate and Grothendieck–Messing deformation theory.

We first consider p -divisible groups (see Appendix B.1 for some terminology). For primes p , set $\mathcal{O}_{F_p} := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Suppose $p \nmid \Delta$ and consider schemes S over $\mathrm{Spf} \mathcal{O}_{F_p}$, i.e. S is a scheme over $\mathrm{Spec} \mathcal{O}_F$ with p locally nilpotent on S . We consider tuples (Y, ι, λ) over S where

$$\begin{aligned}
Y & \text{ is a } p\text{-divisible group over } S \text{ of height } 2n \text{ and dimension } n \\
\iota: \mathcal{O}_{F_p} & \rightarrow \mathrm{End}(Y) \text{ is an action satisfying the } (n-r, r) \text{ Kottwitz signature condition, i.e. for all } a \in \mathcal{O}_{F_p}, \text{ the characteristic polynomial of } \iota(a) \text{ acting on } \mathrm{Lie} Y \text{ is } \\
& (x-a)^{n-r}(x-a^\sigma)^r \in \mathcal{O}_S[x] \\
\lambda: Y & \xrightarrow{\sim} Y^\vee \text{ is a principal polarization whose Rosati involution } \dagger \text{ on } \mathrm{End}(Y) \text{ satisfies } \iota(a)^\dagger = \iota(a^\sigma) \text{ for all } a \in \mathcal{O}_{F_p}.
\end{aligned} \tag{2.3.1}$$

In the signature condition described above, we view \mathcal{O}_S as an \mathcal{O}_{F_p} -algebra via the structure map $S \rightarrow \mathrm{Spf} \mathcal{O}_{F_p}$.

Parts of the next formal smoothness result (Lemma 2.3.1) may exist in some form in the literature, see e.g. discussion about formal smoothness for “unramified Rapoport–Zink data” in [RZ96, 3.82] and the reference to [Kot92, §5] given there.

Following [SProject, Section 04EW], we use the term *thickening* to refer to a closed immersion which is a homeomorphism on underlying topological spaces, and the term *first order thickening* for a thickening defined by a square zero ideal.

Let S be a scheme over $\mathrm{Spf} \mathcal{O}_{F_p}$, and suppose (Y, ι, λ) is a tuple over S as in (2.3.1). There is an associated deformation functor $\mathrm{Def}_{(Y, \iota, \lambda)}$ (possibly non-standard usage, and it will not appear after Lemma 2.3.1) which sends a thickening $S \rightarrow S'$ to the set of (isomorphism classes of) lifts of (Y, ι, λ) to S' . Write $S[\epsilon]$ and $S[\epsilon, \epsilon']$ as shorthand for $S \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Z}[\epsilon]/(\epsilon^2)$ and $S \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Z}[\epsilon, \epsilon']/(\epsilon^2, \epsilon\epsilon', \epsilon'^2)$, respectively. In the proof of Lemma 2.3.1, we will see that the canonical map

$$\mathrm{Def}_{(Y, \iota, \lambda)}(S[\epsilon, \epsilon']) \rightarrow \mathrm{Def}_{(Y, \iota, \lambda)}(S[\epsilon]) \times_{\mathrm{Def}_{(Y, \iota, \lambda)}(S)} \mathrm{Def}_{(Y, \iota, \lambda)}(S[\epsilon']) \tag{2.3.2}$$

is an isomorphism. More generally, if M is a finite rank free \mathcal{O}_S -module and $\mathcal{O}_S \oplus M$ denotes the quasi-coherent \mathcal{O}_S -algebra with M an ideal of square zero, we will see that the functor $M \mapsto \mathrm{Def}_{(Y, \iota, \lambda)}(\mathrm{Spec}_S(\mathcal{O}_S \oplus M))$ preserves fiber products over the base $M = 0$ (note that this holds when $\mathrm{Def}_{(Y, \iota, \lambda)}$ is replaced by any scheme, and this is essentially the method of proof). Here Spec_S denotes relative Spec .

For any scheme S over $\mathrm{Spf} \mathcal{O}_{F_p}$, the above considerations imply that the set $\mathrm{Def}_{(Y, \iota, \lambda)}(S[\epsilon])$ has the natural structure of a $\Gamma(S, \mathcal{O}_S)$ -module in the standard way (as a “tangent space”) as in [SGA3II, Proposition 3.6] or [SProject, Section 06I2].

Lemma 2.3.1. *Let p be a prime which is unramified in \mathcal{O}_F . The deformation problem for triples as in (2.3.1) is formally smooth of relative dimension $(n-r)r$ in the following sense. Let S be any scheme over $\mathrm{Spf} \mathcal{O}_{F_p}$, and let (Y, ι, λ) be a triple over S as in (2.3.1).*

- (1) *The triple (Y, ι, λ) lifts along any first order thickening of affine schemes $S \rightarrow S'$, i.e. the map $\mathrm{Def}_{(Y, \iota, \lambda)}(S') \rightarrow \mathrm{Def}_{(Y, \iota, \lambda)}(S)$ is surjective.*

(2) When $S = \text{Spec } \kappa$ for a field κ , the κ vector space $\text{Def}_{(Y, \iota, \lambda)}(\kappa[\epsilon])$ has dimension $(n - r)r$.

If $(n - r)r = 0$, then (Y, ι, λ) lifts uniquely along any first order thickening of schemes $S \rightarrow S'$.

Proof. We study this lifting problem for p -divisible groups in terms of Grothendieck–Messing deformation theory. Let $S \rightarrow S'$ be a first order thickening of schemes (not necessarily affine). View $S \hookrightarrow S'$ as a PD thickening, with trivial PD structure on the square zero ideal of the thickening.

Write $\mathbb{D}(Y)$ for the covariant Dieudonné crystal of Y , and write $\mathbb{D}(Y)(S)$ and $\mathbb{D}(Y)(S')$ for the evaluation of this crystal on the PD thickenings $\text{id}: S \rightarrow S$ and $S \hookrightarrow S'$ respectively. We have a short exact sequence of \mathcal{O}_S -modules given by the Hodge filtration

$$0 \rightarrow \Omega_{Y^\vee} \rightarrow \mathbb{D}(Y)(S) \rightarrow \text{Lie}_Y \rightarrow 0 \quad (2.3.3)$$

with $\Omega_{Y^\vee} = (\text{Lie}_{Y^\vee})^\vee$ and each \mathcal{O}_S -module above being finite locally free.

We may decompose the Hodge filtration into eigenspaces with respect to the action $\iota: \mathcal{O}_{F_p} \rightarrow \text{End}(Y)$ (and the structure morphism $S \rightarrow \text{Spec } \mathcal{O}_{F_p}$). We use superscripts $(-)^+$ and $(-)^-$ to denote these eigenspaces, where \mathcal{O}_{F_p} acts linearly (resp. σ -linearly) on $(-)^+$ (resp. $(-)^-$) via ι . Then we have short exact sequences

$$0 \rightarrow \Omega_{Y^\vee}^+ \rightarrow \mathbb{D}(Y)(S)^+ \rightarrow \text{Lie}_Y^+ \rightarrow 0 \quad (2.3.4)$$

$$0 \rightarrow \Omega_{Y^\vee}^- \rightarrow \mathbb{D}(Y)(S)^- \rightarrow \text{Lie}_Y^- \rightarrow 0 \quad (2.3.5)$$

where each \mathcal{O}_S -module above is finite locally free and, for example, we have $\mathbb{D}(Y) = \mathbb{D}(Y)^+ \oplus \mathbb{D}(Y)^-$. From left to right, the modules in (2.3.4) have ranks r , n , and $n - r$, and the modules in (2.3.5) have ranks $n - r$, n , and r respectively.

Using the polarization λ , we may identify (2.3.4) with the dual of (2.3.5). There is a choice of sign in this identification, which plays essentially no role in this proof.

We have $\mathbb{D}(Y)(S')|_S \cong \mathbb{D}(Y)(S)$ canonically (as $\mathbb{D}(Y)$ is a crystal), and Grothendieck–Messing theory implies that lifting (Y, ι, λ) to S' is the same as lifting the Hodge filtration (2.3.3) compatibly with the action ι and the polarization λ . Compatibility with the ι action means that we should lift the eigenspace decomposition in (2.3.4) and (2.3.5), and compatibility with the polarization λ means that the resulting exact sequences should again be dual to each other (as determined by λ). It is equivalent to lift either one of the exact sequences of (2.3.4) and (2.3.5) (one determines the other) to a filtration of $\mathbb{D}(Y)(S')^+$ or $\mathbb{D}(Y)(S')^-$ respectively (with no additional restrictions).

Consider the lifting problem for, say, the $+$ eigenspace of the Hodge filtration as in (2.3.4). Zariski locally on S' , this lifting problem may be identified with the problem of lifting an S point to an S' point on the Grassmannian parametrizing rank r subbundles of the rank n trivial bundle. This Grassmannian is smooth of relative dimension $(n - r)r$, which proves the claims in the lemma statement. \square

The next three lemmas are used to prove Lemma 2.3.4. This latter lemma is in turn used in the proof of generic smoothness in Lemma 2.3.5, to reduce to bases where p is locally nilpotent for some unramified prime p . This will allow us to reduce to formal smoothness for deformations of p -divisible groups (with certain additional structure) as proved in Lemma 2.3.1.

Lemma 2.3.2. *Let A be an adic Noetherian ring, and let X be a locally Noetherian scheme over $\mathrm{Spec} A$. If $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ is flat, then $X \rightarrow \mathrm{Spec} A$ is flat at every point of X which lies over $\mathrm{Spf} A$. If $X \rightarrow \mathrm{Spec} A$ is locally of finite type, then the same holds with “flat” replaced by “smooth”.*

Proof. Here, flatness (resp. smoothness) of $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ is equivalent to the requirement that, for every scheme T with a map $T \rightarrow \mathrm{Spf} A$, the base changed map $X_T \rightarrow T$ is flat (resp. smooth).

We first check the flatness assertion. Passing to an affine open of X , we may reduce to the case where $X = \mathrm{Spec} B$ for a Noetherian ring B . Let $I \subseteq A$ be an ideal of definition. Then $X_{\mathrm{Spf} A}$ is described by a completed tensor product, and we have $X_{\mathrm{Spf} A} = \hat{B}$ where \hat{B} is the I -adic completion of B . Since B is a Noetherian ring, the canonical map $B \rightarrow \hat{B}$ is flat. Since $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ is flat, we know that $A \rightarrow \hat{B}$ is a flat ring map. We conclude that B is flat over A at every prime in the image of $\mathrm{Spec} \hat{B} \rightarrow \mathrm{Spec} B$. These are precisely the points of X lying over $\mathrm{Spf} A$.

Next, assume $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ is smooth. By Noetherianity of A , the map $f: X \rightarrow \mathrm{Spec} A$ is locally of finite presentation. We have just shown that $X \rightarrow \mathrm{Spec} A$ is flat at every point $x \in X$ which lies over $\mathrm{Spf} A$. Thus, for such $x \in X$, the map $f: X \rightarrow \mathrm{Spec} A$ is smooth at x if and only if $X_{f(x)} \rightarrow \mathrm{Spec} k(f(x))$ is smooth at x , where $k(f(x))$ denotes the residue field of $f(x)$. But since $\mathrm{Spec} k(f(x)) \rightarrow \mathrm{Spec} A$ factors through $\mathrm{Spf} A \rightarrow \mathrm{Spec} A$, we conclude that $X_{f(x)} \rightarrow \mathrm{Spec} k(f(x))$ is indeed smooth. \square

Lemma 2.3.3. *Let $f: X \rightarrow Y$ be a locally of finite type (resp. locally of finite presentation) morphism of schemes, and assume that Y is a Jacobson scheme. Then f is smooth (resp. flat) if and only if f is smooth (resp. flat) at every point of X which lies over a closed point of Y .*

Proof. Since f is locally of finite type (resp. locally of finite presentation), we know that X is a Jacobson scheme (i.e. closed points are dense in every closed subset). Since f is smooth (resp. flat) on an open subset of X , it is enough to check that f is smooth (resp. flat) at every closed point of X . As f is locally of finite type and Y is Jacobson, we know that f maps closed points to closed points [SProject, Lemma 01TB] which gives the lemma claim. \square

Lemma 2.3.4. *Let \mathcal{X} be an algebraic stack, let Y be a Jacobson locally Noetherian scheme, and let $f: \mathcal{X} \rightarrow Y$ be a morphism which is locally of finite type. For points $y \in Y$, write $\hat{\mathcal{O}}_{Y,y}$ for the completion of the local ring at y . Then $\mathcal{X} \rightarrow Y$ is smooth (resp. flat) if and only if $\mathcal{X}_{\mathrm{Spf} \hat{\mathcal{O}}_{Y,y}} \rightarrow \mathrm{Spf} \hat{\mathcal{O}}_{Y,y}$ is smooth (resp. flat) for every closed point $y \in Y$.*

Proof. Select any scheme U with a surjective smooth morphism $U \rightarrow \mathcal{X}$. Then $U \rightarrow Y$ is a locally of finite type morphism of Jacobson locally Noetherian schemes, and $\mathcal{X} \rightarrow Y$ is smooth (resp. flat) if and only if $U \rightarrow Y$ is smooth (resp. flat). By Lemma 2.3.3, we may check smoothness (resp. flatness) of $U \rightarrow Y$ at points of U lying over closed points of Y . If $x \in U$ and $y = f(x)$, then $U \rightarrow Y$ is smooth (resp. flat) at x if and only if $U_{\mathrm{Spec} \hat{\mathcal{O}}_{Y,y}} \rightarrow \mathrm{Spec} \hat{\mathcal{O}}_{Y,y}$ is smooth at x (first checking flatness, then checking smoothness in the fiber over the closed point). For any $y \in Y$, Lemma 2.3.2 implies that $U \rightarrow Y$ is smooth (resp. flat) at all points $x \in U$ lying over y if and only if $U_{\mathrm{Spf} \hat{\mathcal{O}}_{Y,y}} \rightarrow \mathrm{Spf} \hat{\mathcal{O}}_{Y,y}$ is smooth (resp. flat). By Lemma 2.3.3, we then see that $U \rightarrow Y$ is smooth (resp. flat) if and only if $U_{\mathrm{Spf} \hat{\mathcal{O}}_{Y,y}} \rightarrow \mathrm{Spf} \hat{\mathcal{O}}_{Y,y}$ is smooth (resp. flat) for every closed point $y \in Y$.

This is equivalent to the condition that $\mathcal{X}_{\mathrm{Spf} \widehat{\mathcal{O}}_{Y,y}} \rightarrow \mathrm{Spf} \widehat{\mathcal{O}}_{Y,y}$ is smooth (resp. flat) for all closed points $y \in Y$, since $U \rightarrow \mathcal{X}$ is a smooth surjection. \square

Lemma 2.3.5. *Let L be any non-degenerate Hermitian \mathcal{O}_F -lattice, with associated moduli stack \mathcal{M} . Fix $T \in \mathrm{Herm}_m(F)$.*

Then there exists $N \in \mathbb{Z}$ such that $\mathcal{Z}(T)[1/(Nd_L\Delta)]$ is either empty¹² or smooth of relative dimension $(n - r - \mathrm{rank}(T))r$ over $\mathrm{Spec} \mathcal{O}_F[1/(Nd_L\Delta)]$.

We may take N such that for $p \nmid Nd_L\Delta$, there exists $g \in \mathrm{GL}_m(\mathcal{O}_{F_p})$ with

$${}^t\bar{g}Tg = \begin{pmatrix} \mathrm{Id}_{\mathrm{rank}(T)} & 0 \\ 0 & 0 \end{pmatrix} \quad (2.3.6)$$

where ${}^t\bar{g}$ denotes the conjugate transpose of g .

Proof. Fix a prime $p \nmid d_L\Delta$ such that there exists $g \in \mathrm{GL}_m(\mathcal{O}_{F_p})$ satisfying (2.3.6). By Lemma 2.3.4, it is enough to check that the base change $\mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{F_p}}$ is either empty or smooth of relative dimension $(n - r - \mathrm{rank}(T))r$ over $\mathrm{Spf} \mathcal{O}_{F_p}$.

The morphism $\mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{F_p}} \rightarrow \mathrm{Spf} \mathcal{O}_{F_p}$ is representable by algebraic stacks and locally of finite presentation. Thus $\mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{F_p}} \rightarrow \mathrm{Spf} \mathcal{O}_{F_p}$ is smooth if and only if it is formally smooth [SProject, Lemma 0DP0].

Let $S \rightarrow S'$ be a first order thickening of affine schemes, and assume S' is equipped with a morphism to $\mathrm{Spf} \mathcal{O}_{F_p}$. To check formal smoothness of $\mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{F_p}} \rightarrow \mathrm{Spf} \mathcal{O}_{F_p}$, we need to show that every object $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x}) \in \mathcal{Z}(T)(S)$ admits a lift to S' .

Form $(X_0, \iota_0, \lambda_0)$, where $X_0 = A_0[p^\infty]$ is the p -divisible group of A_0 with induced action $\iota_0: \mathcal{O}_{F_p} \rightarrow \mathrm{End}(X_0)$ and principal polarization $\lambda_0: X_0 \rightarrow X_0^\vee$. Similarly associate (X, ι, λ) to (A, ι, λ) , where $X = A[p^\infty]$ is the p -divisible group of A . Note that the polarization $\lambda: X \rightarrow X^\vee$ is principal because $p \nmid d_L\Delta$. Write also $\underline{x} = [x_1, \dots, x_m]$ for the corresponding m -tuple of morphisms $x_i: X_0 \rightarrow X$. By Serre–Tate, lifting $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x})$ from S to S' is the same as lifting $(X_0, \iota_0, \lambda_0, X, \iota, \lambda, \underline{x})$ from S to S' .

Using an element $g \in \mathrm{GL}_m(\mathcal{O}_{F_p})$ satisfying (2.3.6) as a “change of basis” for X_0^m , we obtain we obtain an \mathcal{O}_F -linear “orthogonal splitting” $X \cong X_0^{\mathrm{rank}(T)} \times Y$. That is, Y is a p -divisible group with action $\iota_Y: \mathcal{O}_{F_p} \rightarrow \mathrm{End}(Y)$, and a principal polarization $\lambda_Y: Y \rightarrow Y^\vee$ whose Rosati involution \dagger satisfies $\iota_Y(a)^\dagger = \iota_Y(a^\sigma)$ for all $a \in \mathcal{O}_F$. Under the described identification $X \cong X_0^{\mathrm{rank}(T)} \times Y$, the polarization λ on X is given by $(\lambda_0)^{\mathrm{rank}(T)} \times \lambda_Y$. The map $\underline{x}: X_0^m \rightarrow X$ may be identified with the projection $X_0^m \rightarrow X_0^{\mathrm{rank}(T)}$ onto the first $\mathrm{rank}(T)$ factors, followed by the canonical inclusion $X_0^{\mathrm{rank}(T)} \rightarrow X_0^{\mathrm{rank}(T)} \times Y$.

Note that the actions of \mathcal{O}_{F_p} on X_0 , X , and Y have signatures $(1, 0)$, $(n - r, r)$, and $(n - r - \mathrm{rank}(T), r)$ respectively (in the sense of (2.3.1)). These considerations also show that $\mathrm{rank}(T) \leq n - r$ if $\mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{F_p}}$ is nonempty.

The triple $(X_0, \iota_0, \lambda_0)$ admits a unique lift to S' as in Lemma 2.3.1. The projection map $\underline{x}: X_0^m \rightarrow X_0^{\mathrm{rank}(T)}$ clearly lifts to S' as well. So it remains only to lift (Y, ι_Y, λ_Y) from S to S' . Such a lift exists by formal smoothness of the corresponding deformation problem described in Lemma 2.3.1.

¹²Following the Stacks project [SProject, Definition 0055], our convention is that $\dim \emptyset = -\infty$.

We apply the same lemma to compute tangent spaces (e.g. after passing to an étale cover by a scheme), which shows that the relative dimension is $(n - r - \text{rank}(T))r$. \square

Remark 2.3.6. Taking $T = \emptyset$ (or $T = 0$) in Lemma 2.3.5 and varying over non-degenerate Hermitian \mathcal{O}_F -lattices L satisfying $L \subseteq L^\vee$ and $|L^\vee/L| = d$, we see that $\mathcal{M}(n-r, r)^{(d)} \rightarrow \text{Spec } \mathcal{O}_F[1/(d\Delta)]$ is smooth of relative dimension $(n-r)r$ for every $d \in \mathbb{Z}_{>0}$. If \mathcal{M} is associated with any non-degenerate Hermitian \mathcal{O}_F -lattice L (not necessarily with $L \subseteq L^\vee$), this then implies that $\mathcal{M} \rightarrow \text{Spec } \mathcal{O}_F[1/(d_L\Delta)]$ is smooth of relative dimension $(n-r)r$.

3. ARITHMETIC CYCLE CLASSES

Retain notation from Section 2. Let L be a non-degenerate Hermitian \mathcal{O}_F -lattice of rank n , with associated moduli stack \mathcal{M} . For Section 3, we assume L has signature $(n-1, 1)$. Consider an $m \times m$ Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$, assume $m \leq n$, and form the associated special cycle $\mathcal{Z}(T) \rightarrow \mathcal{M}$. One expects to be able to construct an associated *arithmetic special cycle class* $[\widehat{\mathcal{Z}}(T)] \in \widehat{\text{Ch}}^m(\mathcal{M})_{\mathbb{Q}}$.

For arbitrary singular T , there is no proposed definition of $[\widehat{\mathcal{Z}}(T)]$ in the literature. In general, $\mathcal{Z}(T)_{\mathcal{H}}$ has larger-than-expected dimension. The stack $\mathcal{Z}(T)$ could also have components with larger-than-expected dimension in positive characteristic (occurs already for nonsingular T). Available methods in the literature for treating the non-Archimedean theory (K -theoretic and derived algebro-geometric) do not incorporate the Archimedean place in general, as needed for arithmetic intersection theory (see introduction). For some additional discussion, see [Che24a, Section 1.2].

The analogue of the “linear invariance” approach of [KRY04, §6.4] (there for Shimura curves) is to first define $[\widehat{\mathcal{Z}}(T^\flat)]$ for nonsingular T^\flat , to consider ${}^t\bar{\gamma}T\gamma = \text{diag}(0, T^\flat)$ for some $\gamma \in \text{GL}_m(\mathcal{O}_F)$ with T^\flat nonsingular, and to define $[\widehat{\mathcal{Z}}(T)]$ by intersecting $[\widehat{\mathcal{Z}}(T^\flat)]$ with a power of some metrized tautological bundle (possibly with additional Archimedean adjustment). This is not literally possible in the unitary setting, where \mathcal{O}_F may have class number $\neq 1$ (in particular, γ as above may not exist). One also needs to verify independence of the choice of γ .

For arbitrary $T \in \text{Herm}_m(\mathbb{Q})$, we propose to define $[\widehat{\mathcal{Z}}(T)]$ as a sum

$$[\widehat{\mathcal{Z}}(T)] := [\widehat{\mathcal{Z}}(T)_{\mathcal{H}}] + \sum_{\substack{p \text{ prime} \\ p \nmid d_L}} [\mathbb{L}\mathcal{Z}(T)_{\mathcal{V}, p}] \in \widehat{\text{Ch}}^m(\mathcal{M})_{\mathbb{Q}}. \quad (3.0.1)$$

We construct $[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}]$ using the horizontal part $\mathcal{Z}(T)_{\mathcal{H}}$ and an appropriate Green current $g_{T, y}$ (3.2.5) with an additional parameter $y \in \text{Herm}_m(\mathbb{R})_{>0}$. The element $[\mathbb{L}\mathcal{Z}(T)_{\mathcal{V}, p}]$ arises from a class $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V}, p} \in \text{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}}$ corresponding to the “vertical part” of $\mathcal{Z}(T)$ at p . The vertical class $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V}, p}$ was defined in [Che24a, Section 3.6]. The classes $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V}, p}$ are zero for all but finitely many primes p . We defined $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V}, p}$ using a “ p -local” variant of the linear invariance strategy above.

We will show that $[\widehat{\mathcal{Z}}(T)]$ satisfies the “linear invariance” property

$$[\widehat{\mathcal{Z}}(T)] = [\widehat{\mathcal{Z}}({}^t\bar{\gamma}T\gamma)] \quad (3.0.2)$$

for all $\gamma \in \text{GL}_m(\mathcal{O}_F)$, where $[\widehat{\mathcal{Z}}(T)]$ is formed with respect to $y \in \text{Herm}_m(\mathbb{R})_{>0}$ and $[\widehat{\mathcal{Z}}(T)]$ is formed with respect to $\gamma^{-1}y{}^t\bar{\gamma}^{-1}$.

In fact, we prove refined statements. We show

$$[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}] = [\widehat{\mathcal{Z}}({}^t\bar{\gamma}T\gamma)_{\mathcal{H}}] \quad (3.0.3)$$

for the Green currents $g_{T,y}$ defined in Section 5.4 (where the current $g_{{}^t\bar{\gamma}T\gamma,\gamma^{-1}y{}^t\bar{\gamma}^{-1}}$ is used on the right-hand side). Moreover, we show $g_{T,y} = g_{{}^t\bar{\gamma}T\gamma,\gamma^{-1}y{}^t\bar{\gamma}^{-1}}$ (Section 5.4); this property is also satisfied for the Garcia–Sankaran currents in [GS19, (4.38)] (but the modified currents in [GS19, Definition 4.7] do not); we do not use the Garcia–Sankaran currents for our arithmetic Siegel–Weil results.

For any $\gamma \in \mathrm{GL}_m(\mathcal{O}_{F,(p)}) \cap M_{m,m}(\mathcal{O}_F)$, we have already shown in our companion paper [Che24a, Section 3.6] that the pullback

$$\mathrm{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}} \leftarrow \mathrm{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}({}^t\bar{\gamma}T\gamma)_{\mathbb{F}_p})_{\mathbb{Q}} \quad (3.0.4)$$

along $\mathcal{Z}(T)_{\mathbb{F}_p} \rightarrow \mathcal{Z}({}^t\bar{\gamma}T\gamma)_{\mathbb{F}_p}$ (defined in [Che24a, (3.2.6)]) sends ${}^{\mathbb{L}}\mathcal{Z}({}^t\bar{\gamma}T\gamma)_{\psi,p}$ to ${}^{\mathbb{L}}\mathcal{Z}(T)_{\psi,p}$ [Che24a, (3.6.11)].

3.1. Arithmetic Chow rings. We fix definitions for arithmetic Chow rings with rational coefficients.

Let (R, Σ, c_{∞}) be an arithmetic ring in the sense of Gillet–Soulé [GS90, §3.1], i.e. R is an excellent regular Noetherian integral domain (e.g. Dedekind domains with fraction field of characteristic 0 or fields), Σ is a finite nonempty set of injective homomorphisms $\tau: R \rightarrow \mathbb{C}$, and $c_{\infty}: \mathbb{C}^{\Sigma} \rightarrow \mathbb{C}^{\Sigma}$ is a conjugate-linear involution of $\mathbb{C} \otimes_{\mathbb{Z}} R$ -algebras. Write K for the fraction field of R .

Suppose X is a scheme which is separated, flat, and finite type over $\mathrm{Spec} R$ with smooth and quasi-projective generic fiber X_K . There are associated Gillet–Soulé *arithmetic Chow groups* $\widehat{\mathrm{Ch}}^m(X)$ in codimensions $m \geq 0$. If X is moreover regular, these groups form an *arithmetic Chow ring* $\widehat{\mathrm{Ch}}^*(X)_{\mathbb{Q}}$ (with \mathbb{Q} -coefficients) [GS90, Theorem 4.2.3].

Let L be any non-degenerate Hermitian \mathcal{O}_F -lattice of rank n , with associated moduli stack \mathcal{M} . Consider the arithmetic ring (R, Σ, c_{∞}) associated with $R = \mathrm{Spec} \mathcal{O}_F[1/d_L]$. We define *arithmetic Chow groups* for \mathcal{M} by limiting over level structure: for any nonzero integer N , set

$$\widehat{\mathrm{Ch}}^*(\mathcal{M}[1/N])_{\mathbb{Q}} := \lim_{K'_f} \widehat{\mathrm{Ch}}^*(\mathcal{M}_{K'_f}[1/N])_{\mathbb{Q}} \quad (3.1.1)$$

where K'_f varies over all small levels as in Section 2.2 (so that each $\mathcal{M}_{K'_f}$ is a scheme). Similar limiting procedures appeared in [BBK07] and [BH21, §4.4]; see also [Gil09] for more on arithmetic Chow rings of Deligne–Mumford stacks.

Since $\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_F[1/d_L]$ is smooth, we know that \mathcal{M} is regular. Hence we obtain an *arithmetic Chow ring* $\widehat{\mathrm{Ch}}^*(\mathcal{M}[1/N])_{\mathbb{Q}}$ via the intersection product for each $\widehat{\mathrm{Ch}}^*(\mathcal{M}_{K'_f}[1/N])_{\mathbb{Q}}$.

Suppose $\mathcal{Z} \rightarrow \mathcal{M}$ is a finite morphism of algebraic stacks with $\mathcal{Z} \rightarrow \mathrm{Spec} \mathcal{O}_F[1/d_L]$ proper and \mathcal{Z} equidimensional of dimension d . Then we define the *height* of \mathcal{Z} with respect to any Hermitian line bundle $\widehat{\mathcal{L}}$ on \mathcal{M} as follows: if $\mathcal{Z}_{K'_f} := \mathcal{Z} \times_{\mathcal{M}} \mathcal{M}_{K'_f}$, the quantity

$$\widehat{\mathrm{deg}}(\widehat{\mathcal{L}}^d|_{\mathcal{Z}}) := \frac{1}{[K'_f(1) : K'_f]} \widehat{\mathrm{deg}}(\widehat{\mathcal{L}}^d|_{\mathcal{Z}_{K'_f}}) \in \mathbb{R}_{d_L N_{K'_f}} = \mathbb{R}/\left(\sum_{p|d_L N_{K'_f}} \mathbb{Q} \cdot \log p\right) \quad (3.1.2)$$

does not depend on the choice of small level K'_f , where $\widehat{\deg}(\widehat{\mathcal{L}}^d|_{\mathcal{Z}_{K'_f}})$ is the arithmetic height as in [BGS94, Proposition 2.3.1, Remarks(ii)] (see also [Zha95]) calculated by replacing $\mathcal{Z}_{K'_f}$ with its pushforward cycle on $\mathcal{M}_{K'_f}$. Varying K'_f , we obtain the *height* $\widehat{\deg}(\widehat{\mathcal{L}}^d|_{\mathcal{Z}}) \in \mathbb{R}_{dL}$. We will be primarily interested in the case where $d = 1$ with \mathcal{Z} reduced and flat over $\text{Spec } \mathcal{O}_F[1/d_L]$. In this case, $\widehat{\deg}(\widehat{\mathcal{L}}|_{\mathcal{Z}})$ is the (stacky) arithmetic degree of $\widehat{\mathcal{L}}$ restricted to \mathcal{Z} , as discussed in [Che24a, Section 3.3].

3.2. Horizontal arithmetic special cycle classes. Consider any $m \times m$ Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$. The horizontal arithmetic special cycle class $[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}]$ should involve some extra Archimedean data, e.g. from a Green current $g_{T,y}$ (which we allow to depend on a parameter $y \in \text{Herm}_m(\mathbb{R})_{>0}$, as is typical in the literature).

Given an equidimensional complex manifold X , recall that a (p, q) -current on X is a continuous linear map $\Omega_c^{\dim X - p, \dim X - q}(X) \rightarrow \mathbb{C}$ on compactly supported smooth forms of degree $(\dim X - p, \dim X - q)$, where $\Omega_c^{\dim X - p, \dim X - q}(X)$ has the usual colimit topology. A (p, p) -current is *real* if it is induced by a continuous real-valued linear map on real (p, p) -forms. Given a top degree current g on X (i.e. a distribution), we say that g is *integrable* or that $\int_X g$ *converges* (possibly non-standard usage) if g extends (necessarily uniquely) to a continuous map $C_{b_1}^\infty(X) \rightarrow \mathbb{C}$, where

$$C_{b_1}^\infty(X) := \{f \in C^\infty(X) : |f(x)| \leq 1 \text{ for all } x \in X\} \quad (3.2.1)$$

with topology given by sup-norms ranging over all compact subsets $K \subseteq X$. In this case, we write $\int_X g$ for the value of g on $1 \in C_{b_1}^\infty(X)$. Suppose α is a (measurable) locally L^1 form of top degree on X . If α is integrable, then the associated distribution $[\alpha]$ on X is integrable, and we have $\int_X [\alpha] = \int_X \alpha$. We use the orientation and sign conventions of [GS90].

Returning to the moduli stack $\mathcal{M} \rightarrow \text{Spec } \mathcal{O}_F$ from above, choose any embedding $F \rightarrow \mathbb{C}$ and form the base changes $\mathcal{M}_{\mathbb{C}} := \mathcal{M} \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathbb{C}$ and $\mathcal{Z}(T)_{\mathbb{C}} := \mathcal{Z}(T) \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathbb{C}$, etc.. By a (p, q) -current on $\mathcal{M}_{\mathbb{C}}$, we mean a system of currents $g = (g_{K'_f})_{K'_f} = (\Omega_c^{n-1-p, n-1-q}(\mathcal{M}_{K'_f, \mathbb{C}}) \rightarrow \mathbb{C})_{K'_f}$ compatible with pullback of currents as we vary K'_f among all small levels. We say a (p, q) -current on $\mathcal{M}_{\mathbb{C}}$ is *real* if the associated current at each small level K'_f is real. If g is a current of top degree on \mathcal{M} its *integral* is defined as

$$\int_{\mathcal{M}_{\mathbb{C}}} g := \frac{1}{[K'_{L,f} : K'_f]} \int_{\mathcal{M}_{K'_f, \mathbb{C}}} g_{K'_f} \quad (3.2.2)$$

for any sufficiently small level K'_f (conditional on convergence). This definition does not depend on the choice of small level.

Suppose $g_{T,y}$ is any real $(m-1, m-1)$ current on $\mathcal{M}_{\mathbb{C}}$ which satisfying a *modified current equation*, i.e. such that

$$-\frac{1}{2\pi i} \partial \bar{\partial} g_{T,y} + \delta_{\mathcal{Z}(T)_{\mathbb{C}}} \wedge [c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^{\vee})^{m-\text{rank}(T)}] \quad (3.2.3)$$

is (represented by) a smooth (m, m) -form (for all small levels K'_f), where $c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^{\vee})$ is the Chern form of the Hermitian line bundle $\widehat{\mathcal{E}}_{\mathbb{C}}^{\vee}$. We call $g_{T,y}$ a *Green current*. We typically write $g_{T,y}$ instead of g_{T,y,K'_f} to lighten notation, for understood level K'_f .

For each small level K'_f , pick a representative (\mathcal{Z}_0, g_0) for the self-intersection arithmetic cycle class $\widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{m-\text{rank}(T)} \in \widehat{\text{Ch}}^{m-\text{rank}(T)}(\mathcal{M}_{K'_f})_{\mathbb{Q}}$. We can assume that \mathcal{Z}_0 intersects $\mathcal{Z}(T)_{K'_f}$ properly in the generic fiber (moving lemma) and that g_0 is a Green form of logarithmic type for \mathcal{Z}_0 (in the sense of [GS90, §4]).

The intersection pairing for Chow groups with supports (as in [GS90, §4]) gives a class $\mathcal{Z}(T)_{K'_f, \mathcal{H}}$. $\mathcal{Z}_0 \in \text{Ch}_{\mathcal{Z}(T)_{K'_f, \mathcal{H}} \cap \mathcal{Z}_0}^m(\mathcal{M}_{K'_f})_{\mathbb{Q}}$. We set

$$[\widehat{\mathcal{Z}}(T)_{K'_f, \mathcal{H}}] := [(\mathcal{Z}(T)_{K'_f, \mathcal{H}} \cdot \mathcal{Z}_0, g_{T,y} + g_0 \wedge \delta_{\mathcal{Z}(T)_{K'_f, \mathbb{C}}})] \in \widehat{\text{Ch}}^m(\mathcal{M}_{K'_f}[1/N])_{\mathbb{Q}} \quad (3.2.4)$$

(where we have suppressed the $1/N$ notation from the left). As in [GS19, (5.158)], a short computation (using well-definedness of arithmetic intersection products) shows that this class does not depend on the choice of (\mathcal{Z}_0, g_0) . One can verify that $g_{T,y} + g_0 \wedge \delta_{\mathcal{Z}(T)_{K'_f, \mathbb{C}}}$ satisfies a Green current equation for $(\mathcal{Z}(T)_{K'_f} \cap \mathcal{Z}_0)_{\mathbb{C}}$ by combining the Green current equation for g_0 with the modified current equation of $g_{T,y}$ (see also [GS19, §5.4]).

These classes $[\widehat{\mathcal{Z}}(T)_{K'_f, \mathcal{H}}]$ thus form a compatible system as K'_f varies, and hence give an element

$$[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}] := ([\widehat{\mathcal{Z}}(T)_{K'_f, \mathcal{H}}])_{K'_f} \in \widehat{\text{Ch}}^m(\mathcal{M}[1/N])_{\mathbb{Q}}. \quad (3.2.5)$$

This construction of $[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}]$ is essentially that of [GS19, §5.4]. If $g_{T,y} = g_{t\bar{\gamma}T\gamma, \gamma^{-1}y^{t\bar{\gamma}-1}}$, note that we automatically have the “linear invariance” equality

$$[\widehat{\mathcal{Z}}(T)_{\mathcal{H}}] = [\widehat{\mathcal{Z}}(t\bar{\gamma}T\gamma)_{\mathcal{H}}]. \quad (3.2.6)$$

Currents $g_{T,y}$ satisfying (3.2.3) were studied by Garcia–Sankaran [GS19, Theorem 1.1 and §4]. We choose to use the star-product approach of Kudla [Kud97b] to define currents $g_{T,y}$ for our main results (for $\text{rank } T \geq n-1$ or $\det T \neq 0$ with T not positive definite). The local version was described in [Che24b, Section 2.4] (on the Hermitian symmetric domain) and the global version $g_{T,y}$ (descended from the local one via uniformization) is discussed in Section 5.4 (5.4.3). Our definition of $g_{T,y}$ is that of [Liu11, Theorem 4.20] in the nonsingular cases. When $T \in \text{Herm}_n(\mathbb{Q})$ is singular with $\text{rank}(T) = n-1$, our definition is new (still based on star products). These Green currents satisfy $g_{T,y} = g_{t\bar{\gamma}T\gamma, \gamma^{-1}y^{t\bar{\gamma}-1}}$ (Section 5.4), so linear invariance (3.2.6) is satisfied.

3.3. Arithmetic degrees without boundary contributions. The moduli stack $\mathcal{M} \rightarrow \text{Spec } \mathcal{O}_F[1/d_L]$ may not be proper. For a robust arithmetic degree theory via arithmetic Chow groups for arbitrary $T \in \text{Herm}_m(\mathbb{Q})$, one might instead consider arithmetic special cycle classes on a suitably compactified moduli space.

If the special cycle $\mathcal{Z}(T)$ is already proper over $\text{Spec } \mathcal{O}_F[1/d_L]$, we defined the *arithmetic degree without boundary contributions* which should result from any reasonable compactification [Che24a,

(1.3.4)]. For the reader's convenience, we recall that the definition is

$$\begin{aligned} \widehat{\deg}([\widehat{\mathcal{Z}}(T)] \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{n-m}) &:= \left(\int_{\mathcal{M}_{\mathbb{C}}} g_{T,y} \wedge c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^\vee)^{n-m} \right) \\ &+ \widehat{\deg}((\widehat{\mathcal{E}}^\vee)^{n-\text{rank}(T)}|_{\mathcal{Z}(T),\mathcal{H}}) \\ &+ \sum_{\substack{p \text{ prime} \\ p \nmid d_L}} \deg_{\mathbb{F}_p}(\mathbb{L}\mathcal{Z}(T)_{\mathcal{Y},p} \cdot (\mathcal{E}^\vee)^{n-m}) \log p \end{aligned} \quad (3.3.1)$$

conditional on convergence of the integral. Since compactification of \mathcal{M} plays no other role in our work, we take this approach. As in Section 3.2, the notation $\mathcal{M}_{\mathbb{C}}$ mean $\mathcal{M} \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathbb{C}$ for a choice of $F \rightarrow \mathbb{C}$ (the choice does not matter).

The quantity in (3.3.1) is an element of $\mathbb{R}_{d_L} = \mathbb{R}/(\sum_{p|d_L} \mathbb{Q} \cdot \log p)$. Here $\widehat{\deg}((\widehat{\mathcal{E}}^\vee)^{n-m}|_{\mathcal{Z}(T),\mathcal{H}})$ is the height of $\mathcal{Z}(T)_{\mathcal{H}}$ with respect to the metrized tautological bundle $\widehat{\mathcal{E}}^\vee$ (Section 3.1). The symbol $\mathbb{L}\mathcal{Z}(T)_{\mathcal{Y},p} \cdot (\mathcal{E}^\vee)^{n-m}$ is shorthand for the intersection product

$$\mathbb{L}\mathcal{Z}(T)_{\mathcal{Y},p} \cdot ([\mathcal{O}_{\mathcal{M}}] - [\mathcal{E}])^{n-m} \in \text{gr}_{\mathcal{M}}^n K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}} = \text{gr}_0 K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}}, \quad (3.3.2)$$

defined in [Che24a, Lemma A.2.1]. With $\mathcal{Z}(T)_{\mathbb{F}_p}$ viewed as a proper scheme over \mathbb{F}_p , the notation $\deg_{\mathbb{F}_p}$ refers to the degree map $\deg: \text{gr}_0 K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}} \rightarrow \mathbb{Q}$ as defined in [Che24a, (A.1.12)].

Certainly $\mathcal{Z}(T) \rightarrow \text{Spec } \mathcal{O}_F[1/d_L]$ is proper if $\mathcal{Z}(T)$ is empty (e.g. if T is not positive semidefinite). In this case, the right-hand side of (3.3.1) consists only of the integral $\int_{\mathcal{M}_{\mathbb{C}}} g_{T,y} \wedge c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^\vee)^{n-m}$ (and we can lift the intersection number to \mathbb{R} rather than \mathbb{R}_{d_L} , i.e. take the value of the integral).

In the rest of Section 3.3, we show that $\mathcal{Z}(T) \rightarrow \text{Spec } \mathcal{O}_F[1/d_L]$ is also proper if $\text{rank}(T) \geq n-1$, so (3.3.1) applies in this case as well.

Lemma 3.3.1. *Fix a Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$ with $\text{rank}(T) \geq n-1$ and $m \geq 0$. Let κ be a field, and consider $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x}) \in \mathcal{Z}(T)(\kappa)$. There exists an \mathcal{O}_F -linear isogeny $(A_0)^{n-1} \times A^- \rightarrow A$ where A^- is an elliptic curve with \mathcal{O}_F -action. After replacing κ by a finite extension, we may take $A^- = A_0^\sigma$ where $A_0^\sigma = A_0$ but with \mathcal{O}_F -action $\iota_0 \circ \sigma$.*

Proof. Write $\underline{x} = [x_1, \dots, x_m]$, where the x_i are \mathcal{O}_F -linear homomorphisms $x_i: A_0 \rightarrow A$. Since T has $\text{rank} \geq n-1$, we may assume (rearranging the elements x_i if necessary) that $\underline{x}^\flat = [x_1, \dots, x_{n-1}]$ has nonsingular Gram matrix $(\underline{x}^\flat, \underline{x}^\flat)$. Then the map

$$f: A \xrightarrow{\sqrt{\Delta} \circ (x_1^\dagger \times \dots \times x_{n-1}^\dagger)} A_0^{n-1} \quad (3.3.3)$$

is a homomorphism and a surjection of fppf sheaves. We form the ‘‘isogeny complement’’ in the standard way, i.e. we let A^- be the reduced connected component of $\ker f$. If $j: A^- \rightarrow A$ is the natural inclusion, then the map $(A_0)^{n-1} \times A^- \xrightarrow{x_1 \times \dots \times x_{n-1} \times j} A$ is an \mathcal{O}_F -linear isogeny.

Note that A^- is \mathcal{O}_F -linearly isogenous to an elliptic curve with \mathcal{O}_F action of signature $(0, 1)$: if $\text{char}(k) = p > 0$ with p nonsplit in \mathcal{O}_F , then A^- is supersingular, so apply Skolem–Noether to $\text{End}(A^-) \otimes \mathbb{Q}$; otherwise, A^- automatically has signature $(0, 1)$ because A has signature $(n-1, 1)$.

If κ is algebraically closed, any two elliptic curves over κ with \mathcal{O}_F -action of the same signature are \mathcal{O}_F -linearly isogenous. This is classical: lift to characteristic zero to reduce to $\kappa = \mathbb{C}$ (the moduli stack $\mathcal{M}_0 \rightarrow \text{Spec } \mathcal{O}_F$ is étale; more classically, see Deuring [Deu41]); recall that elliptic

curves over \mathbb{C} with \mathcal{O}_F -action are defined and isogenous over $\overline{\mathbb{Q}}$. By a standard limiting argument, we conclude that A^- and A_0^σ are \mathcal{O}_F -linearly isogenous over a finite extension of the (not necessarily algebraically closed) original field κ . \square

Remark 3.3.2. If p is a prime which splits in \mathcal{O}_F and if $\text{rank}(T) \geq n$, then $\mathcal{Z}(T)_{\mathbb{F}_p} = \emptyset$. This is a standard argument (e.g. [KR14, Lemma 2.21]): if κ is a field of characteristic p and $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x}) \in \mathcal{Z}(T)(\kappa)$, arguing as in Lemma 3.3.1 shows that A is \mathcal{O}_F -linearly isogenous to A_0^n . This contradicts Lemma 3.3.1, because there is no nonzero \mathcal{O}_F -linear map $A_0 \rightarrow A_0^\sigma$ as A_0 and A_0^σ have \mathcal{O}_F -action of opposite signature (e.g. there are no nonzero maps of the underlying ordinary p -divisible groups).

We say a characteristic $p > 0$ geometric point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ of \mathcal{M} lies in the *supersingular locus* if A_0 and A are supersingular abelian varieties (i.e. the associated p -divisible groups are supersingular). The following corollary also holds in arbitrary signature $(n - r, r)$ (i.e. all but the last sentence of Lemma 3.3.1 is valid for arbitrary signature $(n - r, r)$).

Corollary 3.3.3. *Let p be a prime which is nonsplit in \mathcal{O}_F . Fix $T \in \text{Herm}_m(\mathbb{Q})$ with $\text{rank}(T) \geq n - 1$ and $m \geq 0$. The morphism $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p} \rightarrow \mathcal{M}_{\overline{\mathbb{F}}_p}$ factors (set-theoretically) through the supersingular locus on $\mathcal{M}_{\overline{\mathbb{F}}_p}$.*

Proof. Follows from Lemma 3.3.1 and Deuring's classical results on endomorphisms of elliptic curves in positive characteristic [Deu41] (i.e. over a field of characteristic $p > 0$, the p -divisible group of an elliptic curve with \mathcal{O}_F -action is supersingular (resp. ordinary) if p is nonsplit (resp. split) in \mathcal{O}_F). Here we used the notation $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p} := \mathcal{Z}(T) \times_{\text{Spec } \mathbb{Z}} \text{Spec } \overline{\mathbb{F}}_p$ and similarly for \mathcal{M} . \square

Lemma 3.3.4. *Fix $T \in \text{Herm}_m(\mathbb{Q})$ with $\text{rank}(T) \geq n - 1$ and $m \geq 0$. Then the horizontal special cycle $\mathcal{Z}(T)_{\mathcal{H}}$ is proper and quasi-finite over $\text{Spec } \mathcal{O}_F[1/d_L]$.*

Proof. By Lemma 2.3.5, we know the generic fiber $\mathcal{Z}(T)_{\mathcal{H}, F} \rightarrow \text{Spec } F$ is smooth of relative dimension 0. Hence each generic point of $\mathcal{Z}(T)_{\mathcal{H}}$ is the image of a map $\text{Spec } E \rightarrow \mathcal{Z}(T)$ for some number field E , corresponding to an object $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \underline{x}) \in \mathcal{Z}(T)(E)$. By Lemma 3.3.1, we know that A is isogenous to a product of elliptic curves with complex multiplication by \mathcal{O}_F . It is a classical result of Deuring that such elliptic curves have everywhere potentially good reduction, so A_0 and A have everywhere potentially good reduction [Deu41; ST68]. Enlarging E if necessary, we can thus extend $\text{Spec } E \rightarrow \mathcal{Z}(T)$ to a morphism $\text{Spec } \mathcal{O}_E[1/d_L] \rightarrow \mathcal{Z}(T)$ (the Néron mapping property ensures that the datum $(\iota_0, \lambda_0, \iota, \lambda, \underline{x})$ extends as well; the polarizations must extend to polarizations as in the proof of [FC90, Theorem 1.9]).

Hence each irreducible component of $\mathcal{Z}(T)_{\mathcal{H}}$ may be covered by a morphism $\text{Spec } \mathcal{O}_E[1/d_L] \rightarrow \mathcal{Z}(T)$ for some number field E . Since $\mathcal{Z}(T)$ is quasi-compact and separated, this implies that $\mathcal{Z}(T) \rightarrow \text{Spec } \mathcal{O}_F[1/d_L]$ is proper and quasi-finite. \square

Lemma 3.3.5. *For $m \geq 0$, suppose $T \in \text{Herm}_m(\mathbb{Q})$ has $\text{rank}(T) \geq n - 1$. Then the structure map $\mathcal{Z}(T) \rightarrow \text{Spec } \mathcal{O}_F[1/d_L]$ is proper.*

Proof. We already know that the horizontal part $\mathcal{Z}(T)_{\mathcal{H}}$ is proper over $\text{Spec } \mathcal{O}_F[1/d_L]$, so it suffices to check that every irreducible component of $\mathcal{Z}(T)$ in characteristic $p \nmid d_L$ is proper over $\text{Spec } \mathbb{F}_p$.

It is enough to check that $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p} \rightarrow \mathrm{Spec} \overline{\mathbb{F}}_p$ is proper by fpqc descent (e.g. use away-from- p level structure to replace $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p}$ with a finite cover by a scheme, then use fpqc descent for morphisms of schemes). It is enough to check properness of the map $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p, \mathrm{red}} \rightarrow \mathrm{Spec} \overline{\mathbb{F}}_p$ on reduced substacks (e.g. by local Noetherianity of these algebraic stacks, or because $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p} \rightarrow \mathrm{Spec} \overline{\mathbb{F}}_p$ is locally of finite type).

The supersingular locus on $\mathcal{M}_{\overline{\mathbb{F}}_p}$ is proper (follows from the proof of [Oor74, Theorem 1.1a]); and finiteness of the forgetful map $\mathcal{M}^{(d)} \rightarrow \mathcal{A}_{n,d}$ in Section 2.1). Properness of $\mathcal{Z}(T)_{\overline{\mathbb{F}}_p} \rightarrow \mathrm{Spec} \overline{\mathbb{F}}_p$ now follows from Corollary 3.3.3. \square

Part 2. Uniformization

In Sections 4 and 5, we use global notation as in Sections 2 and 3, e.g. F is an imaginary quadratic field extension of \mathbb{Q} with nontrivial involution $a \mapsto a^\sigma$ and discriminant Δ . The notation \mathbb{A}_f (resp. \mathbb{A}_f^p) will always denote the finite adèle ring (resp. finite adèle ring away from p) for \mathbb{Q} .

For all of Sections 4 and 5, let $L_0 := \mathcal{O}_F$ be the rank one Hermitian \mathcal{O}_F -lattice with pairing $(x, y) := x^\sigma y$. Let L be any non-degenerate Hermitian \mathcal{O}_F -lattice of rank n and signature $(n - r, r)$, with associated moduli stack \mathcal{M} (Definition 2.1.2 and surrounding discussion).

We fix some group-theoretic setup (common in the literature, e.g. [RSZ20; BHKRY20]). Set

$$\begin{aligned} V_0 &:= L_0 \otimes_{\mathcal{O}_F} F & V &:= L \otimes_{\mathcal{O}_F} F \\ G' &:= \{(g_0, g) \in GU(V_0) \times GU(V) : c(g_0) = c(g)\} \subseteq GU(V_0) \times GU(V) \end{aligned}$$

where $c: GU(V_0) \rightarrow \mathbb{G}_m$ and $c: GU(V) \rightarrow \mathbb{G}_m$ are the similitude characters. We use the shorthand

$$L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p \quad V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$$

and use similar notation for local versions of other Hermitian spaces. Given a tuple $\underline{x} \in V^m$, we write $\underline{x}_p \in V_p^m$ and $\underline{x}_{\mathbb{R}} \in V_{\mathbb{R}}^m$ and $\underline{x}_f \in (V \otimes_{\mathbb{Q}} \mathbb{A}_f)^m$ and $\underline{x}^p \in (V \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^m$ for the corresponding projections (and similarly for other Hermitian spaces).

There is an isomorphism

$$\begin{aligned} G' &\longrightarrow GU(V_0) \times U(V) \\ (g_0, g) &\longmapsto (g_0, g_0^{-1}g). \end{aligned} \tag{*}$$

To avoid potential confusion: whenever we write $(g_0, g) \in G'$, we mean $g_0 \in GU(V_0)$ and $g \in GU(V)$ with the same similitude factor.

We use factorizable open compact subgroups $K'_f = K_{0,f} \times K_f \subseteq G'(\mathbb{A}_f)$ as in Section 2.2, where $K_{0,f} \subseteq GU(V_0)(\mathbb{A}_f)$ and $K_f \subseteq U(V)(\mathbb{A}_f)$ (using also (*)).

Recall the moduli stack with level structure $\mathcal{M}_{K'_f}$ defined in Section 2.2. We do not require K'_f to be a small level for non-Archimedean uniformization (Section 4), so $\mathcal{M}_{K'_f}$ is allowed to be a stack. We will take small level to have manifolds rather than orbifolds in complex uniformization (Section 5).

Notation. In Sections 4 and 5, we implicitly fix an open compact subgroup $K'_f \subseteq G'(\mathbb{A}_f)$ as above. We abusively suppress K'_f from notation: we write

$$\mathcal{M} \quad \mathcal{Z}(T) \quad \mathbb{L}\mathcal{Z}(T) \quad \mathbb{L}\mathcal{Z}(T)_{\nu,p}$$

instead of $\mathcal{M}_{K'_f}$, $\mathcal{Z}(T)_{K'_f}$, $\mathbb{L}\mathcal{Z}(T)_{K'_f}$, $\mathbb{L}\mathcal{Z}(T)_{\nu,p,K'_f}$, etc..

For example, given a Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$ (with entries in F) and an appropriate scheme S , our notation entails

$$\mathcal{Z}(T)(S) = \{(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{\eta}_0, \tilde{\eta}, \underline{x}) \text{ over } S\}$$

where $(\tilde{\eta}_0, \tilde{\eta})$ is a K'_f level structure and $\underline{x} \in \text{Hom}_{\mathcal{O}_F}(A_0, A)^m$ satisfies $(\underline{x}, \underline{x}) = T$.

4. NON-ARCHIMEDEAN

Fix a prime p . If p is not inert, we assume the signature is $(n - r, r) = (n - 1, 1)$. We assume that $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is self-dual. If p is ramified, we assume n is even, L is self-dual (for the trace pairing), and $p \neq 2$.

In all cases, we assume the implicit level $K'_f = K_{0,f} \times K_f$ at p is

$$K_{0,p} = K_{L_0,p} \quad K_p = K_{L,p}. \tag{4.0.1}$$

Recall that these denote the stabilizers of L_0 and L , respectively.

Set $\mathcal{O}_{F_p} := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathcal{O}_{F,(p)} := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. We write \check{F}_p for the completion of the maximal unramified extension of F_p if p is nonsplit (resp. $\check{F}_p := \check{\mathbb{Q}}_p$ if p is split, with a choice of morphism $F_p \rightarrow \check{F}_p$). In all cases, $\mathcal{O}_{\check{F}_p}$ (resp. \bar{k}) will denote the ring of integers (resp. residue field) of \check{F}_p .

We discuss Rapoport–Zink uniformization [RZ96, §6], as applied to supersingular loci on special cycles by Kudla–Rapoport [KR14, §5, §6] (there in the inert case, $p \neq 2$). For p inert or ramified, the material in Sections 4.1 to 4.6 is essentially a repackaging of Rapoport–Zink and Kudla–Rapoport. However, we need modified arguments at split places: the abelian varieties will be ordinary. We give a mostly uniform treatment for inert/ramified/split places. We also allow $p = 2$ if p is inert, except in Section 4.9.

Section 4.9 is the newest part of Section 4. Here, we explain how to use uniformization to reduce (global) Faltings or “tautological” heights to quantities expressed in terms of local special cycles and the “local change of heights” from our companion paper [Che24a, Sections 10 and 11] (with the main input being [Che24a, Corollary 11.2.2]).

Section 4.7 is the next newest part of Section 4. We use global special cycles and an “approximation” argument to prove certain properties of local special cycles. Some of these results are available or implicit in the literature (for p nonsplit, sometimes with $p \neq 2$ hypotheses and signature $(n - 1, 1)$ hypotheses); we indicate this where relevant. Our methods of proof are different, based on the approximation argument mentioned above.

Section 4.8 is the next newest part of Section 4. We explain how to reduce global “vertical intersection numbers” to local “vertical intersection numbers”.

Sections 4.7 to 4.9 will need detailed information on the construction of Rapoport–Zink uniformization. This is our other reason for giving an exposition of uniformization in Sections 4.1–4.6, as we need to explain the relevant maps (and fix notation) to give precise statements.

Sections 4.1 to 4.5 state the precise Rapoport–Zink uniformization map for special cycles. The proof of uniformization appears in Section 4.6 (and allows $p = 2$ inert). We differ slightly from [RZ96] by working directly with formal algebraic stacks (rather than requiring sufficient level structure) in the sense of [Eme20]. We occasionally need some notions on formal algebraic stacks which are not defined in [Eme20]; we will define these as needed.

Throughout Section 4, we freely use the relevant Rapoport–Zink spaces and their (Kudla–Rapoport) local special cycles as in [Che24a, Section 4].

4.1. Formal completion. Throughout Section 4, the notation T will always mean an $m \times m$ Hermitian matrix with F -coefficients, i.e. $T \in \text{Herm}_m(\mathbb{Q})$. If p is split in \mathcal{O}_F , we assume $\text{rank}(T) \geq n - 1$. Form the special cycle $\mathcal{Z}(T) \rightarrow \mathcal{M}$.

Suppose p is nonsplit. The *supersingular locus* on $\mathcal{Z}(T)_{\bar{k}} := \mathcal{Z}(T) \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \bar{k}$ is the subset $\mathcal{Z}(T)^{ss} \subseteq |\mathcal{Z}(T)_{\bar{k}}|$ of the underlying topological space¹³ consisting of geometric points $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{\eta}_0, \tilde{\eta}, \underline{x})$ with A supersingular. The supersingular locus $\mathcal{Z}(T)^{ss}$ is a closed subset of $|\mathcal{Z}(T)_{\bar{k}}|$ (by the Katz–Grothendieck theorem on specialization for Newton polygons). The *formal completion of $\mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}} := \mathcal{Z}(T) \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathcal{O}_{\tilde{F}_p}$ along its supersingular locus* is the (strictly full) substack $\check{\mathcal{Z}}(T) \subseteq \mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}$ given by

$$\check{\mathcal{Z}}(T) := \{\alpha \in \mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}(S) : \alpha(|S|) \subseteq \mathcal{Z}(T)^{ss}\} \quad (4.1.1)$$

for schemes S over $\text{Spec } \mathcal{O}_{\tilde{F}_p}$, where the condition $\alpha(|S|) \subseteq \mathcal{Z}(T)^{ss}$ means that the associated map on underlying topological spaces $|S| \rightarrow |\mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}|$ factors through $\mathcal{Z}(T)^{ss}$ (with $\alpha \in \mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}(S)$ “viewed” as a morphism $S \rightarrow \mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}$ by the 2-Yoneda lemma).

If p is split, we define

$$\check{\mathcal{Z}}(T) := \mathcal{Z}(T)_{\text{Spf } \mathcal{O}_{\tilde{F}_p}} := \mathcal{Z}(T) \times_{\text{Spec } \mathcal{O}_F} \text{Spf } \mathcal{O}_{\tilde{F}_p}. \quad (4.1.2)$$

This is also the formal completion of $\mathcal{Z}(T)_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}$ along its special fiber. For any geometric point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{\eta}_0, \tilde{\eta}, \underline{x})$ of $\check{\mathcal{Z}}(T)$, the abelian variety A is ordinary (because Lemma 3.3.1 implies A is isogenous to a product of elliptic curves with \mathcal{O}_F action).

In all cases, $\check{\mathcal{Z}}(T)$ is a locally Noetherian formal algebraic stack in the sense of [Eme20] (formal completion is discussed in [Eme20, Example 5.9]). The structure morphism $\check{\mathcal{Z}}(T) \rightarrow \text{Spf } \mathcal{O}_{\tilde{F}_p}$ is formally smooth,¹⁴ formally locally of finite type,¹⁵ separated, and quasi-compact. If K'_f is a small level, then $\check{\mathcal{Z}}(T)$ is a locally Noetherian formal scheme.

If $\mathcal{M} = \mathcal{Z}(T)$ (e.g. $T = \emptyset$ or $T = 0$), we set $\check{\mathcal{M}} := \check{\mathcal{Z}}(T)$. If p is nonsplit, this is the *formal completion of $\mathcal{M}_{\text{Spec } \mathcal{O}_{\tilde{F}_p}}$ along its supersingular locus \mathcal{M}^{ss}* .

4.2. Local special cycles away from p . Given an m -tuple $\underline{x}^p = [x_1, \dots, x_m] \in (V \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^m$, we consider an “*away-from- p* ” *local special cycle*

$$\mathcal{Z}'(\underline{x}^p) := \{(g_0, g) : G'(\mathbb{A}_f^p)/K_f'^p : g^{-1}g_0x_i \in L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \text{ for all } x_i \in \underline{x}^p\}. \quad (4.2.1)$$

¹³By the *underlying topological space* $|\mathcal{X}|$ of a formal algebraic stack \mathcal{X} , we mean the underlying topological space of its reduced substack \mathcal{X}_{red} . As \mathcal{X}_{red} is an algebraic stack, it has an underlying topological space in the sense of [SProject, Section 04XE].

¹⁴Given a morphism f of categories fibered in groupoids over some base scheme, there is a category of dotted arrows [SProject, Definition 0H18] associated to the infinitesimal lifting problem along each square-zero thickening of affine schemes. We say that f is *formally smooth* (resp. *formally étale*) (resp. *formally unramified*) if each such category of dotted arrows is nonempty (resp. a setoid with exactly one isomorphism class) (resp. either empty or a setoid with exactly one isomorphism class).

¹⁵We say a morphism of locally Noetherian formal algebraic stacks is *formally locally of finite type* if it is locally of finite type on underlying reduced substacks.

We often view $\mathcal{Z}'(\underline{x}^p)$ and $G'(\mathbb{A}_f^p)/K_f^p$ as constant formal schemes over $\mathrm{Spf} \mathcal{O}_{\check{F}_p}$. We also define the “*away-from- p* ” *local special cycle*

$$\mathcal{Z}(\underline{x}^p) := \{g : U(V)(\mathbb{A}_f^p)/K_f^p : g^{-1}x_i \in L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \text{ for all } x_i \in \underline{x}^p\}. \quad (4.2.2)$$

The isomorphism $G'(\mathbb{A}_f^p)/K_f^p \rightarrow GU(V_0)(\mathbb{A}_f^p)/K_{0,f}^p \times U(V)(\mathbb{A}_f^p)/K_f^p (*)$ induces an isomorphism

$$\mathcal{Z}'(\underline{x}^p) \xrightarrow{\sim} GU(V_0)(\mathbb{A}_f^p)/K_{0,f}^p \times \mathcal{Z}(\underline{x}^p). \quad (4.2.3)$$

4.3. Framing objects. To define the uniformization map, we fix an object $(\mathbf{A}_0, \iota_{\mathbf{A}_0}, \lambda_{\mathbf{A}_0}, \mathbf{A}, \iota_{\mathbf{A}}, \lambda_{\mathbf{A}}, \tilde{\eta}_0, \tilde{\eta}) \in \mathcal{M}(\bar{k})$ (“basepoint of the uniformization”). If p is nonsplit (resp. split), we assume \mathbf{A} is supersingular (resp. \mathbf{A} is \mathcal{O}_F -linearly isogenous to $\mathbf{A}_0^{n-r} \times (\mathbf{A}_0^\sigma)^r$); such data exists by Lemma 2.1.6. Let $(\mathbf{X}_0, \iota_{\mathbf{X}_0}, \lambda_{\mathbf{X}_0}, \mathbf{X}, \iota_{\mathbf{X}}, \lambda_{\mathbf{X}})$ be the tuple obtained by passing to p -divisible groups (e.g. \mathbf{X} is the p -divisible group of \mathbf{A}). We use this as the framing object over \bar{k} to define the Rapoport–Zink space \mathcal{N}' (as in [Che24a, Definition 4.1.8]). Set $\mathcal{N} := \mathcal{N}(n-r, r)$ (with the latter as in [Che24a, Definition 4.1.3]).

In the supersingular cases, the abelian variety \mathbf{A} is automatically \mathcal{O}_F -linearly isogenous to $\mathbf{A}_0^{n-r} \times (\mathbf{A}_0^\sigma)^r$, since

$$\mathrm{Hom}_F^0(\mathbf{A}_0^{n-r} \times (\mathbf{A}_0^\sigma)^r, \mathbf{A}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_F^0(\mathbf{X}_0^{n-r} \times (\mathbf{X}_0^\sigma)^r, \mathbf{X}) \quad (4.3.1)$$

by Tate’s isogeny theorem (for any supersingular abelian variety over a finite field, some power of Frobenius will be a power of p , e.g. by [RZ96, Lemma 6.28]); then use uniqueness of the framing object $(\mathbf{X}, \iota_{\mathbf{X}}, \lambda_{\mathbf{X}})$ up to isogeny (see [Che24a, Section 4.1]).

Since $\mathcal{M}_{\mathrm{Spec} \mathcal{O}_{\check{F}_p}} \rightarrow \mathrm{Spec} \mathcal{O}_{\check{F}_p}$ is smooth, this framing object (\mathbf{A}_0, \dots) admits a lift $(\mathfrak{A}_0, \iota_{\mathfrak{A}_0}, \lambda_{\mathfrak{A}_0}, \mathfrak{A}, \iota_{\mathfrak{A}}, \lambda_{\mathfrak{A}}, \tilde{\eta}_0, \tilde{\eta}) \in \mathcal{M}(\mathrm{Spf} \mathcal{O}_{\check{F}_p})$, which we also fix. We fix representatives

$$\eta_0 : T^p(\mathbf{A}_0) \xrightarrow{\sim} L_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \quad \eta : \mathrm{Hom}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p}(T^p(\mathbf{A}_0), T^p(\mathbf{A})) \xrightarrow{\sim} L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \quad (4.3.2)$$

for the K_0^p -orbit $\tilde{\eta}_0$ and the K^p -orbit $\tilde{\eta}$ (see Section 2.2). Recall that η preserves Hermitian pairings but η_0 need not. We also write

$$\eta_0 : T^p(\mathfrak{A}_0) \xrightarrow{\sim} L_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \quad \eta : \mathrm{Hom}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p}(T^p(\mathfrak{A}_0), T^p(\mathfrak{A})) \xrightarrow{\sim} L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \quad (4.3.3)$$

for the identifications induced by η_0 and η .

We define Hermitian F -modules

$$\mathbf{W} := \mathrm{Hom}_F^0(\mathbf{A}_0, \mathbf{A}) \quad \mathbf{W}^\perp := \begin{cases} \mathrm{Hom}_F^0(\mathbf{A}_0^\sigma, \mathbf{A}) & \text{if } p \text{ is split} \\ 0 & \text{if } p \text{ is nonsplit} \end{cases} \quad (4.3.4)$$

$$\mathbf{V}_0 := \mathrm{Hom}_F^0(\mathbf{A}_0, \mathbf{A}_0) \quad \mathbf{V} := \mathbf{W} \oplus \mathbf{W}^\perp \quad (4.3.5)$$

where the direct sum defining \mathbf{V} is orthogonal. In all cases, the Hermitian pairing is $(x, y) := x^\dagger y \in F$. All of these Hermitian spaces are positive definite (positivity of the Rosati involution).

The canonical maps

$$\mathbf{W} \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{F_p}^0(\mathbf{X}_0, \mathbf{X}) \quad \mathbf{W}^\perp \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}_{F_p}^0(\mathbf{X}_0^\sigma, \mathbf{X}) \quad (4.3.6)$$

are isomorphisms of Hermitian spaces. In the nonsplit (hence supersingular) cases, this follows from Tate’s isogeny theorem as above. In the split case, this follows because \mathbf{A} is \mathcal{O}_F -linearly

isogenous to $\mathbf{A}_0^{n-r} \times (\mathbf{A}_0^\sigma)^r$. In particular, the local invariant is $\varepsilon(\mathbf{W}_p) = (-1)^r$ if p is nonsplit (resp. $\varepsilon(\mathbf{W}_p) = 1$ if p is split).

If p is nonsplit, the natural map

$$\mathbf{W} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \rightarrow \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}_f^p}(T^p(\mathbf{A}_0)^0, T^p(\mathbf{A})^0) \quad (4.3.7)$$

is an isomorphism of Hermitian spaces, by similar reasoning.

If p is split, any \mathcal{O}_F -linear isogeny $\mathbf{A}_0^{n-1} \times \mathbf{A}_0^\sigma \rightarrow \mathbf{A}$ defines an F -linear orthogonal decomposition

$$T^p(\mathbf{A})^0 = T^p(\mathbf{A}_0^{n-1})^0 \oplus T^p(\mathbf{A}_0^\sigma)^0. \quad (4.3.8)$$

This decomposition is independent of the choice of isogeny because $\mathrm{Hom}_F^0(\mathbf{A}_0, \mathbf{A}_0^\sigma) = \mathrm{Hom}_F^0(\mathbf{A}_0^\sigma, \mathbf{A}_0) = 0$ (e.g. because $\mathrm{End}^0(\mathbf{A}_0) = F$). Then the natural map

$$\mathbf{W} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \rightarrow \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}_f^p}(T^p(\mathbf{A}_0)^0, T^p(\mathbf{A}_0^{n-1})^0) \quad (4.3.9)$$

is an isomorphism of Hermitian spaces.

Given a tuple $\underline{\mathbf{x}} \in \mathbf{W}^m$, we write

$$\underline{\mathbf{x}}_p \in \mathbf{W}_p^m = \mathrm{Hom}_{F_p}^0(\mathbf{X}_0, \mathbf{X}) \quad (4.3.10)$$

$$\underline{\mathbf{x}}^p \in \mathbf{W}^m \otimes_{\mathbb{Q}} \mathbb{A}_f^p \subseteq \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}_f^p}(T^p(\mathbf{A}_0)^0, T^p(\mathbf{A})^0)^m = V^m \otimes_{\mathbb{Q}} \mathbb{A}_f^p \quad (4.3.11)$$

for the respective images of $\underline{\mathbf{x}}$ (using $\boldsymbol{\eta}$ for the identification with $V^m \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ in the second line).

4.4. Framed stack. We consider the stack $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$ over $\mathrm{Spf} \mathcal{O}_{\check{F}_p}$, given by

$$\check{\mathcal{Z}}(T)_{\mathrm{framed}}(S) := \left\{ (\alpha, \phi_0, \phi) : \begin{array}{l} \alpha = (A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{\eta}_0, \tilde{\eta}, \underline{\mathbf{x}}) \in \check{\mathcal{Z}}(T)(S) \\ \phi_0: A_0 \rightarrow \mathfrak{A}_{0,S} \text{ and } \phi: A \rightarrow \mathfrak{A}_S \text{ quasi-isogenies} \\ \text{such that } \phi_0^* \lambda_{\mathfrak{A}_{0,S}} = b \lambda_0 \text{ and } \phi^* \lambda_{\mathfrak{A},S} = b \lambda \\ \text{for some } b \in \mathbb{Q}_{>0} \end{array} \right\}$$

for schemes S over $\mathrm{Spf} \mathcal{O}_{\check{F}_p}$. The similitude factor b is allowed to vary (and is only required to be locally constant). If $\mathcal{M} = \mathcal{Z}(T)$, we set $\check{\mathcal{M}}_{\mathrm{framed}} := \check{\mathcal{Z}}(T)_{\mathrm{framed}}$. There is a canonical forgetful map

$$\Theta: \check{\mathcal{Z}}(T)_{\mathrm{framed}} \rightarrow \check{\mathcal{Z}}(T) \quad (4.4.1)$$

sending $(\alpha, \phi_0, \phi) \mapsto \alpha$. This will be the uniformization map (Section 4.6).

There is a canonical isomorphism

$$\check{\mathcal{Z}}(T)_{\mathrm{framed}} \xrightarrow{\sim} \coprod_{\substack{\underline{\mathbf{x}} \in \mathbf{W}^m \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} \mathcal{Z}'(\underline{\mathbf{x}}_p) \times \mathcal{Z}'(\underline{\mathbf{x}}^p) \quad (4.4.2)$$

which we now describe. Here $\mathcal{Z}'(\underline{\mathbf{x}}_p)$ is the local special cycle at p from [Che24a, Section 4.2], and $\mathcal{Z}'(\underline{\mathbf{x}}^p)$ is the away-from- p local special cycle from Section 4.2.

Consider $(\alpha, \phi_0, \phi) \in \check{\mathcal{Z}}(T)_{\mathrm{framed}}(S)$ as above. Passing to p -divisible groups gives a datum $(X_0, \iota_0, \lambda_0, X, \iota, \lambda)$ (e.g. X is the p -divisible group of A), along with a framing quasi-isogeny $\rho: X_{\overline{S}} \rightarrow \mathbf{X}_{\overline{S}}$ induced by ϕ (where $\overline{S} := S_{\check{k}}$) and similarly a framing ρ_0 induced by ϕ_0 . We

also obtain $g_0 := \eta_0 \circ \phi_{0,*} \circ \tilde{\eta}_0^{-1} \in G_0(\mathbb{A}_f^p)/K_{0,f}^p$ and $g := \eta \circ (\phi_0^{-1,*} \phi_*) \circ \tilde{\eta}^{-1} \in U(V)(\mathbb{A}_f^p)/K_f^p$ where $\phi_{0,*}: T^p(A_0)^0 \rightarrow T^p(\mathfrak{A}_0)^0$ and

$$\phi_0^{-1,*} \phi_*: \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}_f^p}(T^p(A_0)^0, T^p(A)^0) \rightarrow \mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}_f^p}(T^p(\mathfrak{A}_0)^0, T^p(\mathfrak{A})^0) \quad (4.4.3)$$

is pre- and post-composition (when S is connected, pick any geometric point; there is no dependence on this choice). In general, g_0 and g will be locally constant elements. For any $\underline{x} \in \mathrm{Hom}_F(A_0, A)^m$ over a connected base S , we have $\phi \circ \underline{x} \circ \phi_0^{-1} \in \mathbf{W}^m$ (canonically), by Mumford's rigidity lemma for morphisms of abelian schemes [MFK94, Corollary 6.2]. In the not-necessarily connected case, we obtain a locally constant element of \mathbf{W}^m .

The above constructions give a map

$$\begin{aligned} \check{\mathcal{Z}}(T)_{\mathrm{framed}} &\longrightarrow \mathcal{N}' \times G'(\mathbb{A}_f^p)/K_f'^p \times \mathbf{W}^m \\ (\alpha, \phi_0, \phi) &\longmapsto ((X_0, \iota_0, \lambda_0, \rho_0, X, \iota, \lambda, \rho), (g_0, g_0g), \phi \circ \underline{x} \circ \phi_0^{-1}) \end{aligned} \quad (4.4.4)$$

which induces an isomorphism from $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$ to the open and closed subfunctor

$$\coprod_{\substack{\underline{\mathbf{x}} \in \mathbf{W}^m \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} \mathcal{Z}'(\underline{\mathbf{x}}_p) \times \mathcal{Z}'(\underline{\mathbf{x}}^p) \hookrightarrow \mathcal{N}' \times G'(\mathbb{A}_f^p)/K_f'^p \times \mathbf{W}^m. \quad (4.4.5)$$

One can verify that the map in (4.4.2) is an isomorphism by decomposing the kernels of framing quasi-isogenies (rescale to obtain an isogeny) into their p -power and ℓ -power torsion subgroups. The isomorphism implies that $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$ is a locally Noetherian formal scheme.

4.5. Quotient. Consider the algebraic groups

$$I_0 := GU(\mathbf{V}_0) \quad I_1 := U(\mathbf{W}) \times U(\mathbf{W}^\perp) \quad I' := I_0 \times I_1 \quad (4.5.1)$$

over \mathbb{Q} . Unless specified otherwise, an element $(\gamma_0, \gamma) \in I'$ will mean $\gamma_0 \in I_0$ and $\gamma \in GU(\mathbf{W}) \times GU(\mathbf{W}^\perp)$ with $\gamma_0^{-1}\gamma \in I_1$.

Uniformization will involve the stack quotient $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]$ for an action of $I'(\mathbb{Q})$ on $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$, which we now describe. For \mathbb{Q} -algebras R , there are canonical identifications

$$I_0(R) = \{\gamma_0 \in \mathrm{End}_F^0(\mathbf{A}_0) \otimes_{\mathbb{Q}} R : \gamma_0^\dagger \gamma_0 \in R^\times\} \quad (4.5.2)$$

$$I_1(R) = \{\gamma \in \mathrm{End}_F^0(\mathbf{A}) \otimes_{\mathbb{Q}} R : \gamma^\dagger \gamma = 1\} \quad (4.5.3)$$

(act on \mathbf{V}_0 and \mathbf{V} by post-composition). View $I'(\mathbb{Q})$ as a discrete group. Then $(\gamma_0, \gamma) \in I'(\mathbb{Q})$ acts on $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$ as $(\alpha, \phi_0, \phi) \mapsto (\alpha, \gamma_0 \circ \phi_0, \gamma \circ \phi)$. We are abusing notation: the elements γ_0 and γ lift (uniquely, by Mumford's rigidity lemma or Drinfeld rigidity and Serre–Tate) to quasi-endomorphisms of $\mathfrak{A}_{0,S}$ and \mathfrak{A}_S respectively.

In terms of the isomorphism in (4.4.2), the action of $I'(\mathbb{Q})$ on $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$ admits the following (equivalent) description. By the isomorphism in (4.3.6), the group $I'(\mathbb{Q}_p)$ acts on \mathcal{N}' (discussed in [Che24a, Section 4.3]). By (4.5.3), we have a faithful action of $I(\mathbb{A}_f^p)$ on

$$\mathrm{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{A}_f^p}(T^p(\mathbf{A}_0)^0, T^p(\mathbf{A})^0) = V \otimes_{\mathbb{Q}} \mathbb{A}_f^p \quad (4.5.4)$$

by post-composition. This induces a homomorphism $I_1(\mathbb{A}_f^p) \rightarrow U(V)(\mathbb{A}_f^p)$ and hence an action of $I'(\mathbb{A}_f^p)$ on $G'(\mathbb{A}_f^p)/K_f'^p$ (left multiplication). The group $I'(\mathbb{Q})$ also acts on \mathbf{W} by the projection $I'(\mathbb{Q}) \rightarrow U(\mathbf{W})$.

Hence $I'(\mathbb{Q})$ acts on

$$\mathcal{N}' \times G'(\mathbb{A}_f^p)/K_f'^p \times \mathbf{W}^m. \quad (4.5.5)$$

Under the inclusion (4.4.5), this induces the same action on $\check{\mathcal{Z}}(T)_{\text{framed}}$ described previously. Both descriptions will be useful for us.

We now form the (fppf) stack quotient

$$\coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} \mathcal{Z}'(\mathbf{x}_p) \times \mathcal{Z}'(\mathbf{x}^p) \rightarrow \left[I'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} \mathcal{Z}'(\mathbf{x}_p) \times \mathcal{Z}'(\mathbf{x}^p) \right) \right] \quad (4.5.6)$$

The left-hand side is a locally Noetherian formal scheme, and the right-hand side is a locally Noetherian formal algebraic stack which is formally locally of finite type over $\text{Spf } \mathcal{O}_{\check{F}_p}$. The right-hand side is also $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\text{framed}}]$. The quotient map is representable by schemes, separated, étale, and surjective.

Using [Che24a, (4.2.5)] (and [Che24a, (4.4.4)]) and (4.2.3) (various incarnations of the isomorphism $G' \xrightarrow{\sim} GU(V_0) \times U(V)$) yields a canonical isomorphism from the left-hand side of (4.5.6) to

$$GU(V_0)/K_{0,f} \times \coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} \mathcal{Z}(\mathbf{x}_p) \times U(\mathbf{W}_p^\perp)/K_{1,\mathbf{L}_p^\perp} \times \mathcal{Z}(\mathbf{x}^p) \quad (4.5.7)$$

where $K_{1,\mathbf{L}_p^\perp} \subseteq U(\mathbf{W}_p^\perp)$ is the unique maximal open compact subgroup (since \mathbf{W}^\perp has rank 0 or 1). This is a disjoint union of various local special cycles $\mathcal{Z}(\mathbf{x}_p)$, indexed by the (discrete) set

$$J_p(T) := GU(V_0)/K_{0,f} \times \coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} U(\mathbf{W}_p^\perp)/K_{1,\mathbf{L}_p^\perp} \times \mathcal{Z}(\mathbf{x}^p). \quad (4.5.8)$$

In particular, every element $j \in J_p(T)$ defines a morphism

$$\Theta_j: \mathcal{Z}(\mathbf{x}_p) \rightarrow [I'(\mathbb{Q}) \backslash \mathcal{Z}(T)_{\text{framed}}] \quad (4.5.9)$$

which is étale, separated, and representable by schemes. Given $j \in J_p(T)$, we let $\text{Aut}(j) \subseteq I'(\mathbb{Q})$ be the stabilizer for the action of $I'(\mathbb{Q})$ on $J_p(T)$.

The right-hand side of (4.5.6) is then identified with

$$\left[GU(V_0)(\mathbb{Q}) \backslash \left(GU(V_0)(\mathbb{A}_f)/K_{0,f} \right) \right] \times \left[I_1(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} \mathcal{Z}(\mathbf{x}_p) \times U(\mathbf{W}_p^\perp)/K_{1,\mathbf{L}_p^\perp} \times \mathcal{Z}(\mathbf{x}^p) \right) \right]. \quad (4.5.10)$$

We have

$$\deg[GU(V_0)(\mathbb{Q}) \backslash (GU(V_0)(\mathbb{A}_f)/K_{0,f})] = [K_{L_{0,f}} : K_{0,f}] \cdot h_F / |\mathcal{O}_F^\times| \quad (4.5.11)$$

where the left-hand side denotes (stacky) groupoid cardinality, where $[K_{L_0, f} : K_{0, f}]$ is the index of $K_{0, f}$ in $K_{L_0, f}$, and h_F is the class number of \mathcal{O}_F . In the case where $\text{rank}(T) \geq n - 1$ (we have already assumed $\text{rank}(T) \geq n - 1$ if p is split), the groupoid

$$\left[I_1(\mathbb{Q}) \backslash \left(\prod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x}) = T}} U(\mathbf{W}_p^\perp) / K_{1, \mathbf{L}_p^\perp} \times \mathcal{Z}(\mathbf{x}^p) \right) \right] \quad (4.5.12)$$

has finite automorphism groups and finitely many isomorphism classes, and its groupoid degree is essentially a product of special values of local Whittaker functions away from p [Che24c, Lemma 7.4.1].

In the case $\text{rank}(T) \geq n - 1$, the map Θ_j associated with any $j \in J_p(T)$ is thus representable by schemes and finite étale of constant degree $\deg \Theta_j = |\text{Aut}(j)|$.

4.6. Uniformization. We explain how the uniformization morphism $\Theta: \check{\mathcal{Z}}(T)_{\text{framed}} \rightarrow \check{\mathcal{Z}}(T)$ (4.4.1) descends to an isomorphism of locally Noetherian formal algebraic stacks

$$\tilde{\Theta}: \left[I'(\mathbb{Q}) \backslash \left(\prod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x}) = T}} \mathcal{Z}'(\mathbf{x}_p) \times \mathcal{Z}'(\mathbf{x}^p) \right) \right] \xrightarrow{\sim} \check{\mathcal{Z}}(T). \quad (4.6.1)$$

The main point is surjectivity on \bar{k} -points via the Hasse principle (Lemma 4.6.2).

When p is split, we will allow a change of choice of framing data $(\mathbf{A}_0, \iota_{\mathbf{A}_0}, \lambda_{\mathbf{A}_0}, \mathbf{A}, \iota_{\mathbf{A}}, \lambda_{\mathbf{A}}, \tilde{\eta}_0, \tilde{\eta})$, η_0 , and η , possibly depending on T .

Lemma 4.6.1. *The map $\Theta: \check{\mathcal{Z}}(T)_{\text{framed}} \rightarrow \check{\mathcal{Z}}(T)$ factors uniquely through a monomorphism¹⁶*

$$\tilde{\Theta}: [I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\text{framed}}] \rightarrow \check{\mathcal{Z}}(T) \quad (4.6.2)$$

of formal algebraic stacks. The map $\tilde{\Theta}$ is formally locally of finite type and formally étale.

Proof. Suppose (α, ϕ_0, ϕ) and $(\alpha', \phi'_0, \phi')$ are objects of $\check{\mathcal{Z}}(T)_{\text{framed}}(S)$, and suppose $f': \alpha \rightarrow \alpha'$ is an isomorphism of objects in the groupoid $\check{\mathcal{Z}}(T)(S)$ (for some base scheme S). We claim there is a unique $\gamma' = (\gamma_0, \gamma) \in I'(\mathbb{Q})$ such that f' induces an isomorphism $\gamma' \cdot (\alpha, \phi_0, \phi) \xrightarrow{\sim} (\alpha', \phi'_0, \phi')$ in the setoid $\check{\mathcal{Z}}(T)_{\text{framed}}(S)$.

The map f is given by a pair of isomorphisms $f_0: A_0 \rightarrow A'_0$ and $f: A \rightarrow A'$ (where $\alpha = (A_0, \dots)$ and $\alpha' = (A'_0, \dots)$, with notation as above). Then we take $\gamma_0 = \phi'_0 \circ f_0 \circ \phi_0^{-1}$ and $\gamma = \phi' \circ f \circ \phi^{-1}$. Hence $\tilde{\Theta}$ is a monomorphism.

The map $\tilde{\Theta}$ is a map between locally Noetherian formal algebraic stacks which are formally locally of finite type over $\text{Spf } \mathcal{O}_{\bar{F}_p}$, so $\tilde{\Theta}$ is formally locally of finite type.

The property of being formally étale may be checked “formally étale locally on the source”. The quotient map $\check{\mathcal{Z}}(T)_{\text{framed}} \rightarrow [I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\text{framed}}]$ is representable by schemes and formally étale, so it is enough to check that $\Theta: \check{\mathcal{Z}}(T)_{\text{framed}} \rightarrow \check{\mathcal{Z}}(T)$ is formally étale. This property amounts to the following rigidity statement for abelian schemes: given any first order thickening of schemes $T \rightarrow T'$ on which p is locally nilpotent, and given abelian schemes A_1 and A_2 over T' , any

¹⁶By a *monomorphism* of formal algebraic stacks, we mean a morphism which is fully faithful on underlying fibered categories.

quasi-homomorphism $A_{1,T} \rightarrow A_{2,T}$ lifts uniquely to a quasi-homomorphism $A_1 \rightarrow A_2$ (e.g. by Drinfeld rigidity and Serre–Tate). \square

Lemma 4.6.2. *The map $\Theta(\bar{k}): \check{\mathcal{Z}}(T)_{\text{framed}}(\bar{k}) \rightarrow \check{\mathcal{Z}}(T)(\bar{k})$ (on groupoids of \bar{k} -points) is surjective (resp. surjective for some choice of framing data) on isomorphism classes if p is nonsplit (resp. split).*

Proof. If $\check{\mathcal{Z}}(T)$ is empty, there is nothing to show, so assume $\check{\mathcal{Z}}(T)$ is nonempty. If p is split, we can change the framing object to assume it extends to $(\mathbf{A}_0, \iota_{\mathbf{A}_0}, \lambda_{\mathbf{A}_0}, \mathbf{A}, \iota_{\mathbf{A}}, \lambda_{\mathbf{A}}, \tilde{\eta}_0, \tilde{\eta}, \underline{\mathbf{x}}) \in \check{\mathcal{Z}}(T)(\bar{k})$ (i.e. $\underline{\mathbf{x}} \in \mathbf{W}^m$ with $(\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T$). This implies that T has rank $n-1$ if p is split (we already assumed $\text{rank}(T) \geq n-1$ if p is split, then see Remark 3.3.2). We still know that \mathbf{A} is \mathcal{O}_F -linearly isogenous to $\mathbf{A}_0^{n-1} \times \mathbf{A}_0^\sigma$ (Lemma 3.3.1).

In all cases, the task is to show that any $(\mathbf{A}'_0, \iota_{\mathbf{A}'_0}, \lambda_{\mathbf{A}'_0}, \mathbf{A}', \iota_{\mathbf{A}'}, \lambda_{\mathbf{A}'}, \tilde{\eta}_0, \tilde{\eta}, \underline{\mathbf{x}}') \in \check{\mathcal{Z}}(T)(\bar{k})$ admits a framing (ϕ_0, ϕ) , i.e. quasi-isogenies $\phi_0: \mathbf{A}'_0 \rightarrow \mathbf{A}_0$ and $\phi: \mathbf{A}' \rightarrow \mathbf{A}$ which preserve quasi-polarizations up to the same scalar in $\mathbb{Q}_{>0}$.

Fix any \mathcal{O}_F -linear isogeny $\phi_0: \mathbf{A}'_0 \rightarrow \mathbf{A}_0$, which exists because \mathbf{A}'_0 and \mathbf{A}_0 are elliptic curves with \mathcal{O}_F -action of the same signature (see the proof of Lemma 3.3.1). Let $b \in \mathbb{Q}_{>0}$ be such that $\phi_0^* \lambda_{\mathbf{A}_0} = b \lambda_{\mathbf{A}'_0}$. Set $\mathbf{W}' := \text{Hom}_F^0(\mathbf{A}'_0, \mathbf{A}')$ with the Hermitian pairing $(x, y) = x^\dagger y$.

Case p is nonsplit: There is an isomorphism of F vector spaces

$$\begin{aligned} \text{Hom}_F^0(\mathbf{A}', \mathbf{A}) &\longrightarrow \text{Hom}_F(\mathbf{W}', \mathbf{W}) \\ \phi &\longmapsto (f \mapsto \phi \circ f \circ \phi_0^{-1}). \end{aligned} \tag{4.6.3}$$

An element $\phi \in \text{Hom}_F^0(\mathbf{A}', \mathbf{A})$ satisfies $\phi^\dagger \phi = b$ if and only if ϕ corresponds to an isomorphism of Hermitian spaces $\mathbf{W}' \rightarrow \mathbf{W}$. But we have $\mathbf{W}' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathbf{W} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong V \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ as Hermitian spaces, we have $\varepsilon(\mathbf{W}') = \varepsilon(\mathbf{W}) = (-1)^r$, and we have $\mathbf{W}'_{\mathbb{R}} \cong \mathbf{W}_{\mathbb{R}}$ (both are positive definite of rank n). So we have $\mathbf{W}' \cong \mathbf{W}$ as Hermitian spaces, by the Hasse principle for Hermitian spaces (Landherr's theorem).

Case p is split: Fix \mathcal{O}_F -linear isogenies $\mathbf{B} \times \mathbf{B}^\perp \rightarrow \mathbf{A}$ and $\mathbf{B}' \times \mathbf{B}'^\perp \rightarrow \mathbf{A}'$, where $\mathbf{B} \cong \mathbf{A}_0^{n-1}$, $\mathbf{B}^\perp \cong \mathbf{A}_0^\sigma$, $\mathbf{B}' \cong \mathbf{A}'_0^{n-1}$, and $\mathbf{B}'^\perp \cong \mathbf{A}'_0^\sigma$. Equip $\mathbf{B} \times \mathbf{B}^\perp$ and $\mathbf{B}' \times \mathbf{B}'^\perp$ with the quasi-polarizations pulled back from $\lambda_{\mathbf{A}}$ and $\lambda_{\mathbf{A}'}$ on \mathbf{A} and \mathbf{A}' , respectively.

Any F -linear quasi-isogeny $\phi: \mathbf{A}' \rightarrow \mathbf{A}$ decomposes as a product of quasi-isogenies $\mathbf{B}' \rightarrow \mathbf{B}$ and $\mathbf{B}'^\perp \rightarrow \mathbf{B}^\perp$, since $\text{Hom}_F(\mathbf{B}', \mathbf{B}^\perp) = \text{Hom}_F(\mathbf{B}'^\perp, \mathbf{B}) = 0$ (because of the opposite signatures). We write $\phi^\perp: \mathbf{B}'^\perp \rightarrow \mathbf{B}^\perp$ for the quasi-isogeny induced by ϕ . By similar reasoning, the quasi-polarization on $\mathbf{B} \times \mathbf{B}^\perp$ is the product of a quasi-polarization on \mathbf{B} and a quasi-polarization on \mathbf{B}^\perp .

There is an isomorphism of F vector spaces

$$\begin{aligned} \text{Hom}_F^0(\mathbf{A}', \mathbf{A}) &\longrightarrow \text{Hom}_F(\mathbf{W}', \mathbf{W}) \times \text{Hom}_F^0(\mathbf{B}'^\perp, \mathbf{B}^\perp) \\ \phi &\longmapsto (f \mapsto (\phi \circ f \circ \phi_0^{-1}), \phi^\perp). \end{aligned} \tag{4.6.4}$$

An element $\phi \in \text{Hom}_F^0(\mathbf{A}', \mathbf{A})$ satisfies $\phi^\dagger \phi = b$ if and only if ϕ corresponds to an isomorphism of Hermitian spaces $\mathbf{W}' \rightarrow \mathbf{W}$ and with $\phi^{\perp \dagger} \phi^\perp = b$.

We have $\underline{\mathbf{x}}' \in \mathbf{W}'$ and $\underline{\mathbf{x}} \in \mathbf{W}$ with $(\underline{\mathbf{x}}', \underline{\mathbf{x}}') = (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T$. Since $\text{rank}(T) = \text{rank}(\mathbf{W}') = \text{rank}(\mathbf{W}) = n - 1$, this implies $\mathbf{W}' \cong \mathbf{W}$ as Hermitian spaces.

For every prime $\ell \neq p$, the natural map

$$\text{Hom}_F^0(\mathbf{B}'^\perp, \mathbf{B}^\perp) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow \text{Hom}_{F_\ell}(T_\ell(\mathbf{B}'^\perp)^0, T_\ell(\mathbf{B}^\perp)^0) \quad (4.6.5)$$

is an isomorphism of (one-dimensional) Hermitian spaces. If we set $\mathbf{U}_\ell := \text{Hom}_{F_\ell}(T_\ell(\mathbf{A}_0)^0, T_\ell(\mathbf{B}^\perp)^0)$ and $\mathbf{U}'_\ell := \text{Hom}_{F_\ell}(T_\ell(\mathbf{A}'_0)^0, T_\ell(\mathbf{B}'^\perp)^0)$, there is an isomorphism of F_ℓ vector spaces

$$\begin{aligned} \text{Hom}_{F_\ell}(T_\ell(\mathbf{B}'^\perp)^0, T_\ell(\mathbf{B}^\perp)^0) &\longrightarrow \text{Hom}_{F_\ell}(\mathbf{U}_\ell, \mathbf{U}'_\ell) \\ \phi^\perp &\longmapsto (f \mapsto (\phi^\perp \circ f \circ \phi_0^{-1})). \end{aligned} \quad (4.6.6)$$

An element ϕ^\perp on the left satisfies $\phi^{\perp\dagger}\phi^\perp = b$ if and only if ϕ^\perp corresponds to an isomorphism of Hermitian spaces $\mathbf{U}_\ell \rightarrow \mathbf{U}'_\ell$. We have $V_\ell \cong \mathbf{W}'_\ell \oplus \mathbf{U}'_\ell \cong \mathbf{W}_\ell \oplus \mathbf{U}_\ell$ (orthogonal direct sum) as Hermitian spaces, for all $\ell \neq p$. Hence $\mathbf{U}'_\ell \cong \mathbf{U}_\ell$ for all $\ell \neq p$ (consider the Hermitian space local invariants (in $\{\pm 1\}$) via ε , with conventions as in [Che24a, Section 2.2]).

The preceding discussion produces an element $\phi_\ell^\perp \in \text{Hom}_F^0(\mathbf{B}'^\perp, \mathbf{B}^\perp) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ satisfying $\phi_\ell^{\perp\dagger}\phi_\ell^\perp = b$ for all primes $\ell \neq p$. Since p is split in \mathcal{O}_F , such an element exists for $\ell = p$ as well (i.e. $N_{F_p/\mathbb{Q}_p}(F_p^\times) = \mathbb{Q}_p^\times$). Since $b > 0$, such an element also exists if \mathbb{Q}_ℓ is replaced by \mathbb{R} (positivity of the Rosati involution). By the Hasse principle for Hermitian spaces (or Hasse norm theorem), we obtain $\phi^\perp \in \text{Hom}_F^0(\mathbf{B}'^\perp, \mathbf{B}^\perp)$ satisfying $\phi^{\perp\dagger}\phi^\perp = b$. \square

For the rest of Section 4, we fix framing data as in Lemma 4.6.2 if p is split, so that $\Theta(\bar{k})$ is surjective.

For the supersingular cases, we use the following lifting result to prove surjectivity of $\tilde{\Theta}$ by bootstrapping from surjectivity on \bar{k} points (as in the proof of [RZ96, Theorem 6.30]). Recall that a p -divisible group X over a base scheme S is said to be *isoclinic* if for any geometric point \bar{s} of S , the isocrystal of $X_{\bar{s}}$ has constant slope independent of \bar{s} .

Proposition 4.6.3 (Isoclinic lifting theorem). *For any integer h , there exists an integer c with the following property: Let R be a reduced Noetherian Henselian local ring with residue field κ , and assume that R is an \mathbb{F}_p -algebra. Let X and Y be isoclinic p -divisible groups of heights $\leq h$ over $\text{Spec } R$. For any homomorphism $f: X_\kappa \rightarrow Y_\kappa$, the homomorphism $p^c f$ lifts to a unique homomorphism $X \rightarrow Y$.*

Proof. See [OZ02, Corollary 3.4]. For the statement when $R = \kappa[[t]]$ for an algebraically closed field κ (which is enough for Lemma 4.6.4), see also [Kat79, Theorem 2.7.1] combined with Grothendieck–Messing theory as in [RZ96, pg. 295]. \square

Lemma 4.6.4. *The uniformization map Θ is a surjection¹⁷ of formal algebraic stacks.*

Proof. The reduced substack $\check{\mathcal{Z}}(T)_{\text{red}} \subseteq \check{\mathcal{Z}}(T)$ is Jacobson, Deligne–Mumford, with quasi-compact diagonal, and finite type over $\text{Spec } \bar{k}$. This implies that the closed points of $\check{\mathcal{Z}}(T)_{\text{red}}$ are dense in every closed subset (e.g. [SProject, Lemma 06G2]; the finite type points are the same as closed

¹⁷We say a morphism of formal algebraic stacks is a *surjection* if it is surjective on underlying topological spaces.

points here), each closed point is the image of a map $\mathrm{Spec} \bar{k} \rightarrow \check{\mathcal{Z}}(T)_{\mathrm{red}}$, and every such map has image being a closed point.

Case p is nonsplit: We already know that Θ is surjective on \bar{k} points. It is thus enough to prove the following claim: suppose $\alpha' \rightsquigarrow \alpha$ is an immediate specialization of points in $|\check{\mathcal{Z}}(T)|$ (in the sense of [SProject, Definition 02I9], i.e. α is a points of “codimension one” in the closure of α'). If α is in the image of Θ , we claim that α' is also in the image of Θ . (This specialization process eventually terminates with a \bar{k} point.)

Let κ an algebraically closed field with a morphism $\mathrm{Spec} \kappa[[t]] \rightarrow \check{\mathcal{Z}}(T)$, which sends the closed point to α and the open point to α' .¹⁸ Enlarging κ if necessary, we may lift α to a point $(\alpha, \phi_0, \phi) \in \check{\mathcal{Z}}(T)_{\mathrm{framed}}$. The task is to lift the framing pair (ϕ_0, ϕ) to $\mathrm{Spec} k[[t]]$, which is then a framing pair for α' . Serre–Tate (and formal GAGA as in [EGAIII1, Théorème 5.4.1]) implies that it is enough to lift the induced quasi-isogenies of p -divisible groups to $\mathrm{Spec} k[[t]]$. This is possible by the isoclinic lifting theorem (Proposition 4.6.3).

Case p is split: By [Che24a, Lemma 4.4.2] (finiteness of local special cycles), and since the groupoid $[I'(\mathbb{Q}) \backslash J_p(T)]$ has finite automorphism groups and finitely many isomorphism classes (Section 4.5; we assumed $\mathrm{rank}(T) \geq n - 1$ for p split), we know there is a surjection from finitely many copies of $\mathrm{Spec} \bar{k}$ to $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]$. Since Θ is surjective on \bar{k} -points, the previous considerations show that $|\check{\mathcal{Z}}(T)_{\mathrm{red}}|$ is a finite discrete topological space, and that Θ is a surjection. \square

Lemma 4.6.5. *The map $\tilde{\Theta}$ is proper on underlying reduced substacks, and the reduced substack $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]_{\mathrm{red}}$ is proper over $\mathrm{Spec} \bar{k}$.*

Proof. Since the reduced substack $\check{\mathcal{Z}}(T)_{\mathrm{red}}$ is separated over $\mathrm{Spec} \bar{k}$, it is enough to check that $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]_{\mathrm{red}}$ is proper over $\mathrm{Spec} \bar{k}$, by [SProject, Lemma 0CPT].

We already saw that $\tilde{\Theta}$ is a monomorphism, hence separated. Since $\check{\mathcal{Z}}(T)_{\mathrm{red}}$ is separated, we see that $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]_{\mathrm{red}}$ is also separated over $\mathrm{Spec} \bar{k}$.

We use the description of $\check{\mathcal{Z}}(T)_{\mathrm{framed}}$ in (4.4.2). We know that every irreducible component of the reduced subscheme $\mathcal{N}'_{\mathrm{red}}$ is projective over \bar{k} [Che24a, Section 4.1], hence the same holds for $\mathcal{Z}'(\mathbf{x}_p)$ for any $\mathbf{x} \in \mathbf{W}^m$ (and $\mathcal{Z}'(\mathbf{x}^p)$ is discrete). Hence each irreducible component of $\check{\mathcal{Z}}(T)_{\mathrm{framed}, \mathrm{red}}$ has closed image in $\check{\mathcal{Z}}(T)_{\mathrm{red}}$. Since Θ is surjective, we conclude that finitely many irreducible components of $\check{\mathcal{Z}}(T)_{\mathrm{framed}, \mathrm{red}}$ cover $\check{\mathcal{Z}}(T)_{\mathrm{red}}$ (by Noetherianity of the latter). Since $\tilde{\Theta}$ is a monomorphism, hence injective on underlying topological spaces, we conclude that those finitely many irreducible components cover $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]_{\mathrm{red}}$ as well. Then $[I'(\mathbb{Q}) \backslash \check{\mathcal{Z}}(T)_{\mathrm{framed}}]_{\mathrm{red}}$ is proper over $\mathrm{Spec} \bar{k}$ by [SProject, Lemma 0CQK]. \square

Proposition 4.6.6. *The map $\tilde{\Theta}$ is an isomorphism.*

Proof. We have seen that the morphism $\tilde{\Theta}$ of locally Noetherian formal algebraic stacks is formally étale, surjective, and a monomorphism. The underlying map of reduced substacks is proper. These properties imply that $\tilde{\Theta}$ is an isomorphism. \square

¹⁸The following procedure produces such a morphism $\mathrm{Spec} \kappa[[t]] \rightarrow \check{\mathcal{Z}}(T)$. First, take an étale cover of $\check{\mathcal{Z}}(T)_{\mathrm{red}}$ by a scheme U and lift $x' \rightsquigarrow x$ to an immediate specialization $y' \rightsquigarrow y$ on U . Write Z for the normalization of the integral closed subscheme of U with generic point y' . Note the normalization map is finite, and lift $y' \rightsquigarrow y$ to an immediate specialization $z' \rightsquigarrow z$ on Z . Completion of the local ring at z on Z is a power series ring over a field.

4.7. Global and local. The next lemma (purely linear-algebraic) helps us use uniformization to deduce properties of local special cycles via “approximating” them by global special cycles.

Lemma 4.7.1. *Let $L \subseteq \mathbf{W}_p$ be any non-degenerate Hermitian \mathcal{O}_{F_p} -lattice (of any rank). There exists an element $g_p \in U(\mathbf{W}_p)$ such that $g_p(L)$ admits a basis consisting of elements in \mathbf{W} .*

Proof. Set $W = L \otimes_{\mathcal{O}_{F_p}} F_p$. It is enough to produce $g_p \in U(\mathbf{W}_p)$ such that $g_p(W)$ admits an F_p -basis consisting of elements in \mathbf{W} (since this implies that every full rank \mathcal{O}_{F_p} -lattice in $g_p(W)$ admits a basis consisting of elements of \mathbf{W}).

Select any F_p -basis $\underline{e} = [e_1, \dots, e_d]$ for W . Since \mathbf{W} is dense in \mathbf{W}_p , we may select $\tilde{\underline{e}} = [\tilde{e}_1, \dots, \tilde{e}_d]$ such that each $\|\tilde{e}_i - e_i\|_p \ll 1$ for all i (meaning $\tilde{e}_i - e_i$ lies in a small neighborhood of 0 for the p -adic topology on \mathbf{W}_p). Set $\tilde{W} := \text{span}_{F_p} \{\tilde{e}_1, \dots, \tilde{e}_d\}$. When each $\tilde{e}_i - e_i$ lies in a sufficiently small neighborhood of 0, there exists a (non-canonical) isomorphism of Hermitian spaces $W \cong \tilde{W}$ (the associated Gram matrices $(\underline{e}, \underline{e})$ and $(\tilde{\underline{e}}, \tilde{\underline{e}})$ can be made arbitrarily p -adically close; hence the local invariants $\varepsilon((\underline{e}, \underline{e}))$ and $\varepsilon((\tilde{\underline{e}}, \tilde{\underline{e}}))$ will agree). By Witt’s theorem for Hermitian spaces, any isometry $W \rightarrow \tilde{W}$ extends to an isometry $g_p: \mathbf{W}_p \rightarrow \mathbf{W}_p$. This element $g_p \in U(\mathbf{W}_p)$ satisfies the conditions in the lemma statement. \square

Corollary 4.7.2. *Consider any tuple $\underline{\mathbf{x}}_p \in \mathbf{W}_p^m$ which spans a non-degenerate Hermitian \mathcal{O}_{F_p} -lattice, and write m^b for its rank. Assume $m^b = n - 1$ if p is split.*

For some $T \in \text{Herm}_m(\mathbb{Q})$ (still assuming $\text{rank } T \geq n - 1$ if p is split), and some $j \in J_p(T)$ with associated $\underline{\mathbf{w}} \in \mathbf{W}$, there exists $g_p \in U(\mathbf{W}_p)$ inducing an automorphism $\mathcal{N} \rightarrow \mathcal{N}$ which takes $\mathcal{Z}(\underline{\mathbf{x}}_p)$ isomorphically to $\mathcal{Z}(\underline{\mathbf{w}}_p)$. In particular, there is an induced morphism

$$\mathcal{Z}(\underline{\mathbf{x}}_p) \xrightarrow{\sim} \mathcal{Z}(\underline{\mathbf{w}}_p) \xrightarrow{\Theta_j} \check{\mathcal{Z}}(T). \quad (4.7.1)$$

which is representable by schemes, separated, and étale. If $m^b \geq n - 1$ (equivalently, $\text{rank}(T) \geq n - 1$), this map is finite étale.

Proof. By Lemma 4.7.1, we may pick an element $g_p \in U(\mathbf{W}_p)$ so that $\text{span}_{\mathcal{O}_{F_p}}(g_p \cdot \underline{\mathbf{x}}_p)$ admits an \mathcal{O}_{F_p} -basis $\underline{\mathbf{w}}^b$ of elements in \mathbf{W} . Extend $\underline{\mathbf{w}}^b$ to any m -tuple $\underline{\mathbf{w}} \in \mathbf{W}^m$, and set $T := (\underline{\mathbf{w}}, \underline{\mathbf{w}})$. Recall that $U(\mathbf{W}_p)$ acts on \mathcal{N} , and that g_p gives an automorphism of \mathcal{N} sending $\mathcal{Z}(\underline{\mathbf{x}}_p) \mapsto \mathcal{Z}(\underline{\mathbf{w}}_p)$ [Che24a, Section 4.3]. By uniformization (Proposition 4.6.6), any $j \in J_p(T)$ whose associated tuple is $\underline{\mathbf{w}}$ will satisfy the conditions of the lemma. Replacing $\underline{\mathbf{w}}$ with $a \cdot \underline{\mathbf{w}}$ for suitable $a \in \mathbb{Z}$ with $p \nmid a$ ensures $\mathcal{Z}'(\underline{\mathbf{w}}^p) \neq \emptyset$. Then such $j \in J_p(T)$ will exist. In Section 4.5, we saw that Θ_j is finite étale if $\text{rank}(T) \geq n - 1$. \square

If $p \neq 2$ and in signature $(n - 1, 1)$, the quasi-compactness proved in the next lemma is also [LZ22, Lemma 2.9.] (inert), proved via Bruhat–Tits stratification. In the exotic smooth ramified case, quasi-compactness should be implicit in [LL22], via Bruhat–Tits stratification as discussed in [LL22, §2.3]. In the case when $\underline{\mathbf{x}}_p$ spans a lattice of rank n and signature $(n - 1, 1)$, see [LZ22, Lemma 5.1.1] (inert, $p \neq 2$) and [LL22, Remark 2.26] (ramified, exotic smooth).

Lemma 4.7.3. *Let $\underline{\mathbf{x}}_p \in \mathbf{W}_p^d$ be any tuple which spans a non-degenerate Hermitian \mathcal{O}_{F_p} -lattice of rank $\geq n - 1$. Then the local special cycle $\mathcal{Z}(\underline{\mathbf{x}}_p)$ is quasi-compact and the structure map $\mathcal{Z}(\underline{\mathbf{x}}_p) \rightarrow \text{Spf } \mathcal{O}_{\tilde{F}_p}$ is adic and proper.*

Proof. By Corollary 4.7.2, we obtain $T \in \text{Herm}_m(\mathbb{Q})$ and a map $\mathcal{Z}(\underline{\mathbf{x}}_p) \rightarrow \check{\mathcal{Z}}(T)$ which is representable by schemes, and finite étale. In particular, $\mathcal{Z}(\underline{\mathbf{x}}_p)$ is quasi-compact because the (base-changed) global special cycle $\check{\mathcal{Z}}(T)$ is quasi-compact.

If p is nonsplit, then $\mathcal{Z}(T)_{\bar{k}} \rightarrow \mathcal{M}_{\bar{k}}$ automatically factors through the supersingular locus (Corollary 3.3.3), so we have $\check{\mathcal{Z}}(T) = \mathcal{Z}(T)_{\text{Spf } \mathcal{O}_{\check{F}_p}}$. This formula holds for p split as well, by definition. Hence $\check{\mathcal{Z}}(T) \rightarrow \text{Spf } \mathcal{O}_{\check{F}_p}$ is adic¹⁹ and proper (Lemma 3.3.5), so $\mathcal{Z}(\underline{\mathbf{x}}_p) \rightarrow \text{Spf } \mathcal{O}_{\check{F}_p}$ is adic and proper. \square

We write $\check{\mathcal{Z}}(T)_{\mathcal{H}}$ for the flat²⁰ part (“horizontal”) of $\check{\mathcal{Z}}(T)$, i.e. the largest closed substack which is flat over $\text{Spf } \mathcal{O}_{\check{F}_p}$. We use similar notation $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}}$ for the flat part of the local special cycle $\mathcal{Z}(\underline{\mathbf{x}}_p)$.

Formation of “flat part” is flat local on the source. The quotient map Θ (4.4.1) is representable by schemes and étale, hence flat. So the uniformization result (Proposition 4.6.6) implies that there is an induced uniformization morphism

$$\Theta: \coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} \mathcal{Z}'(\underline{\mathbf{x}}_p)_{\mathcal{H}} \times \mathcal{Z}'(\underline{\mathbf{x}}^p) \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}} \quad (4.7.3)$$

where $\mathcal{Z}'(\underline{\mathbf{x}}_p)_{\mathcal{H}}$ is the flat part of $\mathcal{Z}'(\underline{\mathbf{x}}_p)$. The action of $I'(\mathbb{Q})$ must preserve the flat part, so generalities on stack quotients imply that Θ induces an isomorphism

$$\tilde{\Theta}: \left[I'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} \mathcal{Z}'(\underline{\mathbf{x}}_p)_{\mathcal{H}} \times \mathcal{Z}'(\underline{\mathbf{x}}^p) \right) \right] \xrightarrow{\sim} \check{\mathcal{Z}}(T)_{\mathcal{H}} \quad (4.7.4)$$

¹⁹We say a morphism of formal algebraic stacks is *adic* if the morphism is representable by algebraic stacks in the sense of [Eme20, §3].

²⁰Flatness for morphisms of locally Noetherian formal algebraic stacks was defined in [Eme20, Definition 8.42]. We are using a different definition, since the definition of loc. cit. does not recover the usual notion of flatness for morphisms of schemes (in the situation of [Eme20, Lemma 8.41(1)], consider $\mathcal{X} = \mathcal{Y} = \text{Spec } k$ for a field k and any non-flat morphism of Noetherian affine k -schemes $U \rightarrow V$).

We define flatness in the style of [SProject, Section 06FL] (there for algebraic stacks), which recovers usual flatness for morphisms of locally Noetherian formal schemes. Let $f: X \rightarrow Y$ be a morphism of locally Noetherian formal algebraic spaces. Consider commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow a & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array} \quad (4.7.2)$$

where U and V are locally Noetherian formal schemes and the vertical arrows are representable by schemes, flat, and locally of finite presentation. We say that f is *flat* if it satisfies either of the following equivalent conditions.

- (1) For any diagram as above such that in addition $U \rightarrow X \times_Y V$ is flat, the morphism h is flat.
- (2) For some diagram as above with $a: U \rightarrow X$ surjective, the morphism h is flat.

Next, consider a morphism $f: X \rightarrow Y$ of locally Noetherian formal algebraic stacks. Consider diagrams as above, but assume instead that U and V are locally Noetherian formal algebraic spaces, and that the arrows a and b are representable by algebraic spaces, flat, and locally of finite presentation. We say that f is *flat* if either of the equivalent conditions (1) and (2) as above are satisfied. If the morphism f is adic, then this agrees with the notion of flatness for adic morphisms as in [Eme20, Definition 3.11].

of formal algebraic stacks. For each $j \in J_p(T)$, the map $\Theta_j: \mathcal{Z}(\underline{\mathbf{x}}_p) \rightarrow \check{\mathcal{Z}}(T)$ induces a map $\Theta_j: \mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}}$ (reusing the notation Θ_j). Since Θ_j is flat and since formation of flat part is flat local on the source, the “horizontal” Θ_j arises from the original Θ_j by base-change along $\check{\mathcal{Z}}(T)_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)$.

In the case $p \neq 2$ and for signature $(n-1, 1)$ and $m^b = n-1$, the next lemma is a consequence of [LZ22, Theorem 4.2.1] (decomposition into quasi-canonical lifting cycles via Breuil modules) and is explained in [LL22, Lemma 2.49(1)] (also via decomposition into quasi-canonical lifing cycles). In the case $p \neq 2$, signature $(n-1, 1)$, and $m^b = n$, see again [LZ22, Lemma 5.1.1] (inert, $p \neq 2$) and [LL22, Remark 2.26] (ramified, exotic smooth).

Lemma 4.7.4. *Let $\underline{\mathbf{x}}_p \in \mathbf{W}_p^m$ be a tuple which spans a non-degenerate Hermitian \mathcal{O}_{F_p} -lattice, whose rank we denote m^b . Assume $m^b = n-1$ if F/\mathbb{Q}_p is split. Form the horizontal part $\mathcal{Z}(\underline{\mathbf{x}})_{\mathcal{H}}$ of $\mathcal{Z}(\underline{\mathbf{x}})$.*

- (1) *If $\mathcal{Z}(\underline{\mathbf{x}})_{\mathcal{H}}$ is nonempty, then it is equidimensional of dimension $(n-r)r + 1 - m^b r$.*
- (2) *If $m^b = n-1$ and the signature is $(n-r, r) = (n-1, 1)$, then the structure morphism $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}_p}$ is a finite adic morphism of Noetherian formal schemes. The associated finite scheme over $\mathrm{Spec} \mathcal{O}_{\check{F}_p}$ has reduced generic fiber.*
- (3) *If $m^b = n$ and the signature is $(n-r, r) = (n-1, 1)$, then $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} = \emptyset$.*

Proof.

- (1) By Corollary 4.7.2, we can find $T \in \mathrm{Herm}_m(\mathbb{Q})$ (with $\mathrm{rank}(T) \geq n-1$ if p is split) and a morphism $\mathcal{Z}(\underline{\mathbf{x}}_p) \rightarrow \check{\mathcal{Z}}(T)$ which is representable by schemes and étale. As formation of flat part is flat local on the source, we obtain a morphism $\mathcal{Z}(\underline{\mathbf{x}})_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}}$ which is still representable by schemes and étale. The claim now follows from the corresponding global result for $\mathcal{Z}(T)_{\mathcal{H}}$ (Lemma 2.3.5). Note that we may assume K'_f is a small level (deepen the level away from p) to reduce to the case when $\check{\mathcal{Z}}(T)_{\mathcal{H}}$ is a formal scheme.
- (2) In this case, the map $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}}$ from part (1) is finite étale. We know that $\check{\mathcal{Z}}(T)_{\mathcal{H}} \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}_p}$ (with T as in the proof of loc. cit.) is proper and quasi-finite (Lemma 3.3.4). Since proper and quasi-finite implies finite (for morphisms of schemes) and since $\check{\mathcal{Z}}(\underline{\mathbf{x}}_p) \rightarrow \mathrm{Spf} \mathcal{O}_{\check{F}_p}$ is adic, (already proved in Lemma 4.7.3) i.e. representable by schemes, the claimed finiteness holds. The claim on reducedness in the generic fiber follows because $\mathcal{Z}(T)_{\mathcal{H}} \rightarrow \mathrm{Spec} \mathcal{O}_F$ is étale in the generic fiber (Lemma 2.3.5). We are passing from finite relative schemes over $\mathrm{Spf} \mathcal{O}_{\check{F}_p}$ and $\mathrm{Spec} \mathcal{O}_{\check{F}_p}$ as in Appendix B.3 (i.e. $\mathrm{Spf} R \mapsto \mathrm{Spec} R$).
- (3) If $m^b = n$, then $\check{\mathcal{Z}}(T)_{\mathcal{H}} = \emptyset$ (by Lemma 2.3.5), so $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} = \emptyset$ by existence of the map $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}}$. \square

In the case of $p \neq 2$, signature $(n-1, 1)$, and $m^b = n-1$, the following lemma is [LZ22, §2.9] (there proved differently, using their quasi-compactness result via Bruhat–Tits stratification).

Lemma 4.7.5 (Horizontal and vertical decomposition). *Let $\underline{\mathbf{x}}_p \in \mathbf{W}_p^m$ be a tuple which span a non-degenerate Hermitian \mathcal{O}_{F_p} -lattice of rank m^b . Assume $m^b = n-1$ if F/\mathbb{Q}_p is split. For $e \gg 0$, we have a scheme-theoretic union decomposition*

$$\mathcal{Z}(\underline{\mathbf{x}}_p) = \mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \cup \mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{V}}. \quad (4.7.5)$$

where $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{V}} := \mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}/p^e \mathcal{O}_{\tilde{\mathbb{F}}_p}}$.

Proof. If \mathcal{I} denotes the ideal sheaf of $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}}$ as a closed subscheme of $\mathcal{Z}(\underline{\mathbf{x}}_p)$, it is enough to show that p^e annihilates \mathcal{I} for $e \gg 0$. By Corollary 4.7.2, we can find $T \in \mathrm{Herm}_m(\mathbb{Q})$ (with $\mathrm{rank}(T) \geq n - 1$ if p is split) and a morphism $f: \mathcal{Z}(\underline{\mathbf{x}}_p) \rightarrow \check{\mathcal{Z}}(T)$ which is representable by schemes and étale. We may assume K'_f is small (deepen the level away from p) so that $\check{\mathcal{Z}}(T)$ is a formal scheme.

If \mathcal{S} denotes the ideal sheaf of the flat part $\check{\mathcal{Z}}(T)_{\mathcal{H}} \subseteq \check{\mathcal{Z}}(T)$, then $f^* \mathcal{S} \rightarrow \mathcal{I}$ is surjective (by flatness of f , i.e. formation of flat part is flat local on the source). If p^e annihilates \mathcal{S} , then p^e also annihilates \mathcal{I} . We know that \mathcal{S} consists (locally) of p -power torsion elements in the structure sheaf. Since $\check{\mathcal{Z}}(T)$ is quasi-compact, we know that \mathcal{S} is annihilated by p^e for $e \gg 0$. \square

For the rest of Section 4, we restrict to signature $(n - 1, 1)$ in all cases.

Lemma 4.7.6.

- (1) If p is split, then $\check{\mathcal{Z}}(T) \rightarrow \mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}$ is proper and quasi-finite and we have ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} = 0$.
- (2) Assume $n = 2$ and $\mathrm{rank}(T) \geq 1$. Then $\check{\mathcal{Z}}(T) \rightarrow \mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}$ is proper and quasi-finite. If $\mathrm{rank}(T) = 1$, then we have ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} = 0$.

Proof. (1) Recall our running assumption that $\mathrm{rank}(T) \geq n - 1$ if p is split. Recall also $\check{\mathcal{Z}}(T) := \mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}}$ in the split case, so the map $\check{\mathcal{Z}}(T) \rightarrow \mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}$ is representable by algebraic stacks and locally of finite type. This map is proper on reduced substacks by uniformization (Lemma 4.6.5 and Proposition 4.6.6), hence it is proper.

It remains to check that $\check{\mathcal{Z}}(T) \rightarrow \mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}$ is quasi-finite in the sense of [SProject, Definition 0G2M]. It is enough to check that $\check{\mathcal{Z}}(T)_{\mathrm{red}} \rightarrow \mathrm{Spec} \bar{k}$ is quasi-finite. This follows from the uniformization isomorphism, since $\check{\mathcal{Z}}(T)_{\mathrm{red}}$ may be covered by finitely many copies of $\mathrm{Spec} \bar{k}$ (combine uniformization with the analogous result for local special cycles, which is [Che24a, Lemma 4.4.2]; since we assume $\mathrm{rank}(T) \geq n - 1$ when p is split, the groupoid $[I'(\mathbb{Q}) \backslash J_p(T)]$ has finitely many isomorphism classes, as discussed in Section 4.5).

The derived vertical special cycle class ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} \in \mathrm{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}}$ was defined in [Che24a, Section 3.6]. If $m \geq n$ then $\mathcal{Z}(T)$ is empty. If $m = n - 1$, then $\mathrm{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}} = 0$ because $\mathcal{Z}(T)_{\mathbb{F}_p}$ has dimension 0 (and \mathcal{M} has dimension n).

(2) This may be proved as in part (1). We may assume p is nonsplit. We have $\check{\mathcal{Z}}(T) = \mathcal{Z}(T)_{\mathrm{Spf} \mathcal{O}_{\tilde{\mathbb{F}}_p}}$ (Lemma 3.3.3). Then use quasi-compactness of local special cycles (Lemma 4.7.3), uniformization, and discreteness of $\mathcal{N}_{\mathrm{red}}$ [Che24a, Section 4.4]. Suppose $\mathrm{rank}(T) = 1$. First consider the case $m = 1$. Then ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} \in \mathrm{gr}_{\mathcal{M}}^1 K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}} = 0$ because $\mathcal{Z}(T)_{\mathbb{F}_p}$ has dimension 0 (and \mathcal{M} has dimension 2). If $m = 2$, then ${}^{\mathbb{L}}\mathcal{Z}(t_i)_{\mathcal{V},p} = 0$ for any nonzero diagonal entry t_i of T by the preceding argument, so ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} = 0$ by construction (defined in [Che24a, Section 3.6] as the projection of a product against ${}^{\mathbb{L}}\mathcal{Z}(t_i)_{\mathcal{V},p} = 0$ for some i). \square

Lemma 4.7.7. *Assume p is nonsplit. Assume that K'_f is a small level, so that \mathcal{M} is a scheme. Fix any $j \in J_p(T)$ and consider the map*

$$\mathcal{Z}(\underline{\mathbf{x}}_p) \xrightarrow{\Theta_j} \check{\mathcal{Z}}(T) \rightarrow \mathcal{Z}(T). \tag{4.7.6}$$

The class $\mathbb{L}\mathcal{Z}(T) \in K'_0(\mathcal{Z}(T))_{\mathbb{Q}}$ pulls back to the class $\mathbb{L}\mathcal{Z}(\mathbf{x}_p) \in K'_0(\mathcal{Z}(\mathbf{x}_p))_{\mathbb{Q}}$.

Proof. The maps $\Theta_j: \mathcal{Z}(\mathbf{x}_p) \rightarrow \check{\mathcal{Z}}(T)$ and $\check{\mathcal{Z}}(T) \rightarrow \mathcal{Z}(T)$ are flat maps of locally Noetherian formal schemes, so we may take the non-derived pullback. The lemma may be proved using the fact that the commutative diagrams

$$\begin{array}{ccc} \check{\mathcal{Z}}(T)_{\text{framed}} & \longrightarrow & \check{\mathcal{M}}_{\text{framed}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{Z}(T) & \longrightarrow & \mathcal{M} \end{array} \quad \begin{array}{ccc} \check{\mathcal{Z}}(t_i)_{\text{framed}} & \longrightarrow & \check{\mathcal{M}}_{\text{framed}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{Z}(t_i) & \longrightarrow & \mathcal{M} \end{array} \quad (4.7.7)$$

are 2-Cartesian (where the t_i are the diagonal entries of T), and the fact that the tautological bundle \mathcal{E} on \mathcal{M} pulls back to the tautological bundle \mathcal{E} on \mathcal{N} . \square

Corollary 4.7.8. *Assume p is nonsplit. For any $\mathbf{x}_p \in \mathbf{W}_p^m$, we have $\mathbb{L}\mathcal{Z}(\mathbf{x}_p) \in F_{\mathcal{N}}^m K'_0(\mathcal{Z}(\mathbf{x}_p))_{\mathbb{Q}}$.*

Proof. By Corollary 4.7.2, we can find $T \in \text{Herm}_m(\mathbb{Q})$ (with $\text{rank}(T) \geq n - 1$ if p is split) and a morphism $\mathcal{Z}(\mathbf{x}_p) \rightarrow \check{\mathcal{Z}}(T)$ which is representable by schemes and étale. We can deepen the level K'_f away from p to assume $\check{\mathcal{Z}}(T)$ is a formal scheme. Since $\mathbb{L}\mathcal{Z}(T) \in F_{\mathcal{N}}^m K'_0(\mathcal{Z}(T))_{\mathbb{Q}}$, the corollary follows from Lemma 4.7.7. \square

We previously defined derived vertical (global) special cycles $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V},p} \in \text{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathbb{F}_p})_{\mathbb{Q}}$ [Che24a, Section 3.6]. In the next lemma, we write $\mathcal{Z}(T)_{(p)} := \mathcal{Z}(T) \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}_{(p)}$. We also write $\mathcal{Z}(T)_{\mathcal{V},p} := \mathcal{Z}(T) \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}/p^e\mathbb{Z}$ and $\mathcal{Z}(\mathbf{x}_p)_{\mathcal{V},p} := \mathcal{Z}(\mathbf{x}_p) \times_{\text{Spf } \mathcal{O}_{\check{\mathbb{F}}_p}} \text{Spec } \mathcal{O}_{\check{\mathbb{F}}_p}/p^e\mathcal{O}_{\check{\mathbb{F}}_p}$ for an understood integer $e \gg 0$. We also set $\check{\mathcal{Z}}(T)_{\mathcal{V}} := \check{\mathcal{Z}}(T) \times_{\text{Spf } \mathcal{O}_{\check{\mathbb{F}}_p}} \text{Spec } \mathcal{O}_{\check{\mathbb{F}}_p}/p^e\mathcal{O}_{\check{\mathbb{F}}_p}$.

Lemma 4.7.9. *Fix any $j \in J_p(T)$. Write $\mathbf{x} \in \mathbf{W}^m$ for the associated m -tuple. Fix any $e \gg 0$ such that there are scheme-theoretic union decompositions*

$$\mathcal{Z}(T)_{(p)} = \mathcal{Z}(T)_{(p),\mathcal{H}} \cup \mathcal{Z}(T)_{\mathcal{V},p} \quad \mathcal{Z}(\mathbf{x}_p) = \mathcal{Z}(\mathbf{x}_p)_{\mathcal{H}} \cup \mathcal{Z}(\mathbf{x}_p)_{\mathcal{V},p} \quad (4.7.8)$$

(“horizontal and vertical”). Pullback along the map

$$\mathcal{Z}(\mathbf{x}_p)_{\mathcal{V}} \xrightarrow{\Theta_j} \check{\mathcal{Z}}(T)_{\mathcal{V}} \rightarrow \mathcal{Z}(T)_{\mathcal{V},p} \quad (4.7.9)$$

sends $\mathbb{L}\mathcal{Z}(T)_{\mathcal{V},p} \in \text{gr}_{\mathcal{M}}^m K'_0(\mathcal{Z}(T)_{\mathcal{V},p})_{\mathbb{Q}}$ to $\mathbb{L}\mathcal{Z}(\mathbf{x}_p)_{\mathcal{V}} \in \text{gr}_{\mathcal{N}}^m K'_0(\mathcal{Z}(\mathbf{x}_p)_{\mathcal{V}})_{\mathbb{Q}}$.

Proof. If p is split, the derived vertical special cycles (global and local) are zero (Lemma 4.7.6 (global) and [Che24a, Section 4.5] (local)) and the lemma is trivial. We remind the reader of our running assumption that $\text{rank}(T) \geq n - 1$ if p is split.

We thus assume that p is nonsplit. By the local and global linear invariance results [Che24a, Section 4.5] and [Che24a, (3.6.11)], it is enough to check the case where $T = \text{diag}(0, T^{\flat})$ where $\det T^{\flat} \neq 0$.

First consider the case where T is nonsingular, i.e. $T = T^{\flat}$. If K'_f is a small level, the lemma follows from Lemma 4.7.7, since the projections $\text{gr}_{\mathcal{M}}^{m^{\flat}} K'_0(\mathcal{Z}(T^{\flat}))_{\mathbb{Q}} \rightarrow \text{gr}_{\mathcal{M}}^{m^{\flat}} K'_0(\mathcal{Z}(T^{\flat})_{\mathcal{V},p})_{\mathbb{Q}}$ and $\text{gr}_{\mathcal{N}}^{m^{\flat}} K'_0(\mathcal{Z}(\mathbf{x}_p))_{\mathbb{Q}} \rightarrow \text{gr}_{\mathcal{N}}^{m^{\flat}} K'_0(\mathcal{Z}(\mathbf{x}_p)_{\mathcal{V}})_{\mathbb{Q}}$ are given by (non-derived) pullbacks of coherent sheaves, see [Che24a, Lemma A.1.5] (Deligne–Mumford stacks) and [Zha21, Lemma B.1] (locally Noetherian formal schemes). Note that the codimension graded pieces gr^m are preserved, by étale-ness of Θ_j .

In general, we may reduce to the case where K'_f is a small level by compatibility of ${}^{\mathbb{L}}\mathcal{Z}(T^{\flat})_{\mathcal{V},p}$ with (finite étale) pullback for varying levels.

Next, consider the case where T is possibly singular with $T = \text{diag}(0, T^{\flat})$. If K'_f is a small level, this follows as in the proof of Lemma 4.7.7. That is, the class ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} := (\mathcal{E}^{\vee})^{m-\text{rank}(T)} \cdot \mathcal{Z}(T^{\flat})_{\mathcal{V},p}$ pulls back to ${}^{\mathbb{L}}\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{V}} := (\mathcal{E}^{\vee})^{m-\text{rank}(T)} \cdot \mathcal{Z}(\underline{\mathbf{x}}_p^{\flat})_{\mathcal{V}}$ (use the result for T^{\flat} just proved). In general, we may reduce to the case where K'_f is a small level (deepen level away from p) by compatibility of ${}^{\mathbb{L}}\mathcal{Z}(T^{\flat})_{\mathcal{V},p}$ with (finite étale) pullback for varying levels. \square

4.8. Local intersection numbers: vertical. The main purpose of this section is to reduce “global vertical intersection numbers” to “local vertical intersection numbers” (see end of this section). We continue to assume signature $(n-1, 1)$.

Consider $T' \in \text{Herm}_m(\mathbb{Q}_p)$ (with F_p -coefficients) with $\text{rank}(T') = n-1$, and either $m = n-1$ or $m = n$. For any tuple $\underline{\mathbf{x}}_p \in \mathbf{W}_p^m$ with Gram matrix T' , we define the *local vertical intersection number*

$$\text{Int}_{\mathcal{V},p}(T') := \begin{cases} 2[\check{F}_p : \check{Q}_p]^{-1} \deg_{\bar{k}}({}^{\mathbb{L}}\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{V}} \cdot \mathcal{E}^{\vee}) \log p & \text{if } m = n-1 \\ 2[\check{F}_p : \check{Q}_p]^{-1} \deg_{\bar{k}}({}^{\mathbb{L}}\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{V}}) \log p & \text{if } m = n \end{cases} \quad (4.8.1)$$

Here, \mathcal{E}^{\vee} stands for the class $[\mathcal{O}_{\mathcal{N}}] - [\mathcal{E}] \in K'_0(\mathcal{N})$. If no such $\underline{\mathbf{x}}_p$ exists, we set $\text{Int}_{\mathcal{V},p}(T') := 0$. The definition of $\text{Int}_{\mathcal{V},p}(T')$ does not depend on the choice of $\underline{\mathbf{x}}_p$ (by the action of $U(\mathbf{W}_p)$ on $\mathcal{N}(n-1, 1)$ as in [Che24a, Section 4.3]). The factor $2[\check{F}_p : \check{Q}_p]^{-1}$ will account for total degree of $\text{Spec } \mathcal{O}_{F_p} \rightarrow \text{Spec } \mathbb{Z}_p$ on residue fields (e.g. we need to account for both primes in \mathcal{O}_F over p in the split case). By local linear invariance [Che24a, Section 4.5]), we have

$$\text{Int}_{\mathcal{V},p}(T') = \text{Int}_{\mathcal{V},p}({}^t\bar{\gamma}T'\gamma) \quad (4.8.2)$$

for any $\gamma \in \text{GL}_m(\mathcal{O}_{F_p})$.

Consider any $T \in \text{Herm}_m(\mathbb{Q})$ (with F -coefficients) with $\text{rank}(T) = n-1$, and either $m = n-1$ or $m = n$. Pick any set of representatives $J \subseteq J_p(T)$ for the isomorphism classes of the groupoid $[I'(\mathbb{Q}) \setminus J_p(T)]$. By Lemma 4.7.9, we have

$$\text{Int}_{\mathcal{V},p,\text{global}}(T) := \deg_{\mathbb{F}_p}({}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p} \cdot (\mathcal{E}^{\vee})^{n-m}) \log p \quad (4.8.3)$$

$$= \text{Int}_{\mathcal{V},p}(T) \sum_{j \in J} \frac{1}{|\text{Aut}(j)|} \quad (4.8.4)$$

$$= \text{Int}_{\mathcal{V},p}(T) \frac{[K_{L_0,f} : K_{0,f}]}{|\mathcal{O}_F^{\times}|/h_F} \cdot \deg \left[I_1(\mathbb{Q}) \setminus \left(\prod_{\substack{\underline{\mathbf{x}} \in \mathbf{W}^m \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} U(\mathbf{W}_p^{\perp})/K_{1,\mathbf{L}_p^{\perp}} \times \mathcal{Z}(\underline{\mathbf{x}}^p) \right) \right].$$

For later use in [Che24c, Remark 9.1.2], consider $T \in \text{Herm}_n(\mathbb{Q})$ (with F -coefficients) with $\det T \neq 0$. We consider the *local intersection number*

$$\text{Int}_p(T) := 2[\check{F}_p : \check{Q}_p]^{-1} \deg_{\bar{k}}({}^{\mathbb{L}}\mathcal{Z}(\underline{\mathbf{x}}_p)) \log p \quad (4.8.5)$$

where $\underline{\mathbf{x}}_p \in \mathbf{W}_p^n$ is any n -tuple with Gram matrix T (since $\text{rank } \mathbf{W}_p = n-1$ when p is split, set $\text{Int}_p(T) := 0$ in this case). Note ${}^{\mathbb{L}}\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{V}} = {}^{\mathbb{L}}\mathcal{Z}(\underline{\mathbf{x}}_p)$ by Lemmas 4.7.4 and 4.7.3 (under the dévissage

pushforward identification $K'_0(\mathcal{Z}(\underline{\mathbf{x}}_p)_{\bar{k}}) \xrightarrow{\sim} K'_0(\mathcal{Z}(\underline{\mathbf{x}}_p))$. By Lemma 4.7.9, we have

$$\text{Int}_{p,\text{global}}(T) := \deg_{\mathbb{F}_p}(\mathbb{L} \mathcal{Z}(T)_{\mathcal{Y},p}) \log p \quad (4.8.6)$$

$$= \text{Int}_p(T) \frac{[K_{L_0,f} : K_{0,f}]}{|\mathcal{O}_F^\times / h_F|} \cdot \deg \left[I_1(\mathbb{Q}) \setminus \left(\prod_{\substack{\underline{\mathbf{x}} \in \mathbf{W}^n \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} \mathcal{Z}(\underline{\mathbf{x}}^p) \right) \right]. \quad (4.8.7)$$

4.9. Local intersection numbers: horizontal. The main purpose of this section is to reduce “global horizontal intersection numbers” to “local horizontal intersection numbers” (see end of this section). We continue to assume signature $(n-1, 1)$. In Section 4.9, we require $p \neq 2$ if p is inert (because we required this for our discussion of quasi-canonical lifting cycles, [Che24a, Section 6.3]).

Consider $T' \in \text{Herm}_m(\mathbb{Q}_p)$ (with F_p -coefficients) with $\text{rank}(T') = n-1$, and either $m = n-1$ or $m = n$. Select any $\underline{\mathbf{x}}_p \in \mathbf{W}_p^m$ with Gram matrix T' , and set $L_p^b := \text{span}_{\mathcal{O}_{F_p}}(\underline{\mathbf{x}}_p)$. We define the *local horizontal intersection number*

$$\text{Int}_{\mathcal{H},p}(T') := \sum_{\substack{L_p^b \subseteq M_p^b \subseteq M_p^{b*} \\ t(M_p^b) \leq 1}} \text{Int}_{\mathcal{H},p}(M_p^b)^\circ \quad (4.9.1)$$

where the sum runs over lattices $M_p^b \subseteq L_p^b \otimes_{\mathcal{O}_{F_p}} F_p$, where

$$\text{Int}_{\mathcal{H},p}(M_p^b)^\circ := 2 \cdot \deg \mathcal{Z}(M_p^b)^\circ \cdot \delta_{\text{tau}}(\text{val}'(M_p^b)) \quad (4.9.2)$$

with $\text{val}'(M_p^b) := \lfloor \text{val}(M_p^b) \rfloor$ and with $\delta_{\text{tau}}(-)$ the “local change of tautological height” as defined in [Che24a, (6.2.7)]. Here $\mathcal{Z}(M_p^b)^\circ \subseteq \mathcal{N}(n-1, 1)$ is the quasi-canonical lifting cycle associated with M_p^b [Che24a, Section 6.3]. The local horizontal intersection number should be compared with the decomposition of horizontal local special cycles into quasi-canonical lifting cycles [Che24a, Section 6.3]. The notation $\deg \mathcal{Z}(M_p^b)^\circ$ means the degree of the finite flat adic morphism $\mathcal{Z}(M_p^b)^\circ \rightarrow \text{Spf } \mathcal{O}_{\check{F}_p}$. If no such $\underline{\mathbf{x}}_p$ exists, we set $\text{Int}_{\mathcal{H},p}(T') := 0$.

This definition of $\text{Int}_{\mathcal{H},p}(T')$ does not depend on the choice of $\underline{\mathbf{x}}_p$ (again by the action of $U(\mathbf{W}_p)$ on \mathcal{N} [Che24a, Section 4.3] and Witt’s theorem). The formula for $\deg \mathcal{Z}(M_p^b)^\circ$ (combine [Che24a, (6.2.1)] and [Che24a, (6.3.1)]) shows $\text{Int}_{\mathcal{H},p}(M_p^b)^\circ \in \mathbb{Z}$. The extra factor of 2 in (4.9.2) will account for the fact that $\text{Spf}(\mathcal{O}_F \otimes_{\mathbb{Z}} \check{\mathbb{Z}}_p) \rightarrow \text{Spf } \check{\mathbb{Z}}_p$ has degree 2.

In the above situation, we also set

$$\deg_{\mathcal{H},p}(T') := \deg \mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \quad (4.9.3)$$

where the right-hand side means the degree of the finite flat adic morphism $\mathcal{Z}(\underline{\mathbf{x}}_p)_{\mathcal{H}} \rightarrow \text{Spf } \mathcal{O}_{\check{F}_p}$. If no such $\underline{\mathbf{x}}_p$ exists, we set $\deg_{\mathcal{H},p}(T') = 0$. Again, the definition of $\deg_{\mathcal{H},p}(T')$ does not depend on the choice of $\underline{\mathbf{x}}_p$.

Suppose $T \in \text{Herm}_m(\mathbb{Q})$ (with F -coefficients) with $\text{rank}(T) = n-1$, and either $m = n-1$ or $m = n$. Then (in the notation of Sections [Che24a, Sections 3.3 and 3.5] (adapted to the slightly more general setup as explained in Section 2.1) we have

$$\widehat{\deg}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)_{\mathcal{H}}}) = \widehat{\deg}(\widehat{\Omega}_0^\vee|_{\mathcal{Z}(T)_{\mathcal{H}}}) + \widehat{\deg}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)_{\mathcal{H}}}). \quad (4.9.4)$$

We have

$$\widehat{\deg}(\widehat{\Omega}_0^\vee|_{\mathcal{Z}(T)_{\mathcal{H}}}) = \deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}} \cdot (-h_{\text{Fal}}^{\text{CM}} - \frac{1}{4} \log |\Delta|). \quad (4.9.5)$$

where $\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}}$ means the degree of $\mathcal{Z}(T)_{\mathcal{H}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Q} \rightarrow \text{Spec } \mathbb{Q}$ (stacky degrees as in [Che24a, (A.1.10)] and surrounding discussion).

Pick any set of representatives $J \subseteq J_p(T)$ for the isomorphism classes of the groupoid $[I'(\mathbb{Q}) \backslash J_p(T)]$. Using the finite étale maps $\Theta_j: \mathcal{Z}(\mathbf{x}_p)_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}}$ for $j \in J_p(T)$ (Section 4.5 and (4.7.3)) which cover $\check{\mathcal{Z}}(T)_{\mathcal{H}}$ as j ranges over J , we find

$$\begin{aligned} \deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}} &= \deg_{\mathcal{H},p}(T) \sum_{j \in J_p(T)} \frac{1}{|\text{Aut}(j)|} \\ &= \deg_{\mathcal{H},p}(T) \frac{[K_{L_0,f} : K_{0,f}]}{|\mathcal{O}_F^\times|/h_F} \cdot \deg \left[I_1(\mathbb{Q}) \backslash \left(\prod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} U(\mathbf{W}_p^\perp)/K_{1,\mathbf{L}_p^\perp} \times \mathcal{Z}(\mathbf{x}^p) \right) \right]. \end{aligned} \quad (4.9.6)$$

Combining the following: (1) the finite étale maps $\Theta_j: \mathcal{Z}(\mathbf{x}_p)_{\mathcal{H}} \rightarrow \check{\mathcal{Z}}(T)_{\mathcal{H}}$ for $j \in J_p(T)$ (Section 4.5 and (4.7.3)) which cover $\check{\mathcal{Z}}(T)_{\mathcal{H}}$ as j ranges over J (2) [Che24a, Proposition 6.3.1] (decomposition of horizontal local special cycles into quasi-canonical liftings) and discussion surrounding [Che24a, (6.3.2)], and (3) [Che24a, Corollary 11.2.2] (decomposition of global height into local “change of heights” for p -divisible groups), we find

$$\begin{aligned} \text{Int}_{\mathcal{H},p,\text{global}}(T) &:= \widehat{\deg}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)_{\mathcal{H}}}) - (\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}}) \cdot h_{\text{tau}}^{\text{CM}} \pmod{\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell} \\ &= \text{Int}_{\mathcal{H},p}(T) \sum_{j \in J_p(T)} \frac{1}{|\text{Aut}(j)|} \\ &= \text{Int}_{\mathcal{H},p}(T) \frac{[K_{L_0,f} : K_{0,f}]}{|\mathcal{O}_F^\times|/h_F} \cdot \deg \left[I_1(\mathbb{Q}) \backslash \left(\prod_{\substack{\mathbf{x} \in \mathbf{W}^m \\ (\mathbf{x}, \mathbf{x})=T}} U(\mathbf{W}_p^\perp)/K_{1,\mathbf{L}_p^\perp} \times \mathcal{Z}(\mathbf{x}^p) \right) \right] \end{aligned} \quad (4.9.7)$$

with $h_{\text{tau}}^{\text{CM}}$ and $h_{\widehat{\mathcal{E}}^\vee}^{\text{CM}}$ as in (2.1.13). The notation “ $\pmod{\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell}$ ” means that equality holds as elements of the (additive) quotient $\mathbb{R}/(\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell)$. Note $\text{Int}_{\mathcal{H},p,\text{global}}(T) \in \mathbb{Q} \cdot \log p$. To apply [Che24a, Corollary 11.2.2], we first consider the case of small level K'_f so that $\mathcal{Z}(T)_{\mathcal{H}}$ is a scheme. This immediately implies the case of general (stacky) level, by compatibility of arithmetic degree with finite étale covers, see [Che24a, Section 3.3]. We have

$$\widehat{\deg}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)_{\mathcal{H}}}) - (\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}}) \cdot h_{\widehat{\mathcal{E}}^\vee}^{\text{CM}} = \sum_{\ell} \text{Int}_{\mathcal{H},\ell,\text{global}}(T) \quad (4.9.8)$$

where the sum ranges over all primes ℓ , with all but finitely many terms equal to 0. The preceding expression should be understood modulo $\mathbb{Q} \cdot \log \ell$ for those primes ℓ for which $L \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is not self-dual. If L is not self-dual, we also quotient by $\mathbb{Q} \cdot \log \ell$ for primes $\ell \mid \Delta$. We also quotient by $\mathbb{Q} \cdot \log 2$ unless 2 is split in \mathcal{O}_F .

We define *total “intersection numbers”*

$$\text{Int}_p(T) := \text{Int}_{\mathcal{H},p}(T) + \text{Int}_{\mathcal{V},p}(T) \quad \text{Int}_{p,\text{global}}(T) := \text{Int}_{\mathcal{H},p,\text{global}}(T) + \text{Int}_{\mathcal{V},p,\text{global}}(T) \quad (4.9.9)$$

(local and global) at p . The local version featured in our main non-Archimedean local theorems [Che24a, Section 9], and the global version will appear in the proof of our main global theorem [Che24c, Theorem 9.1.1].

For readers interested in Faltings heights, we record the relation

$$\widehat{\deg}(\widehat{\omega}|_{\mathcal{Z}(T)_{\mathcal{H}}}) - (\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathcal{H}}) \cdot n \cdot h_{\text{Fal}}^{\text{CM}} = -2 \sum_{\ell} \text{Int}_{\mathcal{H}, \ell, \text{global}}(T) \quad (4.9.10)$$

where $\widehat{\omega}$ is the metrized Hodge determinant bundle (see discussion in Section 2.1), the sum again runs over all primes ℓ , and where $\text{Int}_{\mathcal{H}, \ell, \text{global}}(T)$ is the same quantity defined above. This follows by the same argument as above, using [Che24a, Corollary 11.2.2]. The remarks following (4.9.8) (about quotienting by $\mathbb{Q} \cdot \log \ell$ for some primes ℓ) apply here verbatim.

5. ARCHIMEDEAN

We explain complex uniformization for special cycles and Green currents on \mathcal{M} . Aside from our discussion on Green currents for singular T in Section 5.4 (when $\text{rank}(T) = n - 1$), most of the material in Section 5 should be fairly standard, e.g. [KR14, §3] (uniformization of special cycles), [BHKRY20, §2] (including discussion of metrized tautological bundle), [Liu11, §4B] (Green currents via uniformization), etc.. Strictly speaking, however, the references [KR14; BHKRY20] restrict to principal polarizations. We will need non-principal polarizations (this slightly affects how we normalize the metric on the tautological bundle), so we explain the setup.

With notation as explained at the start of Part 2, we also assume L has signature $(n - 1, 1)$. For technical convenience, we assume the implicit level K'_f is small so that \mathcal{M} is a scheme (except at the very end of Section 5.4). Fix one of the two embeddings $F \rightarrow \mathbb{C}$, write $\mathcal{M}_{\mathbb{C}} := \mathcal{M} \times_{\text{Spec } \mathcal{O}_F} \text{Spec } \mathbb{C}$, and let $\mathcal{M}_{\mathbb{C}}^{\text{an}}$ be the analytification (outside of Section 5, we often abuse notation and drop the superscript an). This is a complex manifold of dimension $n - 1$. Given any Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$ (with F -coefficients) with associated special cycle $\mathcal{Z}(T) \rightarrow \mathcal{M}$, we use similar notation $\mathcal{Z}(T)_{\mathbb{C}}$ and $\mathcal{Z}(T)_{\mathbb{C}}^{\text{an}}$. Since $\mathcal{Z}(T)_{\mathbb{C}} \rightarrow \text{Spec } \mathbb{C}$ is smooth (Lemma 2.3.5), we know that $\mathcal{Z}(T)_{\mathbb{C}}^{\text{an}}$ is also a complex manifold.

We view $V_{\mathbb{R}}$ as a complex vector space via the identification $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$ (induced by the choice of $F \rightarrow \mathbb{C}$). We use notation from [Che24b, Section 2] on the Hermitian symmetric space \mathcal{D} and its local special cycles $\mathcal{D}(\underline{x})$ for tuples $\underline{x} \in V_{\mathbb{R}}^m$, etc..

We set $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$ and write $V_{\mathbb{C}} = V_{\mathbb{C}}^+ \oplus V_{\mathbb{C}}^-$ where the F -action on $V_{\mathbb{C}}^+$ (resp. $V_{\mathbb{C}}^-$ is F -linear (resp. σ -linear) with respect to the chosen map $F \rightarrow \mathbb{C}$. We use similar notation for other $F \otimes_{\mathbb{Q}} \mathbb{C}$ -modules.

5.1. Local special cycles away from ∞ . Given an m -tuple $\underline{x}_f = [x_1, \dots, x_m] \in (V \otimes_{\mathbb{Q}} \mathbb{A}_f)^m$, we consider an “*away-from- ∞* ” local special cycle

$$\mathcal{D}'(\underline{x}_f) := \{(g_0, g) \in G'(\mathbb{A}_f)/K'_f : g^{-1}g_0x_i \in L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \text{ for all } x_i \in \underline{x}_f\}. \quad (5.1.1)$$

We view $\mathcal{D}'(\underline{x}_f)$ as a discrete set. We also define the “*away-from- ∞* ” local special cycle

$$\mathcal{D}(\underline{x}_f) := \{g \in U(V)(\mathbb{A}_f)/K_f : g^{-1}x_i \in L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \text{ for all } x_i \in \underline{x}_f\}. \quad (5.1.2)$$

The isomorphism $G'(\mathbb{A}_f)/K'_f \rightarrow GU(V_0)(\mathbb{A}_f)/K_{0,f} \times U(V)(\mathbb{A}_f)/K_f$ (*) induces a bijection

$$\mathcal{D}'(\underline{x}_f) \xrightarrow{\sim} GU(V_0)(\mathbb{A}_f)/K_{0,f} \times \mathcal{D}(\underline{x}_f). \quad (5.1.3)$$

5.2. Framing. Fix the isomorphism of Hermitian \mathcal{O}_F -lattices $\text{Hom}_{\mathcal{O}_F}(L_0, L) \rightarrow L$ sending $x \mapsto x(1)$ (with $1 \in L_0$). This is analogous to $\boldsymbol{\eta}$ from (4.3.2).

Given $\alpha = (A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{\eta}_0, \tilde{\eta}) \in \mathcal{M}_{\mathbb{C}}^{\text{an}}$, a *framing pair* (ϕ_0, ϕ) for α consists of isomorphisms of F vector spaces (singular homology)

$$\phi_0: H_1(A_0, \mathbb{Q}) \rightarrow V_0 \quad \phi: H_1(A, \mathbb{Q}) \rightarrow V \quad (5.2.1)$$

such that the induced map

$$\phi_0^{-1}\phi: \text{Hom}_F(H_1(A_0, \mathbb{Q}), H_1(A, \mathbb{Q})) \rightarrow \text{Hom}_F(V_0, V) = V. \quad (5.2.2)$$

is an isomorphism of Hermitian spaces.

The Hodge structures of weight -1 on $H_1(A_0, \mathbb{Q})$ and $H_1(A, \mathbb{Q})$ induce a Hodge structure of weight 0 on V , with an associated complex line $F^1V_{\mathbb{C}} \subseteq V_{\mathbb{C}}^+$. After pullback along the projection isomorphism $V_{\mathbb{R}} \rightarrow V_{\mathbb{C}}^+$ of $F \otimes_{\mathbb{Q}} \mathbb{R}$ vector spaces, the line $F^1V_{\mathbb{C}} \subseteq V_{\mathbb{R}}$ is a negative definite subspace and hence defines a point $z \in \mathcal{D}$. There is a canonical isomorphism of \mathbb{C} vector spaces

$$\text{Hom}_{F \otimes_{\mathbb{Q}} \mathbb{R}}(\text{Lie } A_0, F^0H_1(A, \mathbb{Q})_{\mathbb{C}}^+) \cong F^1V_{\mathbb{C}}. \quad (5.2.3)$$

We use the fixed choice of $\sqrt{\Delta}$ to pass between Hermitian/alternating/symmetric forms [Che24a, Section 2.1]. This makes $H_1(A_0, \mathbb{Q})$ and $H_1(A, \mathbb{Q})$ into Hermitian F -modules. Using the \mathbb{C} -bilinear extension of the symmetric \mathbb{Q} -bilinear trace pairing on $H_1(A, \mathbb{Q})$, we obtain an induced \mathbb{C} -linear identification $F^0H_1(A, \mathbb{Q})_{\mathbb{C}}^+ \cong \text{Hom}_{\mathbb{C}}((\text{Lie } A)^-, \mathbb{C})$.

We equip $F^1V_{\mathbb{C}} \subseteq V_{\mathbb{R}}$ with the Hermitian metric obtained by restricting the metric on $V_{\mathbb{R}}$. Equip $\text{Lie } A_0$ (resp. $\text{Lie } A$) with the Hermitian metric as normalized in [Che24a, (3.5.2)] (resp. [Che24a, (3.5.3)]); see also discussion in Section 2.1. Then $(\text{Lie } A)^- \subseteq \text{Lie } A$ inherits a Hermitian metric as well. Under the isomorphism

$$\text{Hom}_{\mathbb{C}}(\text{Lie } A_0, \mathbb{C}) \otimes \text{Hom}_{\mathbb{C}}((\text{Lie } A)^-, \mathbb{C}) \cong F^1V_{\mathbb{C}} \quad (5.2.4)$$

induced by (5.2.3), the Hermitian metric on the left is $-(16\pi^3 e^{\gamma})^{-1}$ times the Hermitian pairing on the right.

To the datum (α, ϕ_0, ϕ) , there are associated elements $g_0 \in GU(V_0)/K_{0,f}$ and $g \in U(V)(\mathbb{A}_f)/K_f$ given by $g_0 := \phi_0 \circ \tilde{\eta}_0^{-1}$ and $g := (\phi_0^{-1}\phi) \circ \tilde{\eta}^{-1}$ (strictly speaking, ϕ_0 and ϕ are tensored with \mathbb{A}_f here, with $H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f = T(A)^0$ (rational adèlic Tate module) and similarly for A_0).

5.3. Uniformization. For any Hermitian matrix $T \in \text{Herm}_m(\mathbb{Q})$ (with F -coefficients), define the set

$$\mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}} := \left\{ (\alpha, \underline{x}, \phi_0, \phi) : \begin{array}{l} \alpha \in \mathcal{M}_{\mathbb{C}}^{\text{an}} \text{ with } (\alpha, \underline{x}) \in \mathcal{Z}(T)_{\mathbb{C}}^{\text{an}} \\ \text{and } (\phi_0, \phi) \text{ a framing for } \alpha \end{array} \right\}. \quad (5.3.1)$$

There is a canonical injection of sets

$$\begin{aligned} \mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}} &\longrightarrow \mathcal{D} \times G'(\mathbb{A}_f)/K'_f \times V^m \\ (\alpha, \underline{x}, \phi_0, \phi) &\longmapsto (z, (g_0, g_0g), \phi \circ \underline{x} \circ \phi_0^{-1}) \end{aligned} \quad (5.3.2)$$

where the Hodge structure $z \in \mathcal{D}$ and the elements $g_0 \in GU(V_0)(\mathbb{A}_f)/K_{0,f}$ and $g \in U(V)(\mathbb{A}_f)/K_f$ are associated to (α, ϕ_0, ϕ) as in Section 5.2, and $\phi \circ \underline{x} \circ \phi_0^{-1} \in \text{Hom}_F(V_0, V)^m = V^m$ (using the isomorphism $\text{Hom}_{\mathcal{O}_F}(L_0, L) \cong L$ fixed above). This induces a bijection

$$\mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}} \xrightarrow{\sim} \coprod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_\infty) \times \mathcal{D}'(\underline{x}_f). \quad (5.3.3)$$

There is a forgetful map $\mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}} \rightarrow \mathcal{Z}(T)_{\mathbb{C}}^{\text{an}}$ sending $(\alpha, \phi_0, \phi) \mapsto \alpha$. This is surjective, by the Hasse principle (Landherr's theorem) for Hermitian spaces, and factors through an isomorphism of complex manifolds

$$G'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_\infty) \times \mathcal{D}'(\underline{x}_f) \right) \xrightarrow{\sim} \mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}} \quad (5.3.4)$$

where $G'(\mathbb{Q})$ acts on $\mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}}$ as $(\alpha, \phi_0, \phi) \mapsto (\alpha, \gamma_0 \circ \phi_0, \gamma \circ \phi)$ for $(\gamma_0, \gamma) \in G'(\mathbb{Q})$ with γ_0 and γ having the same similitude factor. The case $T = \emptyset$ (or $T = 0$) gives complex uniformization of $\mathcal{M}_{\mathbb{C}, \text{framed}}^{\text{an}}$.

The isomorphism $G' \xrightarrow{\sim} GU(V_0) \times U(V)$ (see $(*)$) induces an isomorphism

$$\begin{aligned} G'(\mathbb{Q}) \backslash \left(\coprod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_\infty) \times \mathcal{D}'(\underline{x}_f) \right) \\ \xrightarrow{\sim} \left(GU(V_0)(\mathbb{Q}) \backslash (GU(V_0)(\mathbb{A}_f)/K_{0,f}) \right) \times \left(U(V)(\mathbb{Q}) \backslash \left(\coprod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_\infty) \times \mathcal{D}'(\underline{x}_f) \right) \right) \end{aligned} \quad (5.3.5)$$

where $U(V)(\mathbb{Q})$ acts on \mathcal{D} via the $U(V)(\mathbb{R})$ action, and on $U(V)(\mathbb{A}_f)/K_f$ by left multiplication.

5.4. Local intersection numbers: Archimedean. Fix $T \in \text{Herm}_m(\mathbb{Q})$ and $y \in \text{Herm}_m(\mathbb{R})_{>0}$ (i.e. y is any positive definite complex Hermitian matrix). Throughout Section 5.4, we require $m \geq n - 1$ if T is positive definite. If T is singular, we also require $m = n$ and $\text{rank}(T) = n - 1$.

For such T which are nonsingular, we recall Kudla's Green current $g_{T,y}$ for $\mathcal{Z}(T)_{\mathbb{C}}^{\text{an}}$ (i.e. the unitary analogue studied by Liu [Liu11, Proof of Theorem 4.20]), which is defined via uniformization and star products. For the case of singular T , we propose a definition of $g_{T,y}$ by a ‘‘linear invariance’’ method, which has some subtleties in the case where T is not $\text{GL}_m(\mathcal{O}_F)$ -equivalent to $\text{diag}(0, T^{\flat})$ for $\det T^{\flat} \neq 0$ (‘‘not diagonalizable’’). Our treatment of this non-diagonalizable case seems to be new, and is based on the local version which we proposed in [Che24b, Section 2.4].

Allowing T singular or not for the moment, define the set

$$J_\infty(T) := GU(V_0)(\mathbb{A}_f)/K_{0,f} \times \coprod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}'(\underline{x}_f) \quad (5.4.1)$$

In one of our companion papers, we will see that the groupoid $[G'(\mathbb{Q}) \backslash J_\infty(T)]$ has with finite stabilizers and finitely many isomorphism classes [Che24c, Lemma 7.4.1]. Given $j \in J_\infty(T)$, we let $\text{Aut}(j) \subseteq G'(\mathbb{Q})$ be the stabilizer for the action of $G'(\mathbb{Q})$ on $J_\infty(T)$.

For any $\gamma \in \mathrm{GL}_m(\mathcal{O}_F)$, recall that there is an induced isomorphism $\mathcal{Z}(T) \xrightarrow{\sim} \mathcal{Z}({}^t\bar{\gamma}T\gamma)$ (i.e. send the tuple of special homomorphisms \underline{x} to $\underline{x} \cdot \gamma$). Similarly, there is an induced isomorphism $\mathcal{Z}(T)_{\mathbb{C}, \text{framed}}^{\text{an}} \rightarrow \mathcal{Z}({}^t\bar{\gamma}T\gamma)_{\mathbb{C}, \text{framed}}^{\text{an}}$. There is corresponding a bijection $J_\infty(T) \rightarrow J_\infty({}^t\bar{\gamma}T\gamma)$ (which we denote $j \mapsto j \cdot \gamma$) sending $\underline{x} \mapsto \underline{x} \cdot \gamma$ for $\underline{x} \in V^m$ (acting trivially on the remaining data, i.e. view $J_\infty(T)$ as a subset of $G'(\mathbb{A}_f)/K'_f \times V^m$; note $\mathcal{D}(\underline{x}_f \cdot \gamma) = \mathcal{D}(\underline{x}_f)$). Note $\mathrm{Aut}(j) = \mathrm{Aut}(j \cdot \gamma)$.

For each $j \in J_\infty(T)$, there is a corresponding map

$$\Theta_j: \mathcal{D} \rightarrow \mathcal{M}_{\mathbb{C}}^{\text{an}} \quad (5.4.2)$$

induced by the uniformization morphism $\mathcal{D} \times G'(\mathbb{A}_f)/K'_f \rightarrow \mathcal{M}_{\mathbb{C}}^{\text{an}}$ (consider the projection $J_\infty(T) \rightarrow G'(\mathbb{A}_f)/K'_f$; by uniformization of $\mathcal{M}_{\mathbb{C}}^{\text{an}}$, every element of $G'(\mathbb{A}_f)/K'_f$ determines a map $\mathcal{D} \rightarrow \mathcal{M}_{\mathbb{C}}^{\text{an}}$). For any $\gamma \in \mathrm{GL}_m(\mathcal{O}_F)$, we have $\Theta_j = \Theta_{j \cdot \gamma}$.

If $\widehat{\mathcal{E}}_{\mathbb{C}}$ denotes the metrized tautological bundle on $\mathcal{M}_{\mathbb{C}}^{\text{an}}$ (Section 2.1, also [Che24a, Section 3.5]) we have $\Theta_j^* \widehat{\mathcal{E}}_{\mathbb{C}} \cong \widehat{\mathcal{E}}$, where $\widehat{\mathcal{E}}$ is the metrized tautological bundle on \mathcal{D} (as in [Che24b, Section 2.2]). By our normalizations, the metric on $\Theta_j^* \widehat{\mathcal{E}}_{\mathbb{C}}$ is $(16\pi^3 e^\gamma)^{-1}$ times the metric on $\widehat{\mathcal{E}}$ (this normalization constant does not change the Chern form $c_1(\widehat{\mathcal{E}})$).

Given any $\underline{x} \in V$ with $(\underline{x}, \underline{x}) = T$, recall that we previously defined a current $[\xi(\underline{x}, y)]$ on the Hermitian symmetric domain \mathcal{D} [Che24b, Section 2.4].

Definition 5.4.1. For T as above (singular or not), we define the real current

$$g_{T,y} := \sum_{j \in J} \frac{1}{|\mathrm{Aut}(j)|} \Theta_{j,*} [\xi(\underline{x}, y)] \quad (5.4.3)$$

on $\mathcal{M}_{\mathbb{C}}^{\text{an}}$, where the sum runs over a set $J \subseteq J_\infty(T)$ of representatives for the isomorphism classes of $[G'(\mathbb{Q}) \backslash J_\infty(T)]$, where $\underline{x} \in V^m$ is the tuple associated with $j \in J_\infty(T)$.

In the preceding definition, $\Theta_{j,*}$ denotes pushforward of currents along Θ_j (for singular T , see the convergence estimates in [Che24b, Section 2.3]). The current $g_{T,y}$ does not depend on the choice of J , by compatibility of $\mathcal{D}(\underline{x})$ and $[\xi(\underline{x})]$ with the $U(V)(\mathbb{R})$ action on \mathcal{D} [Che24b, Section 2.2]. It is also compatible with pullback of currents for varying (small) levels K'_f . When T is nonsingular, this $g_{T,y}$ agrees with the formulation in [Liu11, Proof of Theorem 4.20] (see also [LZ22, §15.3]) up to our different normalization of the Green current ([Che24b, Footnote 2]).

For any $\gamma \in \mathrm{GL}_m(\mathcal{O}_F)$, we have

$$g_{{}^t\bar{\gamma}T\gamma, \gamma^{-1}y{}^t\bar{\gamma}^{-1}} = g_{T,y} \quad (5.4.4)$$

(“global linear invariance”). This follows from the definition of $g_{T,y}$, from local linear invariance of the currents on \mathcal{D} , and the formulas $\mathrm{Aut}(j) = \mathrm{Aut}(j \cdot \gamma)$ and $\Theta_j = \Theta_{j \cdot \gamma}$.

In all cases, we define the Archimedean intersection number

$$\mathrm{Int}_{\infty, \text{global}}(T, y) := \int_{\mathcal{M}_{\mathbb{C}}^{\text{an}}} g_{T,y} \wedge c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^\vee)^{n-m}. \quad (5.4.5)$$

This is a real number, and the integral is convergent by the estimates in [Che24b, Lemmas 2.3.1 and 2.3.3]. It does not depend on the choice of embedding $F \rightarrow \mathbb{C}$. By the compatibility of $g_{T,y}$ with varying small levels K'_f , we can extend (5.4.5) to the case of not-necessarily small level by (3.2.2) (i.e. cover by a small level and divide by the degree of the cover). In the notation of loc.

cit., the stack \mathcal{M} implicitly has level $K'_{L,f}$ (while we are using the notation \mathcal{M} to mean arbitrary level K'_f in Section 5.4).

In all cases (including possibly K'_f not necessarily small level), we have

$$\mathrm{Int}_{\infty, \mathrm{global}}(T, y) = \mathrm{Int}_{\infty}(T, y) \frac{[K_{L_0} : K_{0,f}]}{|\mathcal{O}_F^\times|/h_F} \cdot \mathrm{deg} \left[U(V)(\mathbb{Q}) \setminus \prod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_f) \right] \quad (5.4.6)$$

by construction, where deg means (stacky) groupoid cardinality, where h_F is the class number of \mathcal{O}_F , and where

$$\mathrm{Int}_{\infty}(T, y) := \int_{\mathcal{D}} [\xi(\underline{x}, y)] \wedge c_1(\widehat{\mathcal{E}}^\vee)^{n-m}. \quad (5.4.7)$$

for any $\underline{x} \in V^m$ satisfying $(\underline{x}, \underline{x}) = T$. If there is no such \underline{x} , we set $\mathrm{Int}_{\infty}(T, y) := 0$.

Appendices

APPENDIX A. QUASI-COMPACTNESS OF SPECIAL CYCLES

Besides fixing notation, the purpose of this appendix is to prove a quasi-compactness statement for special cycles (explicit proofs of other properties, e.g. having finite fibers, are more readily available in the literature, e.g. [KR14, Proposition 2.9]). A similar proof of quasi-compactness (in the context of special divisors on some orthogonal Shimura varieties) is [AGHMP17, Proposition 2.7.2].

A.1. Terminology. Suppose A and B are abelian schemes over a base scheme S . We write $\underline{\text{Hom}}(A, B)$ for the fppf sheaf (on S) of homomorphisms of abelian schemes. Then the sheaf of *quasi-homomorphisms* is $\underline{\text{Hom}}^0(A, B) := \underline{\text{Hom}}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$. We write $\text{Hom}^0(A, B)$ for the space of global sections and call elements $x \in \text{Hom}^0(A, B)$ *quasi-homomorphisms*, sometimes writing $x: A \rightarrow B$. If S is quasi-compact, we have $\text{Hom}^0(A, B) = \text{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$. When $A = B$, we often use the notation $\underline{\text{End}}(A)$, $\underline{\text{End}}^0(A)$, and $\text{End}^0(A)$ instead, and often use the term *quasi-endomorphism*. We write $\text{Isog}(A, B)$ for the set of isogenies $A \rightarrow B$. We write $\underline{\text{Isog}}(A, B) \subseteq \underline{\text{Hom}}(A, B)$ for the subsheaf of sets consisting of isogenies, and $\underline{\text{Isog}}^0(A, B) \subseteq \underline{\text{Hom}}^0(A, B)$ for the subsheaf of *quasi-isogenies*, meaning those quasi-homomorphisms which are locally of the form mf for some isogeny f and some nonzero integer $m \in \mathbb{Z}$. We write $\text{Isog}(A, B)$ (resp. $\text{Isog}^0(A, B)$) for the set of *isogenies* (resp. *quasi-isogenies*), consisting of global sections of $\underline{\text{Isog}}(A, B)$ (resp. $\underline{\text{Isog}}^0(A, B)$). We write $\text{Isog}(A)$ and $\text{Isog}^0(A)$ for the self-isogenies and self quasi-isogenies of A . A *quasi-polarization* of A is a quasi-isogeny $A \rightarrow A^\vee$ which is locally of the form $m\lambda$ for some polarization λ and some positive integer $m \in \mathbb{Z}_{>0}$.

Suppose the abelian schemes A and B are equipped with quasi-polarizations $\lambda_A: A \rightarrow A^\vee$ and $\lambda_B: B \rightarrow B^\vee$. Given any $x \in \text{Hom}^0(B, A)$ with dual $x^\vee \in \text{Hom}^0(A^\vee, B^\vee)$, we set $x^\dagger := \lambda_B^{-1} \circ x^\vee \circ \lambda_A \in \text{Hom}^0(A, B)$, and call the resulting map $\dagger: \text{Hom}^0(B, A) \rightarrow \text{Hom}^0(A, B)$ the *Rosati involution*. Given m -tuples $\underline{x}, \underline{y} \in \text{Hom}^0(B, A)^m$ with $\underline{x} = [x_1, \dots, x_m]$ and $\underline{y} = [y_1, \dots, y_m]$, we write $(\underline{x}, \underline{y})$ for the $m \times m$ matrix whose i, j -th entry is $x_i^\dagger y_j$. We say that $(\underline{x}, \underline{x})$ is the *Gram matrix* of \underline{x} . If $S = \text{Spec } k$ for a field k , the \mathbb{Q} -bilinear pairing

$$\begin{aligned} \text{Hom}^0(B, A) \times \text{Hom}^0(B, A) &\longrightarrow \mathbb{Q} \\ (\underline{x}, \underline{y}) &\longmapsto \text{tr}(\underline{x}^\dagger \underline{y}) \end{aligned} \tag{A.1.1}$$

is symmetric and positive definite (“positivity of the Rosati involution”), where $\text{tr}: \text{End}^0(A) \rightarrow \mathbb{Q}$ is the trace for $\text{End}^0(A)$ acting on the \mathbb{Q} -vector space $\text{End}^0(A)$ by left multiplication.

A.2. Proof. We continue in the setup of Section A.1.

Given any $y \in \text{End}^0(B)$, define a functor $\mathcal{Z}(y): (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ as

$$\mathcal{Z}(y) := \{x \in \underline{\text{Hom}}(B, A) : x^\dagger x = y\}. \tag{A.2.1}$$

We will check that $\mathcal{Z}(y)$ is representable by a scheme which is finite, unramified, and of finite presentation over S .

Lemma A.2.1. *The functor $\mathcal{Z}(y)$ is represented by a scheme over S . The structure morphism $\mathcal{Z}(y) \rightarrow S$ is separated and locally of finite presentation.*

Proof. By a standard limit argument (e.g. using [SProject, Lemma 01ZM]) we may reduce to the case where S is Noetherian, affine, and connected. It is also enough to check the case where λ_A and λ_B are polarizations, not just quasi-polarizations.

Existence of the product polarization $\lambda_B \times \lambda_A$ on $B \times A$ implies that $B \times A$ admits a relatively ample line bundle over S . Thus the Hilbert functor $\text{Hilb}_{B \times A}$ is represented by a scheme, each of whose connected components is locally projective over S (in the sense of [SProject, Definition 01W8]), see [Nit05, Theorem 5.15] and [SProject, Lemma 0DPF]. By [SProject, Lemma 0D1B], we know there is a locally closed immersion

$$\mathcal{Z}(y) \rightarrow \text{Hilb}_{B \times A}$$

which sends $x: B \rightarrow A$ to its graph $(1 \times x): B \rightarrow B \times A$. In particular, $\mathcal{Z}(y)$ is represented by a scheme which is separated and locally of finite presentation over S . \square

Lemma A.2.2. *The structure morphism $\mathcal{Z}(y) \rightarrow S$ is quasi-compact.*

Proof. Again, we may reduce to the case where S is affine, Noetherian, and connected by a standard limit argument. It is also enough to check the case where λ_A and λ_B are polarizations, not just quasi-polarizations.

Consider the graph morphisms

$$\begin{aligned} B &\xrightarrow{1 \times \lambda_B} B \times B^\vee \\ A &\xrightarrow{1 \times \lambda_A} A \times A^\vee. \end{aligned}$$

If \mathcal{P}_B and \mathcal{P}_A denote the Poincaré bundles on $B \times B^\vee$ and $A \times A^\vee$ respectively, we know that $\mathcal{L}_B := (1 \times \lambda_B)^* \mathcal{P}_B$ and $\mathcal{L}_A := (1 \times \lambda_A)^* \mathcal{P}_A$ are relatively ample line bundles on B and A , respectively, over S . If $\pi_B: B \times A \rightarrow B$ and $\pi_A: B \times A \rightarrow A$ are the natural projections, we know $\mathcal{E} := \pi_B^* \mathcal{L}_B \otimes \pi_A^* \mathcal{L}_A$ is a relatively ample line bundle on $B \times A$. Moreover, \mathcal{E} is isomorphic to the pullback of the Poincaré bundle $\mathcal{P}_{B \times A}$ along the graph of the polarization $\lambda_B \times \lambda_A$ of $B \times A$. Let $m \in \mathbb{Z}_{\geq 1}$ be any integer such that $m \cdot \lambda_B$ and $m^2 \cdot \lambda_A$ are both honest homomorphisms (rather than quasi-homomorphisms).

As above, write $\text{Hilb}_{B \times A}$ for the Hilbert scheme associated with $B \times A$. Given a numerical polynomial $P: \mathbb{Z} \rightarrow \mathbb{Z}$, we write $\text{Hilb}_{B \times A}^P \subseteq \text{Hilb}_{B \times A}$ for the open and closed subscheme corresponding to the Hilbert polynomial P with respect to the line bundle $\mathcal{E}^{\otimes m^2}$ on $B \times A$. That is, for a S -scheme T , we have

$$\text{Hilb}_{B \times A}^P(T) := \{Z \in \text{Hilb}_{B \times A}(T) : \chi(Z_t, \mathcal{E}^{\otimes m^2 n}|_{Z_t}) = P(n) \text{ for all } n \in \mathbb{Z} \text{ and } t \in T\}$$

(where Z_t is the fiber of $Z \rightarrow T$ over $t \in T$ and χ denotes Euler characteristic). We know that $\text{Hilb}_{B \times A}^P(T)$ is locally projective over S [Nit05, Theorem 5.15], hence quasi-compact over S .

As in the proof of Lemma A.2.1, there is a locally closed immersion $\mathcal{Z}(y) \rightarrow \text{Hilb}_{B \times A}$ which sends $x \in \mathcal{Z}(y)$ to its graph $1 \times x: B \rightarrow B \times A$. To show that $\mathcal{Z}(y)$ is quasi-compact, it is enough

to check that $\mathcal{Z}(y) \rightarrow \mathrm{Hilb}_{B \times A}$ factors through $\mathrm{Hilb}_{B \times A}^P$ for some fixed numerical polynomial P (possibly depending on y).

Consider the line bundle $\mathcal{F} := \mathcal{L}_B^{\otimes m^2} \otimes ((1 \times \lambda_B)^*(m^2 y \times 1)^* \mathcal{P}_B)$ on B . For any point $s \in S$, there is a numerical polynomial $P: \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$P(n) = \chi(B_s, \mathcal{F}^{\otimes n}|_{B_s}) \quad \text{for all } n \in \mathbb{Z} \quad (\text{A.2.2})$$

as in [SProject, Lemma 0BEM]. The polynomial P does not depend on s because S is connected and the Euler characteristics are locally constant as a function of s (using flatness and properness and the standard facts [SProject, Lemma 0BDJ] and [SProject, Section 07VJ]).

Let T be a scheme over S , and suppose $x \in \mathcal{Z}(y)(T)$. View x as an element of $\mathrm{Hilb}_{B \times A}^P(T)$ as above. We claim that $x \in \mathrm{Hilb}_{B \times A}^P(T)$. By taking a base-change to T , we may assume $T = S$ without loss of generality (to lighten notation). It is enough to check $\mathcal{F} \cong (1 \times x)^* \mathcal{E}^{\otimes m^2}$.

First observe $(1 \times x)^* \mathcal{E}^{\otimes m^2} \cong \mathcal{L}_B^{\otimes m^2} \otimes x^* \mathcal{L}_A^{\otimes m^2}$. It is thus enough to verify the identity $x^* \mathcal{L}_A^{\otimes m^2} \cong (1 \times \lambda_B)^*(m^2 y \times 1)^* \mathcal{P}_B$. Consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{1 \times \lambda_A} & A \times A^\vee & & \\ mx \uparrow & & mx \times mx^{\dagger \vee} \uparrow & \swarrow 1 \times mx^{\dagger \vee} & \\ B & \xrightarrow{1 \times \lambda_B} & B \times B^\vee & \xrightarrow{m \times 1} & A \times B^\vee \\ & & m^2 y \times 1 \downarrow & \swarrow mx^\dagger \times 1 & \\ & & B \times B^\vee & & \end{array} .$$

There exists an isomorphism $(mx^\dagger \times 1)^* \mathcal{P}_B \cong (1 \times mx^{\dagger \vee})^* \mathcal{P}_A$ (this characterizes $mx^{\dagger \vee}$ as the dual of mx^\dagger). Recall also that $m^* \mathcal{L}_A \cong \mathcal{L}_A^{\otimes m^2}$ (consider a similar diagram as above, with $A = B$ and $x = y = 1$, and recall that the pullback of \mathcal{P}_A along $(m \times 1): A \times A^\vee \rightarrow A \times A^\vee$ is isomorphic to $\mathcal{P}_A^{\otimes m^2}$ because $m = m^\vee$). These facts prove the claimed identity $x^* \mathcal{L}_A^{\otimes m^2} \cong (1 \times \lambda_B)^*(m^2 y \times 1)^* \mathcal{P}_B$. \square

Lemma A.2.3. *The functor $\mathcal{Z}(y)$ is represented by a scheme over S , and the structure morphism $\mathcal{Z}(y) \rightarrow S$ is finite, unramified, and of finite presentation.*

Proof. Again, we may reduce to the case where S is Noetherian by a standard limit argument. By Lemmas A.2.1 and A.2.2, we already know that $\mathcal{Z}(y)$ is represented by a scheme which is separated and of finite presentation over S .

To see that $\mathcal{Z}(y) \rightarrow S$ is proper, we can use the valuative criterion for discrete valuation rings [SProject, Lemma 0207] because S is Noetherian. This valuative criterion holds by the Néron mapping property for abelian schemes over discrete valuation rings.

For unramifiedness, it is enough to check that $\mathcal{Z}(y) \rightarrow S$ is formally unramified (i.e. satisfies the infinitesimal lifting criterion of [SProject, Lemma 02HE]). Formal unramifiedness holds because of rigidity for morphisms of abelian schemes as in [MFK94, Corollary 6.2].

Since unramified morphisms of schemes are locally quasi-finite, and since proper locally quasi-finite morphisms of schemes are finite, the lemma is proved. \square

Recall that if \mathcal{X} and \mathcal{Y} are categories fibered in groupoids over the fppf site of some base scheme with \mathcal{Y} being a Deligne–Mumford stack, and if $f: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism which is representable

by algebraic spaces, then \mathcal{X} is also a Deligne–Mumford stack [SProject, Comment 2142]. This can be used in combination with Lemma A.2.3 to verify that various stacks in this work are Deligne–Mumford.

APPENDIX B. MISCELLANY ON p -DIVISIBLE GROUPS

We collect some terminology/notation and miscellaneous facts about p -divisible groups, which we use freely.

B.1. Terminology. Suppose S is a formal scheme²¹ and suppose \mathcal{P} is a property of morphisms of schemes which is fppf local on the target and stable under arbitrary base-change. A sheaf X on $(Sch/S)_{fppf}$ is *represented by a relative scheme with property \mathcal{P} over S* if, for every scheme T over S , the restriction sheaf $X|_T$ is represented by a scheme with property \mathcal{P} over T .

Fix a prime p . A *p -divisible group* over a formal scheme S is a sheaf X of abelian groups on $(Sch/S)_{fppf}$ which satisfies the following conditions.

- (1) (*p -divisibility*) The multiplication by p map $[p]: X \rightarrow X$ is a surjection of sheaves.
- (2) (*p^∞ -torsion*) The natural map $X[p^\infty] := \varinjlim_n X[p^n] \rightarrow X$ is an isomorphism, where $X[p^n] \subseteq X$ are the p^n -torsion subsheaves.
- (3) (*representable p -power-torsion*) The sheaves $X[p^n]$ are represented by finite locally free relative schemes over S for all $n \geq 1$.

If S is an adic (e.g. locally Noetherian) formal scheme and \mathcal{I} is an ideal sheaf of definition on S , giving a p -divisible group over S is the same as giving p -divisible groups X_n over each scheme $S_n := (S, \mathcal{O}_S/\mathcal{I}^n)$ with isomorphisms $X_{n+1}|_{S_n} \xrightarrow{\sim} X_n$.

For a general formal scheme S , we say a p -divisible group X over S has *height h* if $X[p]$ is finite locally free relative scheme over S of degree p^h . In general, h is understood as a locally constant function on S .

If p is locally topologically nilpotent on S (equivalently, S is a formal scheme over $\mathrm{Spf} \mathbb{Z}_p$) and if X is a p -divisible group over S , there is an associated sheaf $\mathrm{Lie} X$ on $(Sch/S)_{fppf}$ (constructed as in [SGA3II, Definition 3.2]). By work of Messing [Mes72, Theorem 3.3.18], it is known that $\mathrm{Lie} X$ is a finite locally free sheaf of modules on $(Sch/S)_{fppf}$. We refer to the dual $\Omega_X := (\mathrm{Lie} X)^\vee$ as a *Hodge bundle*. If r is the rank of $\mathrm{Lie} X$, we say that X has *dimension r* (in general, r is a locally constant $\mathbb{Z}_{\geq 0}$ -valued function). In this case, we write $\omega_X := \bigwedge^r \Omega_X$ for the top exterior power and also call ω_X a *Hodge bundle*.

If p is locally topologically nilpotent on the formal scheme S , a *formal p -divisible group* X over S is a p -divisible group over S such that, fppf (equivalently, Zariski) locally on any $T \in \mathrm{Obj}(Sch/S)_{fppf}$, the pointed fppf sheaf X is isomorphic to $\mathrm{Spf} \mathcal{O}_T[[x_1, \dots, x_r]]$ for some r (possibly varying). See [Mes72, Proposition II.4.4] for equivalent characterizations.

Given p -divisible groups X and Y over a general formal scheme S , a *quasi-homomorphism* is a global section of the sheaf $\underline{\mathrm{Hom}}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ on $(Sch/S)_{fppf}$. We write $\mathrm{Hom}^0(X, Y)$ for the

²¹The formal schemes we use are the “pré-schémas formels” of [EGAI, §10]. Given a formal scheme, the notation $(Sch/S)_{fppf}$ means the site whose objects are morphisms $T \rightarrow S$ for schemes T , where coverings are fppf.

space of quasi-homomorphisms $X \rightarrow Y$, and similarly $\text{End}^0(X) = \text{Hom}^0(X, X)$. Given a quasi-compact scheme T with a map $T \rightarrow S$, we have $\text{Hom}^0(X_T, Y_T) = \text{Hom}(X_T, Y_T) \otimes_{\mathbb{Z}} \mathbb{Q}$. If X and Y are equipped with an action by a ring R , then $\text{Hom}_R^0(X, Y)$ will denote the R -linear quasi-homomorphisms.

A morphism $f: X \rightarrow Y$ of p -divisible groups over S is an *isogeny* if f is a surjection of fppf sheaves and $\ker f$ is represented by a finite locally free relative scheme over S . If $\ker f$ is finite locally free of rank p^r , we say that f has *degree* p^r and *height* r . A *quasi-isogeny* $f: X \rightarrow Y$ is a quasi-homomorphism which, locally on $(\text{Sch}/S)_{\text{fppf}}$, is of the form $f = p^n g$ for $n \in \mathbb{Z}$ and an isogeny g . If the p -divisible group X has height h , such a quasi-isogeny $f = p^n g$ is said to have *degree* $p^{nh} \deg(g)$ and *height* $nh + \text{height}(g)$. We write $\text{Isog}(X, Y)$ (resp. $\text{Isog}^0(X, Y)$) for the isogenies (resp. quasi-isogenies) $X \rightarrow Y$. We write $\text{Isog}(X)$ (resp. $\text{Isog}^0(X)$) for self-isogenies (resp. self quasi-isogenies) of X .

A p divisible group X over S is *étale* if $X[p]$ is an étale relative scheme. This implies that each $X[p^n]$ is an étale relative scheme. If R is a Noetherian Henselian local ring, we say that a p -divisible group X over $\text{Spec } R$ is *connected* if $X[p]$ is connected. This implies that each $X[p^n]$ is connected.

Given any p -divisible group X over a general formal scheme S , there is a *dual* p -divisible group X^\vee . A *polarization* of X is an isogeny $\lambda: X \rightarrow X^\vee$ satisfying $\lambda^\vee = -\lambda$. The polarization is *principal* if λ is an isomorphism. A *quasi-polarization* is a quasi-isogeny $f: X \rightarrow X^\vee$ such that mf is a polarization for some $m \in \mathbb{Q}_p^\times$. Suppose X and Y are p -divisible groups over S with quasi-polarizations $\lambda_X: X \rightarrow X^\vee$ and $\lambda_Y: Y \rightarrow Y^\vee$. Given any $x \in \text{Hom}^0(Y, X)$ with dual $x^\vee \in \text{Hom}^0(X^\vee, Y^\vee)$, we set $x^\dagger := \lambda_Y^{-1} \circ x^\vee \circ \lambda_X \in \text{Hom}^0(X, Y)$, and call the resulting map $\dagger: \text{Hom}^0(Y, X) \rightarrow \text{Hom}^0(X, Y)$ the *Rosati involution*.

Over an algebraically closed field, we say that a p -divisible group is *supersingular* if all slopes of its isocrystal are equal to $1/2$, and we say that it is *ordinary* if all slopes of its isocrystal are either 0 or 1. A p -divisible group over an arbitrary formal scheme is *supersingular* (resp. *ordinary*) if it is supersingular (resp. ordinary) for every geometric fiber.

Over any algebraically closed field, there is a unique étale p -divisible group of height r (namely the constant sheaf $(\mathbb{Q}_p/\mathbb{Z}_p)^r$). Over any algebraically closed field of characteristic p , there is also a unique p -divisible group of height r with all slopes of its isocrystal being 1 (namely $\mu_{p^\infty}^r := (\varinjlim_e \mu_{p^e})^r \cong (\mathbb{Q}_p/\mathbb{Z}_p^\vee)^r$, given by p -th power roots of unity). Since the connected étale exact sequence of any p -divisible group over a perfect field is (canonically) split, we conclude that $\mu_{p^\infty}^{n-r} \times (\mathbb{Q}_p/\mathbb{Z}_p)^r$ is the unique ordinary p -divisible group of height n and dimension $n - r$ over any algebraically closed field.

By *Drinfeld rigidity* we mean the following phenomenon: if $S_0 \rightarrow S$ is a finite order thickening of schemes over $\text{Spf } \mathbb{Z}_p$, and X, Y are p -divisible groups over S , any quasi-homomorphism of $X \rightarrow Y$ over S_0 lifts uniquely to a quasi-homomorphism over S [And03, Theorem 2.2.3] (alternative proof: Grothendieck–Messing theory).

If A is a relative abelian scheme over a general formal scheme S , there is an associated p -divisible group $A[p^\infty] := \varinjlim_n A[p^n]$, where $A[p^n]$ is the p^n -torsion subfunctor of A . If p is locally topologically nilpotent on S , there is a canonical identification $\text{Lie } A \cong \text{Lie } A[p^\infty]$.

Given a p -divisible group X over a formal scheme S and given a finite free \mathbb{Z}_p -module M of some rank $d \geq 0$, there is the *Serre tensor construction* p -divisible group $X \otimes_{\mathbb{Z}_p} M$ given by the functor

$$(X \otimes_{\mathbb{Z}_p} M)(T) := X(T) \otimes_{\mathbb{Z}_p} M \quad (\text{B.1.1})$$

for schemes T over S . Any choice of \mathbb{Z}_p -basis for M gives an isomorphism $X \otimes_{\mathbb{Z}_p} M \cong X^d$ as p -divisible groups. This construction is functorial in M : in particular, any \mathbb{Z}_p -algebra R acting on M also acts on $X \otimes_{\mathbb{Z}_p} M$. The resulting R -action on $X \otimes_{\mathbb{Z}_p} M$ is the *Serre tensor R -action*. There is a canonical identification $(X \otimes_{\mathbb{Z}_p} M)^\vee \cong X^\vee \otimes_{\mathbb{Z}_p} M^\vee$ where $M^\vee := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$. More generally, see [Con04, §7].

B.2. Isogeny criterion. We explain a criterion for a morphism of p -divisible groups to be an isogeny (Lemma B.2.2). This should be well-known.²²

Lemma B.2.1. *Let S be a scheme, and let H , G , and Q be commutative group schemes over S which are locally of finite presentation. Suppose*

$$0 \rightarrow H \rightarrow G \xrightarrow{f} Q \rightarrow 0$$

is an exact sequence of fppf sheaves of abelian groups. If $G \rightarrow S$ is finite locally free and $Q \rightarrow S$ is separated, then

- (1) *The map $f: G \rightarrow Q$ is finite locally free.*
- (2) *The group schemes Q and H are finite locally free over S .*

Proof. Since f is a surjection of fppf sheaves, it is a surjection on underlying topological spaces. We also know that f is locally of finite presentation because both G and Q are locally of finite presentation over S [SProject, Lemma 00F4]. Since $G \rightarrow S$ is flat, the fibral flatness criterion [EGAIV3, 11.3.11] implies that flatness of f may be checked fiberwise over S , i.e. it is enough to check flatness of the base-change $G_{k(s)} \rightarrow Q_{k(s)}$ for each $s \in S$. The exact sequence

$$0 \rightarrow H_{k(s)} \rightarrow G_{k(s)} \rightarrow Q_{k(s)} \rightarrow 0$$

Since $H = \ker(f)$ and f is an fppf morphism, we know $H \rightarrow S$ is fppf as well. Since $Q \rightarrow S$ is separated, the identity section $S \rightarrow Q$ is a closed immersion, hence $H = \ker(f)$ is a closed subscheme of G . Since $G \rightarrow S$ is finite, we conclude that $H \rightarrow S$ is also finite, hence finite locally free.

We have already seen that $Q \rightarrow S$ is flat, proper, and locally of finite presentation. To check that $Q \rightarrow S$ is finite, it is enough to check that it has finite fibers, which follows because $G \rightarrow Q$ is surjective and $G \rightarrow S$ is finite. \square

Lemma B.2.2. *Let S be a formal scheme. Let $f: X \rightarrow Y$ be a homomorphism of p -divisible groups over S . Then f is an isogeny if and only if, locally on $(\text{Sch}/S)_{\text{fppf}}$, there exists a homomorphism $g: Y \rightarrow X$ such that*

$$g \circ f = [p^N] \quad f \circ g = [p^N]$$

for some integer $N \geq 0$, where $[p^N]$ denotes multiplication by p^N .

²²The only reference I know is the sketch in [Far05, Lemme 9]. We spell out the argument for completeness.

Moreover, given an isogeny f , such g, N will exist globally on S if S is quasi-compact or has finitely many connected components. If f is an isogeny of constant degree p^n , we may take $N = n$.

Proof. If $f: X \rightarrow Y$ is an isogeny, then Y is the fppf sheaf quotient of X by $\ker(f)$. If S is a quasi-compact formal scheme or if S has finitely many connected components, we have $\ker f \subseteq X[p^N]$ for N large, so $g \circ f = [p^N]$ for some homomorphism $g: Y \rightarrow X$. We also have $f \circ g \circ f = [p^N] \circ f$. Since f is an epimorphism of fppf sheaves, we conclude that $f \circ g = [p^N]$.

Conversely, suppose that locally on $(Sch/S)_{fppf}$ there exists a homomorphism $g: Y \rightarrow X$ and an integer $N \geq 0$ as in the lemma statement. Since the property of being an isogeny may be checked locally on $(Sch/S)_{fppf}$, we may assume that S is a scheme and that g, N exist globally on S . Since $f \circ g = [p^N]$, we see that f is a surjection of fppf sheaves. It remains only to check that $\ker f$ is representable by a finite locally free group scheme over S .

We know that $\ker(f) \subseteq X[p^N]$ and $\ker(g) \subseteq Y[p^N]$. We have $\ker(f) = \ker(X[p^N] \rightarrow Y[p^N])$ and $\ker(g) = \ker(Y[p^N] \rightarrow X[p^N])$. Since $X[p^N]$ and $Y[p^N]$ represented by finite locally free group schemes over S , we see that $\ker(f)$ and $\ker(g)$ are represented by schemes which are finite and locally of finite presentation over S .

We have short exact sequences

$$\begin{aligned} 0 \rightarrow \ker(f) \rightarrow X[p^N] \xrightarrow{f} \ker(g) \rightarrow 0 \\ 0 \rightarrow \ker(g) \rightarrow Y[p^N] \xrightarrow{g} \ker(f) \rightarrow 0 \end{aligned}$$

of fppf sheaves of abelian groups. By Lemma B.2.1, we conclude that $\ker(f)$ and $\ker(g)$ are finite locally free group schemes over S . \square

Lemma B.2.3. *Let S be a formal scheme. Let X and Y be p -divisible groups over S . Then $f \in \text{Hom}^0(X, Y)$ is a quasi-isogeny if and only if it is invertible, meaning there exists $g \in \text{Hom}^0(Y, X)$ (necessarily unique) with $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$.*

Proof. Invertibility and the property of being a quasi-isogeny can both be checked locally on $(Sch/S)_{fppf}$, so the lemma follows from Lemma B.2.2. \square

B.3. p -divisible groups over $\text{Spec } A$ and $\text{Spf } A$. The following facts are implicitly used (including in our companion paper [Che24a]).

Lemma B.3.1. *Let A be an adic Noetherian ring. There are equivalences of categories*

$$\begin{aligned} \{\text{finite schemes over } \text{Spec } A\} &\rightarrow \{\text{finite relative schemes over } \text{Spf } A\} \\ \{\text{finite locally free schemes over } \text{Spec } A\} &\rightarrow \{\text{finite locally free relative schemes over } \text{Spf } A\} \\ \{p\text{-divisible groups over } \text{Spec } A\} &\rightarrow \{p\text{-divisible groups over } \text{Spf } A\} \end{aligned}$$

given by base change, i.e. restriction of fppf sheaves along the inclusion $(Sch/\text{Spf } A)_{fppf} \rightarrow (Sch/\text{Spec } A)_{fppf}$.

Proof. For the statements about finite relative schemes, the quasi-inverse functor is given by $\text{Spf } R \mapsto \text{Spec } R$ for finite A -algebras R (topologized so that R is an adic ring and the map $A \rightarrow R$ is adic). This also gives the quasi-inverse functor for finite locally free relative schemes (check using

the local criterion for flatness). For the statement about p -divisible groups (which follows from the other statements), see [Mes72, 4.15, Lemma II.4.16] or [dJo95, Lemma 2.4.4]. \square

Lemma B.3.2. *Let A be an adic Noetherian ring, and let $\phi: X \rightarrow Y$ be a homomorphism of p -divisible groups over $\mathrm{Spec} A$. Then ϕ is an isogeny if and only if $\phi_{\mathrm{Spf} A}: X_{\mathrm{Spf} A} \rightarrow Y_{\mathrm{Spf} A}$ is an isogeny.*

Proof. Follows from Lemma B.3.1 and the isogeny criterion from Lemma B.2.2. \square

For adic Noetherian rings A , we may thus pass between p -divisible groups over $\mathrm{Spec} A$ and $\mathrm{Spf} A$ without loss of information, and similarly for finite locally free relative schemes. We abuse notation in this way: for example, if A is a domain, the *generic fiber* of a p -divisible group over $\mathrm{Spf} A$ will refer to its generic fiber as a p -divisible group over $\mathrm{Spec} A$.

To avoid potential confusion, we remark on three situations where p -divisible groups may have different properties when considered over $\mathrm{Spec} A$ versus over $\mathrm{Spf} A$.

Remark B.3.3. Let A be an adic Noetherian ring, and suppose p is topologically nilpotent in A . Let X be a p -divisible group over $\mathrm{Spec} A$. By work of Messing, [Mes72, §II], the sheaf $\mathrm{Lie}(X_{\mathrm{Spf} A})$ (in the sense of [SGA3II]) is locally free of finite rank on $(\mathrm{Sch}/\mathrm{Spf} A)_{\mathrm{fppf}}$. However, $\mathrm{Lie} X$ (viewed as a sheaf on $(\mathrm{Sch}/\mathrm{Spec} A)_{\mathrm{fppf}}$) is *not* necessarily locally free.

For example, consider $A = \mathbb{Z}_p$ and $X = \mu_{p^\infty} := \varinjlim \mu_{p^n}$, where μ_{p^n} is the group scheme of p^n -th roots of unity. Then the p -divisible group X over $\mathrm{Spec} \mathbb{Z}_p$ is étale in the generic fiber, but connected of dimension 1 in the special fiber. We find that $\mathrm{Lie} X|_{\mathrm{Spec} \mathbb{Q}_p} = 0$ but $\mathrm{Lie} X|_{\mathrm{Spec} \mathbb{F}_p}$ is free of rank 1, so $\mathrm{Lie} X$ cannot be a locally free sheaf of modules on $(\mathrm{Sch}/\mathrm{Spec} A)_{\mathrm{fppf}}$.

Thus, when writing $\mathrm{Lie} X$ in this situation, we always mean (by abuse of notation) to view X as a p -divisible group over $\mathrm{Spf} A$, so that $\mathrm{Lie} X$ will be a finite locally free sheaf on $(\mathrm{Sch}/\mathrm{Spf} A)_{\mathrm{fppf}}$. Similarly, if we say X has dimension r , we mean that the finite locally free sheaf $\mathrm{Lie} X$ on $(\mathrm{Sch}/\mathrm{Spf} A)_{\mathrm{fppf}}$ has rank r .

Remark B.3.4. Let A be an adic Noetherian ring, and let X be a p -divisible group over $\mathrm{Spec} A$. In general, there are sections of $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ which do not arise as sections of $X \rightarrow \mathrm{Spec} A$. Indeed, sections of $X \rightarrow \mathrm{Spec} A$ correspond precisely to torsion sections of $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ (use quasi-compactness of $\mathrm{Spec} A$). But $X_{\mathrm{Spf} A} \rightarrow \mathrm{Spf} A$ may have many non-torsion sections, e.g. when $A = \mathbb{Z}_p$ and $X_{\mathrm{Spf} A}$ is a formal p -divisible group, hence $X_{\mathrm{Spf} A} \cong \mathrm{Spf} \mathbb{Z}_p[[X_1, \dots, X_r]]$ as pointed fppf sheaves on $(\mathrm{Sch}/\mathrm{Spf} \mathbb{Z}_p)_{\mathrm{fppf}}$. There will be uncountably many non-torsion sections in this situation. This makes a difference in [Che24a, Section 5.1], for example, where some statements are correct over $\mathrm{Spf} R$ (which is the written version) but incorrect over $\mathrm{Spec} R$.

Remark B.3.5. Let A be an adic Noetherian ring. By our conventions, it is *not* true that any quasi-homomorphism of p -divisible groups over $\mathrm{Spf} A$ necessarily lifts to a quasi-homomorphism of p -divisible groups over $\mathrm{Spec} A$. See [Che24a, Example 6.1.1]. On the other hand, homomorphisms and isogenies will lift (uniquely) by the preceding lemmas.

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