CO-RANK 1 ARITHMETIC SIEGEL–WEIL IV:
ANALYTIC LOCAL-TO-GLOBAL

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ABSTRACT. This is the fourth in a sequence of four papers, where we prove the arithmetic Siegel–Weil formula in co-rank 1 for Kudla–Rapoport special cycles on exotic smooth integral models of unitary Shimura varieties of arbitrarily large even arithmetic dimension. Our arithmetic Siegel–Weil formula implies that degrees of Kudla–Rapoport arithmetic special 1-cycles are encoded in the first derivatives of unitary Eisenstein series Fourier coefficients.

In this paper, we pin down precise normalizations for some $U(m, m)$ Siegel Eisenstein series, give local Siegel–Weil special value formulas with explicit constants, and record a geometric Siegel–Weil result for degrees of complex 0-cycles. Using this, we complete the proof of our arithmetic Siegel–Weil results by patching together the local main theorems from our companion papers.

CONTENTS

1. INTRODUCTION

This paper is a continuation of our companion papers [Che24a; Che24b; Che24c]. In this paper, we complete the proof of our global arithmetic Siegel–Weil results using our local main theorems from [Che24a; Che24b] and the geometric local-to-global reduction procedure from [Che24c]. We refer the reader to the introduction of [Che24a] for additional motivation, overview, and strategy for our four-part sequence of papers.

For the reader's convenience, we recall the statements of our global arithmetic Siegel–Weil results (Section 1.3); this section is an abridged version of the analogous [Che24a, Sections 1.2 and 1.3].

In the present paper, the main new ingredients are (1) precise normalizations for Eisenstein series and their relation with singular Fourier coefficients of co-rank 1, (2) classical local Siegel–Weil formulas with precise constants, and (3) a geometric Siegel–Weil formula for complex 0-cycles, which will be treated in Section 8 (along with an observation about complex volumes of unitary Shimura varieties, which may be of independent interest).

For introductory purposes, Section 1.2 contains some background on classical and geometric Siegel–Weil. This is included for comparison with arithmetic Siegel–Weil, and also helps us fix some notation. The material in Section 1.2 is mostly expository, but some of our formulations may be new, particularly in our normalizations for Eisenstein series. The same normalization choices play an amplified role in our main arithmetic Siegel–Weil results. We also mention some results (comparison of complex volume and degrees of complex zero cycles to Eisenstein series) which seem to be new or at least not explicit in the literature; see discussion following (1.2.6) and (1.2.10).

In Section 1.4, we outline the structure of this paper and its relation with our companion papers [Che24a; Che24b; Che24c].

1.1. Eisenstein series.
In our work, we focus on the unitary/Hermitian case. For the introduction, fix an imaginary quadratic field $F/\mathbb{Q}$ with ring of integers $\mathcal{O}_F$ and odd discriminant Δ. Given $m \in \mathbb{Z}_{\geq 0}$ and an even integer $n \in \mathbb{Z}$, we consider the (normalized) *Siegel Eisenstein series*

$$E^*(z,s)_n^\circ := \Lambda_m(s)_n^\circ \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P_1(\mathbb{Z}) \backslash SU(m,m)(\mathbb{Z})} \frac{\det(y)^{s-s_0}}{\det(cz+d)^n |\det(cz+d)|^{2(s-s_0)}} \tag{1.1.1}$$

for the group

$$U(m,m) := \left\{ h \in \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \, \mathrm{GL}_{2m} : {}^t\bar{h} \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} h = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} \right\} \tag{1.1.2}$$

where $\Lambda_m(s)_n^\circ$ is the normalizing factor

$$\Lambda_m(s)_n^\circ := \frac{(2\pi)^{m(m-1)/2}}{(-2\pi i)^{nm}} \pi^{m(-s+s_0)} |\Delta|^{m(m-1)/4+\lfloor m/2 \rfloor(s+s_0)} \tag{1.1.3}$$

$$\cdot \left(\prod_{j=0}^{m-1} \Gamma(s-s_0+n-j) \cdot L(2s+m-j, \eta^{j+n}) \right).$$

In (1.1.2), the notation 1_m stands for the $m \times m$ identity matrix, we wrote $SU(m,m) \subseteq U(m,m)$ for the determinant 1 subgroup, and we set $P_1 := P \cap SU(m,m)$ for the Siegel parabolic $P \subseteq U(m,m)$

(consisting of $m \times m$ block upper triangular matrices). The variable $s \in \mathbb{C}$ is a complex parameter, we set $s_0 = (n-m)/2$, and the element $z = x + iy$ lies in Hermitian upper-half space (i.e. $x \in \mathrm{Herm}_m(\mathbb{R})$ and $y \in \mathrm{Herm}_m(\mathbb{R})_{>0}$; the latter means that y is positive definite).[1] The symbol η denotes the quadratic character associated to $F/\mathbb{Q}$ (via class field theory). The sum in (1.1.1) is convergent for $\mathrm{Re}(s) > m/2$, and admits meromorphic continuation to all $s \in \mathbb{C}$. When $m = 1$, the expression in (1.1.1) is a classical Eisenstein series on the usual upper-half plane.

The normalized Eisenstein series has a symmetric functional equation

$$E^*(z,s)_n^\circ = (-1)^{m(m-1)(n-m-1)/2} E^*(z,-s)_n^\circ, \tag{1.1.4}$$

see Section 6.1. Our definition of the normalizing factor $\Lambda_m(s)_n^\circ$ is motivated by symmetry of global and local functional equations, along with certain local special value formulas; see Sections 2 to 6 for further discussion. The function $\Lambda_m(s)_n^\circ$ should be closely related with the L-function of an Artin–Tate motive attached to the group $U(m,m)$, in the sense of Gross [Gro97] (see [BH21, Remark 1.1.1]).

Given $T \in \mathrm{Herm}_m(\mathbb{Q})$, the Eisenstein series $E^*(z,s)_n^\circ$ has T-th Fourier coefficient

$$E_T^*(y,s)_n^\circ := 2^{m(m-1)/2} |\Delta|^{-m(m-1)/4} \int_{\mathrm{Herm}_m(\mathbb{Z}) \backslash \mathrm{Herm}_m(\mathbb{R})} E^*(z,s)_n^\circ e^{-2\pi i \mathrm{tr}(Tz)} \, dx \tag{1.1.5}$$

for $z = x + iy$ in Hermitian upper-half space, where this integral is taken with respect to the Euclidean measure[2] on $\mathrm{Herm}_m(\mathbb{R})$. The integral is convergent for $\mathrm{Re}(s) > m/2$, and admits meromorphic continuation to all $s \in \mathbb{C}$. When $\det T \neq 0$, there is a factorization into normalized *local Whittaker functions*

$$E_T^*(y,s)_n^\circ = W_{T,\infty}^*(y,s)_n^\circ \prod_p W_{T,p}^*(s)_n^\circ \tag{1.1.6}$$

over all places, see Part 1. The preceding setup is as in [Che24a, Section 1.1], and is taken from loc. cit. verbatim.

For example, if $n = 2$ and $m = 1$, we have

$$W_{T,p}^*(s)_2^\circ = p^{v_p(T)(s+1/2)} \sigma_{-2s}(p^{v_p(T)}) \qquad \sigma_s(r) := \sum_{d|r} d^s \tag{1.1.7}$$

$$W_{T,\infty}^*(y,s)_2^\circ = \Gamma(s-1/2)^{-1} |4\pi T y|^{s-1/2} \int_a^\infty e^{-4\pi |T| y u}(u \pm 1)^{s+1/2} u^{s-3/2}$$

for any nonzero $T \in \mathbb{Z}$, where $a = 0$ and the sign $\pm$ is $+$ (resp. $a = 1$ and the sign $\pm$ is $-$) if $T > 0$ (resp. if $T < 0$). Here $W_{T,\infty}^*(y,s)_2^\circ$ (resp. $W_{T,p}^*(s)_2^\circ$) is a certain normalized Archimedean (resp. non-Archimedean) local Whittaker function.

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[1] Here, the notation Herm_m denotes a scheme over $\mathrm{Spec}\,\mathbb{Z}$, e.g. $\mathrm{Herm}_m(\mathbb{R})$ denotes $m \times m$ complex Hermitian matrices, and $\mathrm{Herm}_m(\mathbb{Q})$ denotes $m \times m$ Hermitian matrices with entries in F.

[2] The factor $2^{m(m-1)/2} |\Delta|^{-m(m-1)/4}$ disappears in the (usual) equivalent adèlic formulation, upon taking a certain self-dual Haar measure. The adèlic formulations of (1.1.1) and (1.1.5) are used in Section 2.

1.2. Classical and geometric Siegel–Weil.

Let V be an n-dimensional F-vector space, equipped with a non-degenerate Hermitian pairing $(-,-)$. Set $G = U(V)$ and assume $n > 0$. Fix a full-rank $\mathcal{O}_F$-lattice $L \subseteq V$. For simplicity, we assume in the introduction that L is self-dual.[3] Write $K_{L,f} \subseteq G(\mathbb{A}_f)$ for the stabilizer of $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, where $\mathbb{A}_f$ denotes the finite adèle ring of $\mathbb{Q}$.

First consider the case where V is positive definite. Since we assumed L is self-dual, this forces $n \equiv 0 \pmod 4$ (by the global product formula for local invariants of Hermitian spaces). Given any positive definite Hermitian $\mathcal{O}_F$-lattice $\mathcal{L}$, we set

$$\mathcal{Z}_{T,\mathcal{L}} := \{\underline{x} \in \mathcal{L}^m : (\underline{x}, \underline{x}) = T\}, \tag{1.2.1}$$

where $(\underline{x}, \underline{x})$ denotes the Gram matrix[4] of $\underline{x}$. When $m \leq n$, we have

$$2\Lambda_n(0)_n^\circ = \sum_{\substack{\mathcal{O}_F\text{-lattices } \mathcal{L} \\ \text{self-dual, rank } n, \\ \text{positive definite}}} \frac{1}{|\operatorname{Aut}(\mathcal{L})|} = \#[G(\mathbb{Q})\backslash(G(\mathbb{A}_f)/K_{L,f})] \tag{1.2.2}$$

$$\frac{2\Lambda_n(0)_n^\circ}{\kappa\Lambda_m(s_0)_n^\circ} E_T^*(y, s_0)_n^\circ = \sum_{\substack{\mathcal{O}_F\text{-lattices } \mathcal{L} \\ \text{self-dual, rank } n, \\ \text{positive definite}}} \frac{|\mathcal{Z}_{T,\mathcal{L}}|}{|\operatorname{Aut}(\mathcal{L})|} \qquad \text{for any } T \in \operatorname{Herm}_m(\mathbb{Q}), \tag{1.2.3}$$

where $\kappa = 2$ (resp. $\kappa = 1$) if $m = n$ (resp. if $m < n$). The sums run over isomorphism classes of positive definite rank n self-dual $\mathcal{O}_F$-lattices, the notation $\operatorname{Aut}(\mathcal{L})$ means the (unitary) automorphism group of $\mathcal{L}$. The symbols $\#[-]$ and $|-|$ mean groupoid and set cardinality, respectively. That is, we have

$$\frac{2\Lambda_n(0)_n^\circ}{\kappa\Lambda_m(s_0)_n^\circ} E^*(z, s_0)_n^\circ = \sum_{\substack{\mathcal{O}_F\text{-lattices } \mathcal{L} \\ \text{self-dual, rank } n, \\ \text{positive definite}}} \frac{1}{|\operatorname{Aut}(\mathcal{L})|}\Theta_{\mathcal{L}}(z) \tag{1.2.4}$$

which re-expresses the Eisenstein series at $s = s_0$ as a weighted sum of theta series for the lattices $\mathcal{L}$.

Equations (1.2.2) and (1.2.3) are special cases of (unitary analogues of) the classical *Siegel mass formula* and *Siegel–Weil formula* respectively. For (1.2.2), see Proposition 8.2.1. Equation (1.2.3) follows from [Ich04, Proposition 6.2], [Ich07, Theorem 1.1], and [Yam11, Theorem 2.2] (in combination with (1.2.2)).

Next, consider the case where V has arbitrary signature $(n-r, r)$. Since L was assumed self-dual, this forces $n \equiv 2^r \pmod 4$. There is an associated Hermitian symmetric domain $\mathcal{D}$ which parameterizes maximal negative definite subspaces of the complex Hermitian space $V_{\mathbb{R}}$. For sufficiently small open compact $K_f \subseteq K_{L,f}$ (so that we have manifolds instead of orbifolds, for simplicity), there is an associated complex Shimura variety

$$\operatorname{Sh}_{K_f,\mathbb{C}}(G) = [G(\mathbb{Q})\backslash(\mathcal{D} \times G(\mathbb{A}_f)/K_f)] \tag{1.2.5}$$

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[3]We always mean self-dual for the bilinear *trace pairing* $\operatorname{tr}_{F/\mathbb{Q}}(v, w)$ unless otherwise specified; see conventions in [Che24a, Section 2.2].

[4]If $\underline{x}$ is the m-tuple $[x_1, \ldots, x_m]$, the notation $(\underline{x}, \underline{x})$ will mean the matrix with i, j-th entry (x_i, x_j). We often write e.g. $[x_1, \ldots, x_m]$ instead of $(x_1, \ldots, x_m)$ for tuples, to avoid confusion with Hermitian pairings $(-,-)$.

of dimension $(n-r)r$ (analytification suppressed from notation). In the signature $(n,0)$ and $(n-1,1)$ cases respectively, we have "geometric Siegel mass formulas"

$$2\Lambda_n(0)_n^\circ = \frac{\mathrm{vol}(\mathrm{Sh}_{K_f,\mathbb{C}}(G))}{[K_{L,f}:K_f]} \qquad -2\Lambda_n(0)_n^\circ = \frac{\mathrm{vol}(\mathrm{Sh}_{K_f,\mathbb{C}}(G))}{[K_{L,f}:K_f]} \tag{1.2.6}$$

where $\mathrm{vol}(\mathrm{Sh}_{K_{L,f},\mathbb{C}}(G))$ is the volume with respect to the Chern form of a certain dual tautological bundle. The case of signature $(n-1,1)$ may be extracted from [BH21, Theorem A], see Proposition 8.2.3.[5] The case of signature $(n,0)$ is an equivalent reformulation of the classical Siegel mass formula (1.2.2): if we allow the (stacky) level $K_f = K_{L,f}$, then there is a canonical equivalence of groupoids

$$\mathrm{Sh}_{K_L,\mathbb{C}}(G) \cong \left\{ \begin{array}{c} \text{Hermitian } \mathcal{O}_F\text{-lattices } \mathcal{L} \text{ which are} \\ \text{self-dual and signature } (n,0) \end{array} \right\} \tag{1.2.7}$$

in that case.

In *geometric Siegel–Weil formulas*, the sets $\mathscr{Z}_{T,\mathcal{L}}$ (from classical Siegel–Weil) are replaced by *special cycles* $\mathscr{Z}_{T,\mathbb{C}}$ over the Shimura variety, and the theta series $\Theta_{\mathcal{L}}(z)$ become *generating series of special cycles*. One can define $\mathscr{Z}_{T,\mathbb{C}}$ by the complex uniformization

$$\mathscr{Z}_{T,\mathbb{C}} := \left[G(\mathbb{Q}) \backslash \left(\coprod_{\substack{\underline{x} \in V^m \\ (\underline{x},\underline{x})=T}} \mathcal{D}(\underline{x}_\infty) \times \mathcal{D}(\underline{x}_f) \right) \right] \tag{1.2.8}$$

where $\mathcal{D}(\underline{x}) \subseteq \mathcal{D}$ is the closed complex submanifold consisting of those complex lines perpendicular to all elements of the m-tuple $\underline{x}$, and

$$\mathcal{D}(\underline{x}_f) := \{ g \in G(\mathbb{A}_f)/K_f : g^{-1}x_i \in L \otimes_\mathbb{Z} \hat{\mathbb{Z}} \text{ for all } x_i \in \underline{x}_f \} \subseteq G(\mathbb{A}_f)/K_f. \tag{1.2.9}$$

Here $\underline{x}_\infty$ and $\underline{x}_f$ denote the image of $\underline{x}$ in $V(\mathbb{R})^m$ and $V(\mathbb{A}_f)^m$, respectively. The definition in (1.2.8) is (a reformulation of) a definition due to Kudla [Kud04] (there for GSpin), with unitary analogue as in [Liu11, §3]. We call $\mathcal{D}(\underline{x}_\infty)$ an Archimedean *local special cycle* and $\mathcal{D}(\underline{x}_f)$ an "away-from-∞" local special cycle. There is a natural map $\mathscr{Z}_{T,\mathbb{C}} \to \mathrm{Sh}_{K_f,\mathbb{C}}$, which is a disjoint union of closed immersions of complex manifolds after possibly shrinking K_f.

A *geometric Siegel–Weil formula* for signature $(n-1,1)$ is an identity of the shape

$$-\frac{2\Lambda_n(0)_n^\circ}{\kappa \Lambda_m(s_0)_n^\circ} E_T^*(y,s_0)_n^\circ = \frac{\mathrm{vol}(\mathscr{Z}_{T,\mathbb{C}})}{[K_{L,f}:K_f]}. \tag{1.2.10}$$

for $m \leq n-1$ (so $\kappa = 1$). In the case of signature $(n,0)$, the expression in (1.2.10) (without the minus sign on the left) is an equivalent reformulation of the classical Siegel–Weil formula (1.2.3): if we allow the (stacky) level $K_f = K_{L,f}$, there is a canonical equivalence of groupoids

$$\mathscr{Z}_{T,\mathbb{C}} \cong \left\{ \begin{array}{c} \text{pairs } (\mathcal{L},\underline{x}), \text{ where } \mathcal{L} \text{ is a self-dual Hermitian } \mathcal{O}_F\text{-lattice} \\ \text{of signature } (n,0) \text{ and } \underline{x} \in \mathcal{L}^m \text{ is an } m\text{-tuple with } (\underline{x},\underline{x})=T \end{array} \right\}. \tag{1.2.11}$$

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[5]In Proposition 8.2.3, note that we took $\mathrm{vol}(-)$ with respect to the tautological bundle. Here we are taking volume with respect to the dual bundle, which produces the minus sign in (1.2.6).

Our presentation of the geometric Siegel–Weil formula in (1.2.10) may be nonstandard. Its appearance is intended to highlight the similarity with our formulation of arithmetic Siegel–Weil in (1.3.3).

Strictly speaking, geometric Siegel–Weil formulas in literature typically restrict to V satisfying Weil's convergence condition (meaning V anisotropic or $m < n - 1$ in the signature $(n-1, 1)$ Hermitian setup), see remarks following [Kud04, Theorem 4.1] and [Li24, Theorem 3.6.1]. It is also typical to phrase geometric Siegel–Weil formulas in terms of "coherent" Eisenstein series, while our $E^*(z, s)_n^\circ$ is described in terms of an incoherent adèlic Hermitian space (positive definite at ∞ and self-dual at all finite places), see Part 1. Outside of those cases available in the literature, geometric Siegel–Weil formulas may need additional care. For example, when $m = 1$ and $n = 2$ and $T = 0$ (which is essentially about "complex volume of modular curve"), the formula in (1.2.10) is only valid up to a non-holomorphic correction term $\frac{2h_F}{w_F} \cdot \frac{1}{8\pi y}$ on the left, where h_F (resp. w_F) is the class number of (resp. number of roots of unity in) $\mathcal{O}_F$. In this case, the right-hand side is $\frac{2h_F}{w_F} \cdot \frac{\zeta(-1)}{2} = \frac{-h_F}{12 w_F}$.

We will need the following geometric Siegel–Weil result which does not seem to be covered by the literature discussed in the previous paragraph. We prove (1.2.10) when T is nonsingular of rank $m = n - 1$, see Proposition 8.1.1 (also complex uniformization from [Che24c, Section 5.3], as well as (9.1.2)); in that case, $\mathcal{Z}_{T,\mathbb{C}}$ is 0 dimensional. For example, when $n = 2$ and $\mathcal{O}_F^\times = \{\pm 1\}$, the special cycle $\mathcal{Z}_{T,\mathbb{C}}$ can be described in terms of Hecke translates of CM elliptic curves Section 9.2, and (1.2.10) is then the (well-known) statement that the T-th Hecke correspondence (over the modular curve) has bidegree

$$-\frac{1}{h_F} \frac{2\Lambda_2(0)_2^\circ}{\Lambda_1(1/2)_2^\circ} E_T^*(y, 1/2)_2^\circ = \sigma_1(T) \tag{1.2.12}$$

for $T \in \mathbb{Z}_{>0}$. The extra factor of h_F accounts for multiple connected components in the Shimura variety, see Section 9.2.

We remark that our proof of (1.2.10) (for T nonsingular of rank $m = n - 1$) is inspired by [LZ22, Remark 4.6.2], and may be carried out using either complex or non-Archimedean (Rapoport–Zink) uniformization. We need that case of (1.2.10) as an ingredient for our main arithmetic Siegel–Weil results.

1.3. **Arithmetic Siegel–Weil.** Since the work of Kudla–Rapoport [KR14] (also Rapoport–Smithling–Zhang [RSZ21]), it has been customary to define special cycles $\mathcal{Z}(T) \to \mathcal{M}$ over (stacky) integral models $\mathcal{M} \to \operatorname{Spec} \mathcal{O}_F$ for Shimura varieties associated to $G' := \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_m \times G$. After adding enough level $K_f' \subseteq G'(\mathbb{A}_f)$ to $\mathcal{M}_\mathbb{C}$, we have a finite covering map $\mathcal{M}_{K_f', \mathbb{C}} \to \operatorname{Sh}_{K_f, \mathbb{C}}(G)$. In this paper, we mainly take $\mathcal{M} \to \operatorname{Spec} \mathcal{O}_F$ to be the "exotic smooth" Rapoport–Smithling–Zhang (RSZ) integral model of odd relative dimension $n - 1$ [RSZ21, §6] (empty if $n \equiv 0 \pmod 4$). When $n = 2$, the stack $\mathcal{M}$ is essentially a disjoint union of (stacky) modular curves (Section 9.2, also [Che24a, Example 3.1.2]).

The stack $\mathcal{M}$ admits a moduli description: it parameterizes tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ where A_0 and A are abelian schemes (dimensions 1 and n respectively) with $\mathcal{O}_F$-actions ι_0 and ι, and with compatible quasi-polarizations λ_0 and λ. The datum $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ satisfies a few additional

conditions, which we suppress in the introduction (see [Che24a, Section 3.1] and [Che24c, Section 2.1]). We are able to prove versions of our main global results for more general $\mathcal{M}$ (including odd arithmetic dimension n) at the price of discarding finitely many primes (particularly ramified primes for odd n); see Remark 9.1.5.

The moduli stack $\mathcal{M}$ carries a natural family of Hermitian $\mathcal{O}_F$-lattices

$$\mathcal{L}at \to \mathcal{M} \qquad \mathcal{L}at := \underline{\mathrm{Hom}}_{\mathcal{O}_F}(A_0, A). \tag{1.3.1}$$

Given any $T \in \mathrm{Herm}_m(\mathbb{Q})$, the associated *Kudla–Rapoport* special cycle $\mathcal{Z}(T) \to \mathcal{M}$ is defined as the substack

$$\mathcal{Z}(T) := \{\underline{x} \in \mathcal{L}at^m : (\underline{x}, \underline{x}) = T\} \subseteq \mathcal{L}at^m \tag{1.3.2}$$

consisting of m-tuples with Gram matrix T. More precisely, see [Che24c, Section 2.1]. This is in close analogy with classical Siegel–Weil: there we considered $\mathcal{O}_F$-lattices varying in a given *genus*,[6] and here we are considering $\mathcal{O}_F$-lattices varying over the moduli stack $\mathcal{M}$. In the complex fiber, the special cycles $\mathcal{Z}(T)_\mathbb{C}$ recover the special cycles $\mathcal{Z}_{T,\mathbb{C}}$ appearing in (1.2.8), up to $\mathcal{M}_{K'_f,\mathbb{C}}$ being a finite cover of $\mathrm{Sh}_{K_f,\mathbb{C}}(G)$ (for suitable K'_f); see [Che24c, Section 5.3]. The morphism $\mathcal{Z}(T) \to \mathrm{Spec}\,\mathcal{O}_F$ is smooth of relative dimension $n - 1 - \mathrm{rank}(T)$ in the generic fiber over $\mathrm{Spec}\,F$. If T is not positive semi-definite, then $\mathcal{Z}(T)$ is empty.

An *arithmetic Siegel–Weil formula* is a (in general conjectural) identity roughly of the shape

$$\frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=s_0} \frac{2\Lambda_n(s-s_0)_n^\circ}{\kappa \Lambda_m(s)_n^\circ} E_T^*(y,s)_n^\circ \stackrel{?}{=} \widehat{\mathrm{vol}}_{\widehat{\mathcal{E}}^\vee}([\widehat{\mathcal{Z}}(T)]). \tag{1.3.3}$$

with κ as in Section 1.2. As discussed in [Che24a, Section 1.2], a precise formulation of arithmetic Siegel–Weil has not been proposed in full generality (on both the analytic and geometric sides). Our normalization on the analytic side is already nonstandard; we would like to highlight the similarity with normalizations for geometric and classical Siegel–Weil, as presented in (1.2.3) and (1.2.10). The normalization is in general delicate for arithmetic Siegel–Weil formulas, where both the derivative and special value at $s = s_0$ may have meaning (we do observe this for our main theorem, see [Che24a, Remark 1.3.1] and discussion below).

In [Che24c, Section 3] and [Che24a, Section 3.6], we proposed a new candidate definition of arithmetic cycle classes

$$[\widehat{\mathcal{Z}}(T)] := [\widehat{\mathcal{Z}}(T)_{\mathscr{H}}] + \sum_{p \text{ prime}} [{}^{\mathbb{L}}\mathcal{Z}(T)_{\mathscr{V},p}] \in \widehat{\mathrm{Ch}}^m(\mathcal{M})_\mathbb{Q} \tag{1.3.4}$$

associated to arbitrary (possibly singular) T, where $\widehat{\mathrm{Ch}}^m(\mathcal{M})_\mathbb{Q}$ is an arithmetic Chow group associated to $\mathcal{M}$. Here, $[\widehat{\mathcal{Z}}(T)_{\mathscr{H}}]$ should describe "horizontal" contributions and ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathscr{V},p}$ should describe "vertical" contributions.

Due to non-properness of $\mathcal{M} \to \mathrm{Spec}\,\mathcal{O}_F$ in general, one should likely modify $[\widehat{\mathcal{Z}}(T)]$ on a suitable compactification of $\mathcal{M}$. If $\mathcal{Z}(T) \to \mathrm{Spec}\,\mathcal{O}_F$ is proper, however, we consider certain "arithmetic

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[6]In our previous setup, this meant the set of isomorphism classes of positive definite rank n self-dual $\mathcal{O}_F$-lattices.

degrees without boundary contributions" (a real number)

$$\widehat{\deg}([\widehat{\mathcal{Z}}(T)] \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{n-m}) := \left(\int_{\mathcal{M}_\mathbb{C}} g_{T,y} \wedge c_1(\widehat{\mathcal{E}}_\mathbb{C}^\vee)^{n-m} \right) \tag{1.3.5}$$
$$+ \widehat{\deg}((\widehat{\mathcal{E}}^\vee)^{n-\mathrm{rank}(T)}|_{\mathcal{Z}(T)_{\mathscr{H}}})$$
$$+ \sum_{p \text{ prime}} \deg_{\mathbb{F}_p}(^{\mathbb{L}}\mathcal{Z}(T)_{\mathscr{V},p} \cdot (\mathcal{E}^\vee)^{n-m}) \log p$$

conditional on convergence of the integral, for a certain metrized tautological bundle $\widehat{\mathcal{E}}$ on $\mathcal{M}$ ([Che24a, Sections 3.4 and 3.5]) (we do check convergence of the integral in the settings of our arithmetic Siegel–Weil results). Here we set $\mathcal{M}_\mathbb{C} := \mathcal{M} \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathrm{Spec}\,\mathbb{C}$ for either embedding $F \to \mathbb{C}$. The middle term is mixed characteristic in nature: for $\mathrm{rank}\,T = n-1$, it is (essentially) a weighted sum of Faltings heights of abelian varieties (Remark 9.1.4). For proper $\mathcal{Z}(T) \to \mathcal{O}_F$, the quantity in (1.3.5) should coincide with the arithmetic degree (without boundary contributions) of a version of $[\widehat{\mathcal{Z}}(T)]$ on any reasonable compactification of $\mathcal{M}$.

The following is the main global theorem for our four-paper sequence.

Theorem A (Co-rank 1 arithmetic Siegel–Weil). *Assume the prime 2 splits in $\mathcal{O}_F$.*

(1) For any $T \in \mathrm{Herm}_n(\mathbb{Q})$ with $\mathrm{rank}(T) = n-1$ and any $y \in \mathrm{Herm}_n(\mathbb{R})_{>0}$, we have

$$\frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=0} E_T^*(y,s)_n^\circ = \widehat{\deg}([\widehat{\mathcal{Z}}(T)]). \tag{1.3.6}$$

(2) For any $T^\flat \in \mathrm{Herm}_{n-1}(\mathbb{Q})$ with $\det T^\flat \neq 0$ and any $y^\flat \in \mathrm{Herm}_{n-1}(\mathbb{R})_{>0}$, we have

$$2\frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=0} \left(\frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ} E_{T^\flat}^*(y^\flat, s+1/2)_n^\circ \right) = \widehat{\deg}([\widehat{\mathcal{Z}}(T^\flat) \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)]). \tag{1.3.7}$$

This appears below as Theorem 9.1.1. Note that part (1) concerns the central derivative of a $U(n,n)$ Eisenstein series, while part (2) concerns a non-central derivative of a $U(n-1,n-1)$ Eisenstein series. For $n \equiv 0 \pmod 4$, Theorem A(1) also holds in the sense that there is no self-dual $\mathcal{O}_F$-lattice of signature $(n-1,1)$ and the right-hand side is 0 (Remark 9.1.3).

Remark 1.3.1. Our result in Theorem A(2) combined with our geometric Siegel–Weil result ((1.2.10) and surrounding discussion, also Proposition 8.1.1) shows that, for the relevant (normalized) $U(n-1,n-1)$ Eisenstein series, both the derivative and special value at $s = 1/2$ *simultaneously* have arithmetic-geometric meaning (also mentioned in [Che24a, Remark 1.3.1]). This phenomenon amplifies the sensitivity of Theorem A(2) to the choice of normalization for the Eisenstein series, and is one reason for the importance of Part 1 in this paper (where we discuss our normalizations for Eisenstein series and local Whittaker functions).

Question (Arithmetic Siegel–Weil). *Let $T \in \mathrm{Herm}_n(\mathbb{Q})$ be arbitrary. For a suitable current $g_{T,y}$, a suitable compactification of $\mathcal{M}$, and a possibly modified class $[\widehat{\mathcal{Z}}(T)]$ on the compactification, do we have*

$$\frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=0} E_T^*(y,s)_n^\circ \stackrel{?}{=} \widehat{\deg}([\widehat{\mathcal{Z}}(T)]). \tag{1.3.8}$$

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Our theorem verifies this proposed arithmetic Siegel–Weil formula for all singular $T \in \mathrm{Herm}_n(\mathbb{Q})$ of rank $n-1$, in the sense of "arithmetic degrees without boundary contributions". The formula also holds (in the same sense) for all nonsingular $T \in \mathrm{Herm}_n(\mathbb{Q})$. This latter case ("central derivative nonsingular arithmetic Siegel–Weil") is possibly considered known to experts up to a volume constant by collecting the local theorems in [Liu11; LZ22; LL22]. This particular global statement does not appear in the literature, though other variants are available (e.g. for unramified CM fields F/F_0 with $F_0 \neq \mathbb{Q}$ [LZ22] or on integral models with bad reduction and correction terms by special values of other Eisenstein series [HLSY23]). In our setup, we will compute the volume constant and explain how to extract the $\det T \neq 0$ case of (1.3.8) from the literature (Remark 9.1.2).

Remark 1.3.2. Part (2) of Theorem A is the special case of part (1) when $T = \mathrm{diag}(0, T^\flat)$ and $y = \mathrm{diag}(1, y^\flat)$. The geometric sides agree essentially by definition (1.3.5). On the analytic side, the relation is provided by the formula

$$E_T^*(y, s)_n^\circ = \frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ} E_{T^\flat}^*(y^\flat, s+1/2)_n^\circ - \frac{\Lambda_n(-s)_n^\circ}{\Lambda_{n-1}(-s+1/2)_n^\circ} E_{T^\flat}^*(y^\flat, s-1/2)_n^\circ \qquad (1.3.9)$$

from Corollary 6.2.2, along with the functional equation $E_{T^\flat}^*(y^\flat, s)_n^\circ = E_{T^\flat}^*(y^\flat, -s)_n^\circ$. The general case of Theorem A is proved in a similar way as the special case $T = \mathrm{diag}(0, T^\flat)$, with an additional "local diagonalizability argument" (Proof of Theorem 9.1.1) where the identity is proved modulo $\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell$ for any given p (varying p removes the ambiguity).

Unlike $E_{T^\flat}^*(y, s)_n^\circ$, the Fourier coefficient $E_T^*(y, s)_n^\circ$ does not admit an obvious Euler product decomposition into local Whittaker functions as T is singular. Since our proof of Theorem A is local (via local Whittaker functions and local special cycles), the decomposition (1.3.9) is crucial for our method (to use local Whittaker functions to describe $E_T^*(y, s)_n^\circ$). This decomposition is sensitive to the normalization used to define $E^*(z, s)_n^\circ$, and is another reason for the importance of Part 1 in this paper, where we treat normalized Eisenstein series and local Whittaker functions.

It is also possible to formulate and prove Theorem A in terms of Faltings heights (i.e. replacing the middle term in (1.3.5) with the degree of the metrized Hodge bundle). The formulation in Theorem A seems more natural to us, but the version with Faltings heights is in Remark 9.1.4.

Since our proof of Theorem A will be local in nature, we also have a version for more general moduli stacks $\mathcal{M}$ (including odd arithmetic dimension n) at the price of discarding finitely many primes (particularly the ramified ones). This is explained in Remark 9.1.5.

The simplest case of Theorem A is the case $n = 2$. When $\mathcal{O}_F^\times = \{\pm 1\}$, the Serre tensor construction gives an open and closed embedding $\mathcal{M}_0 \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathcal{M}_{\mathrm{ell}} \to \mathcal{M}$, where $\mathcal{M}_0$ is the moduli stack of elliptic curves with signature $(1, 0)$ action by $\mathcal{O}_F$ and $\mathcal{M}_{\mathrm{ell}}$ is the moduli stack of all elliptic curves, base-changed to $\mathcal{O}_F$ (Section 9.2). In this case, the special cycle $\mathcal{Z}(j) \to \mathcal{M}$ for $j \in \mathbb{Z}_{>0}$ pulls back to the j-th Hecke correspondence. Then the proof of Theorem 9.1.1(2) gives the following corollary (appearing below as Corollary 9.2.2). One might think of this corollary as reformulating a result of Nakkajima–Taguchi [NT91] (they compute Faltings heights of elliptic curves with CM by possibly non-maximal orders) by averaging over Hecke translates and expressing the result in terms of Eisenstein series Fourier coefficients.

Corollary 1.3.3. *Assume 2 is split in $\mathcal{O}_F$. Fix any elliptic curve E_0 over $\mathbb{C}$ with $\mathcal{O}_F$-action. For any integer $j > 0$, we have*

$$\sum_{E \xrightarrow{w} E_0} (h_{\mathrm{Fal}}(E) - h_{\mathrm{Fal}}(E_0)) = \frac{1}{2} \frac{d}{ds}\bigg|_{s=1/2} \left(j^{s+1/2} \sigma_{-2s}(j) \right) \tag{1.3.10}$$

where the sum runs over degree j isogenies $w \colon E_0 \to E$ of elliptic curves.

The notation $h_{\mathrm{Fal}}(E)$ denotes the (stable) Faltings height of the elliptic curve E after descent to any number field, and similarly for E_0. The quantity $j^{s+1/2} \sigma_{-2s}(j)$ is the product of the normalized non-Archimedean local Whittaker functions in the j-th Fourier coefficient $E_j^*(y, s)_2^\circ$ (with $m = 1$) as in (1.1.7). The derivative of the Archimedean local Whittaker function $W_{j,\infty}^*(y, s)_2^\circ$ at $s = 1/2$ was calculated explicitly and compared with its geometric counterpart (integral of Green function wedge Chern form on upper half-plane) in our companion paper [Che24b, Section 4.2].

Our purely Archimedean result (for arbitrary n and $m^\flat \geq 1$) is the following.

Theorem B (Archimedean arithmetic Siegel–Weil, nonsingular)**.** *Consider any integer $m^\flat$ with $1 \leq m^\flat \leq n$, and consider any $T^\flat \in \mathrm{Herm}_{m^\flat}(\mathbb{Q})$ which is nonsingular and not positive definite.*

(3) For any $y^\flat \in \mathrm{Herm}_{m^\flat}(\mathbb{R})_{>0}$, we have an equality of real numbers

$$\widehat{\deg}([\widehat{\mathcal{Z}}(T^\flat)] \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{n-m^\flat}) := \int_{\mathcal{M}_{\mathbb{C}}} g_{T^\flat, y^\flat} \wedge c_1(\widehat{\mathcal{E}}_{\mathbb{C}}^\vee)^{n-m^\flat} = (-1)^{n-m^\flat} C \cdot \frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=s_0^\flat} E_{T^\flat}^*(y^\flat, s)_n^\circ \tag{1.3.11}$$

where $s_0^\flat := (n - m^\flat)/2$. Here $C \in \mathbb{Q}_{>0}$ is the volume constant from Lemma 7.4.1(1), for the Hermitian space V and $v_0 = \infty$ in the notation of loc. cit.. The constant C may depend on n and $m^\flat$ (and F), but does not otherwise depend on $T^\flat$.

This appears below (in stronger form) as Theorem 9.1.6. That version applies for all n (even or not) and arbitrary level, as it is a statement about the complex Shimura variety. We gave the weaker version here to avoid more notation in the introduction. Due to non-properness of $\mathcal{M}_{\mathbb{C}} \to \mathrm{Spec}\,\mathbb{C}$ for $n > 2$, the corresponding Archimedean result of [GS19] does not apply here if $n > 2$.

When $m^\flat = n$, the preceding Archimedean theorem follows from Liu's result [Liu11, Theorem 4.17]. We do not have a new proof of this case. Instead, we deduce our general result from his by a certain limiting argument. This is also our method at non-Archimedean places (replacing Liu's Archimedean results with the non-Archimedean results of Li–Zhang [LZ22] and Li–Liu [LL22]). These local theorems were the main results of our companion papers [Che24a; Che24b]. In this paper, we explain how to combine our local theorems at all places to prove the (global) Theorems A and B.

1.4. **Outline.** We briefly summarize the remaining content in this paper, and discuss the relation with our companion papers [Che24a; Che24b; Che24c]. Further explanations may be found at the beginning of some sections.

The remaining sections are divided into Parts 1 and 2.

In Part 1 "Eisenstein series", we study $U(m, m)$ Siegel–Weil Eisenstein series. To formulate and prove our main results, it is extremely important that we normalize the Eisenstein series and local

Whittaker functions (e.g. by certain L-factors). As in our main local theorems from [Che24a; Che24b], it seems that these normalized versions correspond more naturally to geometric objects (e.g. global and local special cycles). We pin down explicit precise normalizations, guided by special value formulas and symmetric functional equations. We also study (normalized) Fourier coefficients for singular T (focusing on rank $m-1$ and size $m \times m$), and give formulas needed for our main results.

In Part 2 "Siegel–Weil", we first give some special value formulas (local and geometric Siegel–Weil, Sections 7 and 8) which are needed to prove our arithmetic Siegel–Weil theorems. Our treatment of our geometric Siegel–Weil result is inspired by [LZ22, Remark 4.6.2], and the argument may be carried out using complex (Archimedean) uniformization or Rapoport–Zink (non-Archimedean) uniformization in similar fashions. We also compare complex volumes of unitary Shimura varieties with the special value of our Eisenstein series normalizing factor ("geometric Siegel mass formula"), which may be of independent interest Section 8.2. The finale occurs in Section 9.1, where we collect our local main theorems to prove our (global) arithmetic Siegel–Weil theorems. This proof relies on results from almost all preceding sections, including our three other companion papers. The key inputs from those papers are our non-Archimedean local arithmetic Siegel–Weil theorems from [Che24a], our Archimedean local arithmetic Siegel–Weil theorem from [Che24b], and the geometric local-to-glocal reduction process (via Archimedean and non-Archimedean uniformization) from [Che24c]. Section 9.2 contains a reformulation of our arithmetic Siegel–Weil results in the special case $n = 2$, via an exceptional comparison with Hecke translates of CM elliptic curves.

Part 1. Eisenstein series

2. Setup

2.1. The group $U(m, m)$.

We fix notation for the unitary group $U(m, m)$.

Let $A \to B$ be a finite locally free morphism of (commutative) rings, and suppose B is given an involution $b \mapsto \bar{b}$ ("conjugation") over A. We are mostly interested in the case where F/F^+ is a CM extension of number fields (with F^+ the index 2 totally real subfield) and $B/A = \mathcal{O}_F/\mathcal{O}_{F^+}$ for the corresponding rings of integers (also the local analogues) etc..

Fix an integer $m \geq 0$. Write 1_m for the $m \times m$ identity matrix (sometimes we drop the subscript m), and let $H = U(m, m)$ be the unitary group

$$H = U(m, m) := \left\{ h \in \mathrm{Res}_{B/A} \, \mathrm{GL}_{2m} : {}^t\bar{h} \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} h = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} \right\} \tag{2.1.1}$$

where ${}^t\bar{h}$ denotes conjugate transpose (with H the trivial group if $m = 0$, by convention). Equivalently, H consists of block matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{satisfying} \quad {}^t\bar{a}c = {}^t\bar{c}a \qquad {}^t\bar{a}d - {}^t\bar{c}b = 1_m \qquad {}^t\bar{b}d = {}^t\bar{d}b \tag{2.1.2}$$

with $a, b, c, d \in \mathrm{Res}_{B/A} M_{m \times m}$. We refer to H as the group $U(m, m)$ (for signature reasons when B/A is $\mathbb{C}/\mathbb{R}$).

Given an integer j with $0 \leq j \leq m$, we consider the injection

$$\mu_j^m : U(j, j) \to U(m, m) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1_{m-j} & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1_{m-j} & 0 \\ 0 & c & 0 & d \end{pmatrix}. \tag{2.1.3}$$

Consider the subgroups

$$P := \left\{ h = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in H \right\} \tag{2.1.4}$$

$$M := \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{Res}_{B/A} \, \mathrm{GL}_m \right\} \tag{2.1.5}$$

$$N := \left\{ n(b) = \begin{pmatrix} 1_m & b \\ 0 & 1_m \end{pmatrix} : b \in \mathrm{Herm}_m \right\} \tag{2.1.6}$$

of H. We have $P(R) = M(R)N(R)$ for all A-algebras R. We occasionally write P_m, M_m, N_m to emphasize dependence on m.

Set

$$w_j = \begin{pmatrix} 1_{m-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_j \\ 0 & 0 & 1_{m-j} & 0 \\ 0 & -1_j & 0 & 0 \end{pmatrix} \tag{2.1.7}$$

for j with $0 \leq j \leq m$. We also write $w = w_m$ when $j = m$ and m is understood.

Let F_v be a finite étale algebra of degree 2 over a local field F_v^+. Consider $B/A = \mathcal{O}_{F_v}/\mathcal{O}_{F_v^+}$ for the respective rings of integers (with $\mathcal{O}_{F_v^+} := F_v^+$ and $\mathcal{O}_{F_v} := F_v$ if F_v^+ is Archimedean).

If $F_v/F_v^+ = \mathbb{C}/\mathbb{R}$, we consider the standard maximal compact subgroup

$$K_v^\circ := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in H(\mathbb{R}) : a^t\overline{a} + b^t\overline{b} = 1_m \text{ and } a^t\overline{b} = b^t\overline{a} \right\} \subseteq H(\mathbb{R}) \tag{2.1.8}$$

We write $U(m) \subseteq \mathrm{GL}_m(\mathbb{C})$ for (the real points of) the unitary group for the usual positive definite rank m complex Hermitian space (specified by the Gram matrix 1_m). There is an isomorphism $K_v \to U(m) \times U(m)$ sending the displayed matrix to $(a+ib, a-ib) \in U(m) \times U(m)$ (see e.g. [GS19, §2.5.1]).

If F_v^+ is non-Archimedean, we consider the standard open compact subgroup

$$K_v^\circ := H(\mathcal{O}_{F_v^+}) \subseteq H(F_v^+). \tag{2.1.9}$$

If $F_v/F_v^+ = \mathbb{C}/\mathbb{R}$ or if F_v^+ is non-Archimedean, we have $H(F_v^+) = P(F_v^+)K_v^\circ$. If F_v^+ is non-Archimedean and

$$w^{-1}n(b)w = m(a)k \tag{2.1.10}$$

with $n(b) \in N(F_v^+)$ and $m(a) \in M(F_v^+)$ and $k \in K_v^\circ$, we have $|\det a|_{F_v} < 1$ and moreover $\det a \in F_v^+$ (see [Shi97, §13.4]).

If F/F^+ is a CM extension of number fields and $B/A = \mathcal{O}_F/\mathcal{O}_{F^+}$, we write

$$K^\circ = \prod_v K_v^\circ \qquad K_\infty^\circ = \prod_{v|\infty} K_v^\circ \qquad K_f^\circ = \prod_{v<\infty} K_v^\circ \tag{2.1.11}$$

where the products run over places v of F^+.

For places v of F^+, we use the notation $F_v := \prod_{w|v} F_w$ where w runs over places of F, similarly $\mathcal{O}_{F_v} := \prod_{w|v} \mathcal{O}_{F_w}$, as well as $F_\infty^+ = \prod_{v|\infty} F_v^+$ and $F_\infty = \prod_{w|\infty} F_w$, etc..

2.2. Adèlic and classical Eisenstein series.

Characters are assumed continuous and unitary unless specified otherwise. Let F_v be a degree 2 étale algebra over a local field F_v^+, and form the corresponding unitary group $H = U(m,m)$ as in Section 2.1. If F_v^+ is Archimedean, we assume in Section 2.2 that F_v/F_v^+ is $\mathbb{C}/\mathbb{R}$.

Given $s \in \mathbb{C}$ and a character $\chi_v \colon F_v^\times \to \mathbb{C}^\times$, we may form the local *degenerate principal series*

$$I(s, \chi_v) := \mathrm{Ind}_{P(F_v^+)}^{H(F_v^+)}(\chi_v| - |_{F_v}^{s+m/2}). \tag{2.2.1}$$

This is an unnormalized induction, consisting of smooth and K_v-finite functions $\Phi_v \colon H(F_v^+) \to \mathbb{C}$ satisfying

$$\Phi_v(m(a)n(b)h, s) = \chi_v(a)|\det a|_{F_v}^{s+m/2} \tag{2.2.2}$$

for all $m(a) \in M(F_v^+)$ and $n(b) \in N(F_v^+)$ and $h \in H(F_v^+)$. Here we wrote $\chi_v(a) := \chi_v(\det a)$ for short. A section $\Phi_v(h, s)$ of $I(s, \chi_v)$ is *standard* if $\Phi(k, s)$ is independent of s for any fixed $k \in K_v$. We say Φ_v is *spherical* if $\Phi_v(hk, s) = \Phi_v(h, s)$ for any $k \in K_v$. We write Φ_v° for the unique spherical standard section satisfying $\Phi_v^\circ(1, s) = 1$ for all s, and call Φ_v° the *normalized spherical section*.

Next, suppose F/F^+ is a CM extension of number fields. We write $\mathbb{A}_F$ for the adèle ring of F and $\mathbb{A}$ for the adèle ring of F^+. Given $s \in \mathbb{C}$ and a character $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ and $s \in \mathbb{C}$, we similarly form the global *degenerate principal series*

$$I(s,\chi) := \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\chi|-|_F^{s+m/2}) \tag{2.2.3}$$

which is an unnormalized induction, consisting of smooth and K-finite functions $\Phi \colon H(\mathbb{A}) \to \mathbb{C}$ satisfying

$$\Phi(m(a)n(b)h,s) = \chi(a)|\det a|_F^{s+m/2} \tag{2.2.4}$$

for all $m(a) \in M(\mathbb{A})$ and $n(b) \in N(\mathbb{A})$ and $h \in H(\mathbb{A})$. Given characters $\chi_f \colon \mathbb{A}_{F,f}^\times \to C^\times$ and $\chi_\infty \colon \mathbb{A}_{F,\infty}^\times \to \mathbb{C}^\times$, we similarly form $I(s,\chi_f)$ and $I(s,\chi_\infty)$. We also speak of *spherical sections* and the *spherical standard section*, as above. We sometimes write $I_m(s,\chi)$ etc. to indicate dependence on m.

Given a standard section $\Phi(h,s)$ of the global degenerate principal series $I(s,\chi)$, we form the *Siegel Eisenstein series*

$$E(h,s,\Phi) = \sum_{\gamma \in P(F^+)\backslash H(F^+)} \Phi(\gamma h, s) \tag{2.2.5}$$

which is absolutely convergent for $\mathrm{Re}(s) > m/2$. We also form $E(h,\Phi,s)$ when Φ is a finite meromorphic linear combination of standard sections by extending linearly.

Define another character $\check{\chi} \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ as $\check{\chi}(a) := \chi(\bar{a})^{-1}$. There is a functional equation

$$E(h,-s,M(\chi,s)\Phi) = E(h,s,\Phi) \tag{2.2.6}$$

where $M(\chi,s) \colon I(s,\chi) \to I(-s,\check{\chi})$ is the intertwining operator

$$(M(s,\chi)\Phi)(h) = \int_{N(\mathbb{A})} \Phi(w^{-1}n(b)h,s) \; dn(b) \tag{2.2.7}$$

for $\mathrm{Re}(s) > m/2$ (see e.g. [Tan99]). We occasionally write $M_m(s,\chi)$ to emphasize the understood m (in $U(m,m)$).

Fix an identification of F_v^+-algebras $F_v \cong \mathbb{C}$ for each Archimedean place v of F^+. We consider classical Eisenstein series on the *Hermitian upper-half space*

$$\mathcal{H}_m := \{z \in M_{m,m}(F_\infty) : (2i)^{-1}(z - {}^t\bar{z}) > 0\} \tag{2.2.8}$$

$$= \{z = x + iy : x, y \in \mathrm{Herm}_m(F_\infty^+) \text{ with } y > 0\}, \tag{2.2.9}$$

where the latter expression means that x and y are $m \times m$ Hermitian matrices with y positive definite (at every place $v \mid \infty$ of F^+). Given $z = x + iy \in \mathcal{H}_m$, we write $h_z \in H(F_\infty^+) \subseteq H(\mathbb{A})$ for any element $h_z = n(x)m(a)$ where $a \in \mathrm{GL}_m(F_\infty)$ satisfies $a^t\bar{a} = y$. Note $h_z \cdot i1_m = z$.

We restrict to $\Phi = \Phi_\infty \otimes \Phi_f$ for standard sections $\Phi_\infty \in I(s,\chi_\infty)$ and $\Phi_f \in I(s,\chi_f)$. Fix an integer n_v for each place $v \mid \infty$ of F_v^+, and assume $\chi_v|_{F_v^{+\times}} = \mathrm{sgn}(-)^{n_v}$ for every $v \mid \infty$. We also let $k(\chi_v) \in \mathbb{Z}$ be the integer satisfying

$$\chi_v(z) = (z/|z|_{F_v}^{1/2})^{k(\chi_v)} \qquad \text{where } z \in F_v, \tag{2.2.10}$$

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for each place $v \mid \infty$ of F_v^+. For such v, we let $\Phi_v = \Phi_v^{(n_v)}$ be the unique standard section of $I(s, \chi_v)$ of scalar weight

$$(n_1, n_2) \quad \text{where} \quad n_1 = \frac{n + k(\chi)}{2} \quad \text{and} \quad n_2 = \frac{-n + k(\chi)}{2} \quad (2.2.11)$$

such that $\Phi_v^{(n_v)}(1, s) = 1$ (as in [GS19, §3.2, §3.3]). The scalar weight condition means that $\Phi_v^{(n_v)}(hk, s) = \det(k_1)^{n_1} \det(k_2)^{n_2} \Phi_v^{(n_v)}(h, s)$ for all $h \in H(F_v^+)$ and $k \in K_v$ where $n_1 = (n_v + k(\chi_v))/2$ and $n_2 = (-n_v + k(\chi_v))/2$ and

$$k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K_v \qquad k_1 = a + ib \qquad k_2 = a - ib. \quad (2.2.12)$$

Note that $\Phi_v^{(n_v)}$ does not depend on the choice of identification $F_v \cong \mathbb{C}$.

If $y = a^t \overline{a}$ for some $a \in \mathrm{GL}_m(F_v)$, a computation (omitted) shows

$$\chi_v(a)^{-1}(\det y)^{-n_v/2} \Phi_v^{(n_v)}(w^{-1} n(b) m(a)) = (\det y)^{s-s_0} \det(-iy + b)^{-(s-s_0)} \det(iy + b)^{-(s-s_0)-n_v} \quad (2.2.13)$$

for any $b \in \mathrm{Herm}_m(F_v^+)$, where $s_0 = (n_v - m)/2$ (reduce to the case $a = 1_m$ and write $w^{-1} n(b) = n(-b(1_m + b^2)^{-1}) m(b + i1_m)^{-1} k$ for $k \in K_v$). Equation (2.2.13) may be used to translate various statements from [Shi82] to statements about Archimedean Whittaker functions, etc. (see [Che24b, Section 4.3] for more on this).

Remark 2.2.1. Given $g = x_g + iy_g \in M_{m,m}(\mathbb{C})$ with x_g, y_g Hermitian and x_g positive definite, we define $\log \det(g)$ by the "principal branch" (such that $g \mapsto \log \det g$ is holomorphic, and $\log \det g \in \mathbb{R}$ if $y_g = 0$) as in [Shi82, (1.11)] and the surrounding discussion of loc. cit.. If y_g is positive definite and x_g is only assumed Hermitian, we also take

$$\log \det g = \log \det(-ig) + m \log i \qquad \log \det \overline{g} = \log \det(ig) - m \log i \quad (2.2.14)$$

where $\log i := \pi i/2$ (as in [Shi82, (1.11)]). This convention is implicit in (2.2.13).

We take $\Phi_\infty = \otimes_{v \mid \infty} \Phi_v^{(n_v)}$. We write $n = (n_v)_{v \mid \infty}$ for the collection of Archimedean weights (and will eventually focus on the case where all n_v are equal to some fixed integer n). In the above situation, we write $E(h, s, \Phi)_n := E(h, s, \Phi)$ and consider an associated *classical Eisenstein series*

$$E(z, s, \Phi)_n := E(z, s, \Phi) := \chi_\infty(a)^{-1} \det(y)^{-n/2} E(h_z, s, \Phi)_n \quad (2.2.15)$$

where $z = x + iy$ and $h_z = n(x)m(a)$ with $a^t \overline{a} = y$ as above, and where $\det(y)^{-n/2}$ stands for $\prod_{v \mid \infty} \det(y_v)^{-n_v/2}$. This does not depend on the choice of h_z, i.e. $E(h_z k_\infty, s, \Phi)_n = E(h_z, s, \Phi)_n$ for any $k_\infty \in K_\infty$.

When $F^+ = \mathbb{Q}$ and $s_0 := (n - m)/2$ (setting $n = n_\infty$ and $k(\chi) = k(\chi_\infty)$), a computation (omitted) gives the more classical form

$$E(z, s, \Phi)_n = \sum_{\gamma \in P(\mathbb{Z}) \backslash U(m,m)(\mathbb{Z})} \frac{\det(y)^{s-s_0} \det(\gamma)^{(n+k(\chi))/2}}{\det(cz+d)^n |\det(cz+d)|^{2(s-s_0)}} \Phi_f(\gamma, s) \quad (2.2.16)$$

$$= \sum_{\gamma \in P_1(\mathbb{Z}) \backslash SU(m,m)(\mathbb{Z})} \frac{\det(y)^{s-s_0}}{\det(cz+d)^n |\det(cz+d)|^{2(s-s_0)}} \Phi_f(\gamma, s) \quad (2.2.17)$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.2.18}$$

where $SU(m,m) \subseteq U(m,m)$ is the determinant 1 subgroup, and $P_1 := SU(m,m) \cap P$. We have $P(\mathbb{Q})\backslash H(\mathbb{Q}) = P(\mathbb{Z})\backslash H(\mathbb{Z}) = P_1(\mathbb{Q})\backslash H_1(\mathbb{Q}) = P_1(\mathbb{Z})\backslash H_1(\mathbb{Z})$ (e.g. [Ike08, Proposition 12.6]). When $m = 1$, the exceptional isomorphism $\mathrm{SL}_2 \to SU(1,1)$ (over $\mathrm{Spec}\,\mathbb{Q}$) implies that the above expression is a classical Eisenstein series for SL_2 on the upper-half plane.

Our main theorem (Theorem 9.1.1) concerns Fourier coefficients of $E(z,s,\Phi)_n$ (normalized as in Section 6.1), but the variant

$$\tilde{E}(a,s,\Phi)_n := \chi(a)^{-1}|\det(a)|_F^{-n/2} E(m(a),s,\Phi)_n \quad \text{for } a \in \mathrm{GL}_n(\mathbb{A}_F). \tag{2.2.19}$$

will be useful for studying Fourier coefficients of $E(z,s,\Phi)_n$ for singular T (see below). If $a \in \mathrm{GL}_m(F_\infty)$ is any element satisfying $a^t\bar{a}$ for $y \in \mathrm{Herm}_m(F_\infty^+)_{>0}$, we have

$$E(iy,s,\Phi)_n = \tilde{E}(a,s,\Phi)_n. \tag{2.2.20}$$

2.3. **Fourier expansion and local Whittaker functions.** Take notation as in Section 2.2, e.g. F/F^+ is a CM extension of number fields. Choose a nontrivial additive character $\psi \colon F^+\backslash\mathbb{A} \to \mathbb{C}^\times$. We have a Fourier expansion

$$E(h,s,\Phi) = \sum_{T \in \mathrm{Herm}_m(F^+)} E_T(h,s,\Phi) \tag{2.3.1}$$

where

$$E_T(h,s,\Phi) = \int_{N(F^+)\backslash N(\mathbb{A})} E(n(b)h,s,\Phi)\psi(-\mathrm{tr}(Tb))\,dn(b) \tag{2.3.2}$$

for $\mathrm{Re}(s) > m/2$, and where $dn(b)$ is the Haar measure on $N(\mathbb{A})$ which is self-dual with respect to the pairing $(b,b') \mapsto \psi(\mathrm{tr}(bb'))$. We refer to $E_T(h,s,\Phi)$ as the T-*th Fourier term*.

For any $a \in \mathrm{GL}_m(F)$, a change of variables gives

$$E_T(m(a)h,s,\Phi) = E_{t\bar{a}Ta}(h,s,\Phi). \tag{2.3.3}$$

We also have

$$E_T(m(a)h,s,\Phi) = E_T(h,s,\Phi) \quad \text{for any} \quad \begin{pmatrix} 1_{m-m^\flat} & * \\ 0 & 1_{m^\flat} \end{pmatrix} \in \mathrm{GL}_m(\mathbb{A}_F) \quad \text{if} \quad T = \begin{pmatrix} 0 & 0 \\ 0 & T^\flat \end{pmatrix} \tag{2.3.4}$$

with the block matrix $T^\flat \in \mathrm{Herm}_{m^\flat}(F^+)$ having $\det T^\flat \neq 0$ (here $m^\flat$ is arbitrary) (follows from [GS19, Lemma 5.4, (5.56)]).

Allowing arbitrary T again, assume there is a factorization $\Phi = (\otimes_{v|\infty}\Phi_v) \otimes \Phi_f$. For each $v \mid \infty$, assume $\Phi_v = \Phi_v^{(n_v)}$ is the scalar weight standard section as in Section 2.2, for some $n_v \in \mathbb{Z}$. Write $n = (n_v)_{v|\infty}$ for the resulting tuple of integers.

Consider $a = a_\infty a_f \in \mathrm{GL}_m(\mathbb{A}_F)$, with $a_\infty \in \mathrm{GL}_m(F_\infty)$ and $a_f \in \mathrm{GL}_m(\mathbb{A}_{F,f})$. Set $y = a_\infty{}^t\bar{a}_\infty$ (temporary). We then have T-th *Fourier coefficients* $E_T(y,s,\Phi)_n$ and $\tilde{E}_T(a,s,\Phi)_n$ characterized

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by the relations

$$E_T(y, s, \Phi)_n q^T = \chi_\infty(a_\infty)^{-1} \det(y)^{-n/2} E_T(n(x)m(a), s, \Phi) \tag{2.3.5}$$

$$\tilde{E}_T(a, s, \Phi)_n \psi_f(\text{tr}(Tb))q^T = \chi(a)^{-1}|\det a|_F^{-n/2} E_T(n(x+b)m(a), s, \Phi) \tag{2.3.6}$$

for any $x \in \text{Herm}_m(F_\infty)$ and $b \in \text{Herm}_m(\mathbb{A}_f)$, with $z := x + iy$, and with $q^T := \psi_\infty(\text{tr}(Tz))$. These correspond to the classical Eisenstein series and its variant in (2.2.15) and (2.2.19).

When $\det T \neq 0$ and $\Phi = \otimes_v \Phi_v$ is factorizable over all places, we have a factorization

$$E_T(h, s, \Phi) = \prod_v W_{T,v}(h_v, s, \Phi_v) \tag{2.3.7}$$

into *local Whittaker functions* defined below (2.3.8).

We switch to local notation: let F_v be a degree 2 étale algebra over a local field F_v^+, with nontrivial involution $a \mapsto \bar{a}$. We assume F_v^+ has characteristic 0 (because Karel assumes this [Kar79]). If F_v^+ is Archimedean, we also assume $F_v/F_v^+ = \mathbb{C}/\mathbb{R}$.

Let $\chi_v \colon F_v^\times \to \mathbb{C}^\times$ and $\psi_v \colon F_v^+ \to \mathbb{C}^\times$ be characters with ψ_v nontrivial, and suppose $\Phi_v \in I(s, \chi_v)$ is a standard section. Given $T \in \text{Herm}_m(F_v^+)$ with $\det T \neq 0$, there is a *local Whittaker function* defined by the absolutely convergent integral

$$W_{T,v}(h, s, \Phi_v) := \int_{N(F_v^+)} \Phi_v(w^{-1}n(b)h, s)\psi_v(-\text{tr}(Tb)) \, dn(b) \tag{2.3.8}$$

for $h \in H(F_v^+)$ and $s \in \mathbb{C}$ with $\text{Re}(s) > m/2$, where $dn(b)$ is the Haar measure which is self-dual with respect to the pairing $(b, b') \mapsto \psi_v(\text{tr}(bb'))$ on $\text{Herm}_m(F_v^+) \cong N(F_v^+)$. For each fixed h, the function $W_{T,v}(h, s, \Phi_v)$ admits holomorphic continuation to $s \in \mathbb{C}$ [Kar79, Corollary 3.6.1][KS97][Ich04, §6]. Extending linearly defines $W_{T,v}(h, s, \Phi_v)$ whenever Φ_v is a finite meromorphic linear combination of standard sections. For any $a \in \text{GL}_m(F_v)$, a change of variables shows

$$W_{T,v}(m(a)h, s, \Phi_v) = \check{\chi}_v(a)|\det a|_{F_v}^{-s+m/2} W_{{}^t\bar{a}Ta,v}(h, s, \Phi_v) \tag{2.3.9}$$

for $\check{\chi}_v(a) := \chi_v(\bar{a})^{-1}$ as above. We use the shorthand $W_{T,v}(s, \Phi_v) := W_{T,v}(1, s, \Phi_v)$.

Lemma 2.3.1. *With notation as above, assume that F_v^+ is non-Archimedean with residue field of cardinality q_v. Suppose $\Phi_v \in I(s, \chi_v)$ is a standard section and $h \in H(F_v^+)$ is a fixed element.*

(1) We have $W_{T,v}(h, s, \Phi_v) \in \mathbb{C}[q_v^{-s}, q_v^s]$.

(2) If $h \in K_v$, we have $W_{T,v}(h, s, \Phi_v) \in \mathbb{C}[q_v^{-2s}]$.

(3) Suppose $\chi_v' \colon F_v^\times \to \mathbb{C}^\times$ is another character satisfying $\chi_v'|_{F_v^{+\times}} = \xi_v \chi_v|_{F_v^{+\times}}$ for an unramified character $\xi_v \colon F_v^{+\times} \to \mathbb{C}^\times$. Assume $h \in K_v$, and suppose $\Psi_v \in I(s, \chi_v')$ is a standard section satisfying $\Psi_v(w^{-1}h) = \Phi_v(w^{-1}h)$. If $f(X) \in \mathbb{C}[X]$ is the polynomial satisfying $f(q_v^{-2s}) = W_{T,v}(h, s, \Phi_v)$, then we have $f(\xi_v(\varpi_0)q^{-2s}) = W_{T,v}(h, s, \Psi_v)$, where $\varpi_0 \in F_v^+$ is a uniformizer.

Proof. A general result of Karel [Kar79, Corollary 3.6.1] states that $W_{T,v}(h, s, \Phi_v) \in \mathbb{C}[q_v^{-s}, q_v^s]$, and that $W_{T,v}(h, s, \Phi_v)$ may be computed for all s as the integral over a sufficiently large open compact subgroup of $N(F_v^+)$. Recall that we have $\Phi_v(m(a)h, s) = \chi_v(\det a)|\det a|_{F_v}^{s+m/2}\Phi_v(h, s)$ for all $a \in \text{GL}_m(F_v)$ and all $h \in H(F_v^+)$. Then apply the discussion surrounding (2.1.10). $\square$

In the case where F_v^+ is non-Archimedean, consider the case where χ_v is unramified and $\chi_v|_{F_v^{+\times}} = \eta_v^n$ for some integer n, where $\eta_v \colon F_v^{+\times} \to \{\pm 1\}$ is the quadratic character associated to F_v/F_v^+. Consider the normalized spherical standard section $\Phi_v^\circ \in I(s, \chi_v)$. We temporarily write $W_{T,v}(h, s, \Phi_v^\circ)_n$ for the associated local Whittaker function, emphasizing the possible dependence on n. By Lemma 2.3.1(3), the implicit χ_v-dependence of $W_{T,v}(h, s, \Phi_v^\circ)_n$ is only on the restriction $\chi_v|_{F_v^{+\times}}$. If F_v/F_v^+ is not inert, then $W_{T,v}(h, s, \Phi_v^\circ)_n$ does not depend on n (note n must be even if F_v/F_v^+ is ramified). If F_v/F_v^+ is inert, then $W_{T,v}(h, s, \Phi_v^\circ)_n$ depends only the parity of n. The ring endomorphism of $\mathbb{C}[q_v^{-2s}]$ sending $q_v^{-2s} \mapsto -q_v^{-2s}$ swaps $W_{T,v}(h, s, \Phi_v^\circ)_n$ and $W_{T,v}(h, s, \Phi_v^\circ)_{n+1}$, by Lemma 2.3.1(3).

2.4. **Singular Fourier coefficients.** Retain notation from Section 2.3 (switching back to global notation). The Fourier terms $E_T(h, s, \Phi)$ for singular $T \in \mathrm{Herm}_m(F^+)$ are known to be closely related with Fourier terms of Eisenstein series on smaller groups (e.g. [GS19, §5.2]). We focus on the case where $\mathrm{rank}\, T = m - 1$ (assume this throughout Section 2.4). On account of (2.3.3), it will be enough to describe the case where T is block diagonal of the form

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T^\flat \end{pmatrix} \tag{2.4.1}$$

with $\det T^\flat \neq 0$.

Assume $m \geq 1$, and fix an integer $n \in \mathbb{Z}$. Let $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a character satisfying $\chi|_{\mathbb{A}^\times} = \eta^n$, where η is the quadratic character associated with F/F^+. Note $\tilde\chi = \chi$ in this case.

Take a factorizable standard section $\Phi = \otimes_v \Phi_v \in I(s, \chi)$, and assume $\Phi_v = \Phi_v^{(n)}$ is the normalized scalar weight standard section (Section 2.2) for every Archimedean place v, with n the fixed integer from above (same for every $v \mid \infty$).

Take T as in (2.4.1). Given $a \in \mathrm{GL}_m(\mathbb{A}_F)$, we study the Fourier coefficient $\tilde{E}_T(a, s, \Phi)_n$. By the Iwasawa decomposition, every $a \in \mathrm{GL}_m(\mathbb{A}_F)$ admits a decomposition

$$a = \begin{pmatrix} 1_1 & * \\ 0 & 1_{m-1} \end{pmatrix} \begin{pmatrix} a^\# & 0 \\ 0 & a^\flat \end{pmatrix} k \tag{2.4.2}$$

with $a^\# \in \mathrm{GL}_1(\mathbb{A}_F)$, with $a^\flat \in \mathrm{GL}_{m-1}(\mathbb{A}_F)$, and with $k \in \prod_{v \mid \infty} U(m) \times \prod_{v < \infty} \mathrm{GL}_m(\mathcal{O}_{F_v})$. We will be eventually interested in the case when Φ_f is spherical, which implies $\tilde{E}_T(ak, s, \Phi)_n = \tilde{E}_T(a, s, \Phi)$ for any $k \in \prod_{v \mid \infty} U(m) \times \prod_{v < \infty} \mathrm{GL}_m(\mathcal{O}_{F_v})$ (also using the fact that Φ_v is a scalar weight standard section for each $v \mid \infty$). In light of the invariance property in (2.3.4), it is thus harmless to restrict to the case of block diagonal $a = \mathrm{diag}(a^\#, a^\flat)$. Assume this for the rest of Section 2.4 (but we do not assume Φ_f is spherical for now).

Set $m^\flat := m - 1$. Arguing as in the proof of [KR88, Lemma 2.4] (see also [GS19, Lemma 5.4] and [HSY21, Theorem 2.2]) gives

$$\tilde{E}_T(a, s, \Phi)_n = |\det a^\#|_F^{s-s_0} \tilde{E}_{T^\flat}(a^\flat, s + 1/2, \mu_{m^\flat}^{m*}(s, \chi)\Phi)_n \tag{2.4.3}$$

$$+ |\det a^\#|_F^{-s-s_0} \tilde{E}_{T^\flat}(a^\flat, s - 1/2, U_{m^\flat}^m(s, \chi)\Phi)_n$$

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where $s_0 := (n - m)/2$, where

$$I_m(s, \chi) \xrightarrow{\mu_{m^\flat}^{m*}(s, \chi)} I_{m^\flat}(s + 1/2, \chi)$$

$$\Psi \longmapsto \Psi \circ \mu_{m^\flat}^m$$

(2.4.4)

(with $\mu_{m^\flat}^m \colon U(m^\flat, m^\flat) \to U(m, m)$ as in Section 2.1), and where

$$I_m(s, \chi) \xrightarrow{\quad U_{m^\flat}^m(s, \chi) \quad} I_{m^\flat}(s - 1/2, \chi)$$

$$\Psi \longmapsto \left(h \mapsto \int_{\substack{b_1 \in \mathrm{Herm}_{m - m^\flat}(\mathbb{A}) \\ b_{12} \in M_{m - m^\flat, m^\flat}(\mathbb{A}_F)}} \Psi \left(w_m^{-1} \cdot n \begin{pmatrix} b_1 & b_{12} \\ {}^t\bar{b}_{12} & 0 \end{pmatrix} w_{m^\flat} \mu_{m^\flat}^m(h), s \right) db_1 \, db_{12} \right)$$

(2.4.5)

for $\mathrm{Re}(s) > m/2$ (with meromorphic continuation to $s \in \mathbb{C}$). A calculation shows

$$M_{m^\flat}(s - 1/2, \chi) \circ U_{m^\flat}^m(s, \chi) = \mu_{m^\flat}^{m*}(-s, \chi) \circ M_m(s, \chi),$$

(2.4.6)

compare [GS19, Lemma 5.5(iii)].

In Corollary 6.2.2, we rewrite (2.4.3) more explicitly when Φ_v is the normalized spherical standard section for every non-Archimedean v.

3. Weil representation

3.1. **Weil index.** We recall *Weil indices*, which are certain constants appearing in the Weil representation and other calculations below. We compute the instances which we need.

Suppose F_v^+ is a local field (arbitrary characteristic) with nontrivial additive character $\psi_v \colon F_v^+ \to \mathbb{C}^\times$, and suppose V_v is a (finite dimensional) F_v^+ vector space equipped with a non-degenerate quadratic form $Q(-)$. The map $V \to \mathbb{C}^\times$ given by $x \mapsto \psi_v(Q(x))$ is a "non-degenerate character of the second degree" in the sense of [Wei64] [Rao93, Appendix], so there is an associated *Weil index* $\gamma_{\psi_v}(V_v) \in \mathbb{C}^\times$ (which is an eighth root of unity). The quantity $\psi_{\psi_v}(V_v)$ depends only on ψ_v and the isomorphism class of V_v, and we have

$$\gamma_{\overline{\psi}_v}(V_v) = \overline{\gamma_{\psi_v}(V_v)} \qquad \gamma_{\psi_v}(V_v \oplus V_v') = \gamma_{\psi_v}(V_v)\gamma_{\psi_v}(V_v')$$

(3.1.1)

for orthogonal direct sums $V_v \oplus V_v'$ (follows from the definition, see [Rao93, Theorem A.2]). The Weil index also satisfies a global product formula [Wei64, Proposition 5].

When F_v^+ has characteristic $\neq 2$ and V_v has a bilinear pairing $(-, -)$, our convention is that $x \mapsto (x, x)$ is the associated quadratic form (and vice-versa).

Lemma 3.1.1. *Let F_v^+ be a local field of characteristic $\neq 2$, let $\psi_v \colon F_v^+ \to \mathbb{C}^\times$ be a nontrivial additive character, and let V_v be a finite dimensional F_v^+ vector space with non-degenerate bilinear pairing. Assume any of the following situations holds.*

(1) The bilinear pairing on V_v is given by

$$\begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}.$$

(3.1.2)

(2) *The field F_v^+ is non-Archimedean with residue characteristic $\neq 2$, there exists a self-dual lattice in V_v, and ψ_v is unramified.*

Then the Weil index is $\gamma_{\psi_v}(V_v) = 1$.

Proof. (1) By compatibility with orthogonal direct sums, we reduce to the case $d = 1$. Given a nonzero element $a \in F_v^{+\times}$, we use the temporary notation $\gamma_{\psi_v}(a)$ for the Weil index of the one-dimensional quadratic space containing an element x with $(x, x) = a$. We have $\gamma_{\psi_v}(V_v) = \gamma_{\psi_v}(a)\gamma_{\psi_v}(-a^{-1})$ for some $a \in F_v^{+\times}$. We have $\gamma_{\psi_v}(a)\gamma_{\psi_v}(-a^{-1}) = 1$ (follows from [Rao93, Theorem A.4], which relates Weil indices and the Hilbert symbol).

(2) By compatibility with orthogonal direct sums, it is enough to show $\gamma_{\psi_v}(a) = 1$ for $a \in \mathcal{O}_{F_v^+}^\times$. This follows from [Rao93, Proposition A.11]. $\qquad\square$

Remark 3.1.2. The explicit formula of [Rao93, Proposition A.12] shows that Lemma 3.1.1(2) is false if $F_v^+ = \mathbb{Q}_2$ (e.g. if V_v has rank one).

Next, let F_v be an étale algebra of degree 2 over a local field F_v^+ of characteristic $\neq 2$ (residue characteristic 2 allowed). Write $\eta_v : F_v^{+\times} \to \{\pm 1\}$ for the quadratic character associated to F_v/F_v^+ (trivial if F_v/F_v^+ is split), and write $a \mapsto \bar{a}$ for the nontrivial involution of F_v over F_v^+. If F_v^+ is non-Archimedean, we write $\mathfrak{d}$ (resp. Δ) for the different (resp. discriminant) ideal for the extension F_v/F_v^+ (where $\mathfrak{d} = \mathcal{O}_{F_v}$ and $\Delta = \mathcal{O}_{F_v^+}$ in the split case). We sometimes abuse notation and write $\mathfrak{d}$ and Δ for understood/chosen generators of these ideals. We write q_v for the residue cardinality of F_v^+ if F_v^+ is non-Archimedean.

Any non-degenerate F_v/F_v^+ Hermitian space V_v has an associated F_v^+-bilinear pairing $\frac{1}{2}\mathrm{tr}_{F_v/F_v^+}(-, -)$ and quadratic form $x \mapsto \frac{1}{2}\mathrm{tr}_{F_v/F_v^+}(x, x)$. (Elsewhere, we typically normalize the trace bilinear pairing without the factor of $\frac{1}{2}$.) We write $\gamma_{\psi_v}(V_v)$ for the Weil index of this quadratic space with respect to a nontrivial additive character $\psi_v : F_v^+ \to \mathbb{C}^\times$. We know $\gamma_{\psi_v}(V_v)^4 = 1$ (see e.g. [Rao93, Corollary A.5(4)] and [Kud94, Theorem 3.1]).

We write $\gamma_{\psi_v}(F_v)$ for the Weil index associated to the one-dimensional Hermitian space F_v with pairing $(x, y) = \bar{x}y$. We write $\epsilon_v(s, \xi_v, \psi_v)$ for the local epsilon factor associated to a quasi-character $\xi_v : F_v^{+\times} \to \mathbb{C}^\times$ (as in [Tat79, §3][Tat67], for the quasi-character $\xi_v|-|^s$ and the self-dual Haar measure for ψ_v).

If F_v^+ is non-Archimedean with uniformizer ϖ_0, we have

$$\epsilon_v(s, \eta_v, \psi_v) = |\varpi_0^{c(\psi_v)}\Delta|_{F_v^+}^{s-1/2}\gamma_{\psi_v}(F_v) \tag{3.1.3}$$

where

$$c(\psi_v) = \max\{j \in \mathbb{Z} : \psi_v|_{\varpi_0^{-j}\mathcal{O}_{F_v^+}} \text{ is trivial}\}. \tag{3.1.4}$$

If F_v^+ is Archimedean, we have

$$\epsilon_v(s, \eta_v, \psi_v) = |a|_{F_v^+}^{s-1/2}\gamma_{\psi_v}(F_v). \tag{3.1.5}$$

where $a \in F_v^{+\times}$ is such that

$$\psi_v(x) = e^{2\pi iax} \quad \text{if } F_v^+ = \mathbb{R} \quad \text{and} \quad \psi_v(z) = e^{2\pi i\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(az)} \quad \text{if } F_v^+ = \mathbb{C}. \tag{3.1.6}$$

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These identities follow from [JL70, Lemma 1.2(iii),(iv)] and properties of epsilon factors. For the reader's convenience, we recall $\gamma_{\psi_v}(\mathbb{C}) = i$ if $F_v^+ = \mathbb{R}$ and $\psi_v(x) = e^{2\pi i x}$.

In all cases, we have

$$\gamma_{\psi_v}(F_v)^2 = \epsilon_v(1/2, \eta_v, \psi_v)^2 = \eta_v(-1). \tag{3.1.7}$$

If F_v^+ is non-Archimedean, recall that $\epsilon_v(s, \xi_v, \psi_v) = 1$ if ξ_v and ψ_v are unramified. If $F_v^+ = \mathbb{R}$ and $\psi_v(x) = e^{2\pi i x}$, recall $\epsilon_v(s, \mathrm{sgn}^j, \overline{\psi}_v) = 1$ (resp. $= -i$) if j is even (resp. odd) where $\mathrm{sgn} \colon \mathbb{R}^\times \to \{\pm 1\}$ is the sign character (these formulas will be used implicitly in Section 5.2). Recall our convention that self-duality for Hermitian lattices is understood with respect to the trace pairing (unless otherwise specified) [Che24a, Section 2.2].

For Hermitian lattices, we always use the term *self-dual* to mean self-dual with respect to the trace pairing (i.e. $L = L^\vee$) unless specified otherwise. If F_v/F_v^+ is ramified and L is a self-dual Hermitian lattice, then L must have even rank (see e.g. [Shi97, Lemma 13.3]).

Lemma 3.1.3. *Let F_v^+ be a local field of characteristic $\neq 2$, let $\psi_v \colon F_v^+ \to \mathbb{C}^\times$ be a nontrivial additive character, and let F_v/F_v^+ be a degree 2 étale algebra. Let V_v be a finite dimensional non-degenerate F_v/F_v^+ Hermitian space. Assume any of the following situations hold.*

(1) The Hermitian space V_v admits a basis with Gram matrix.

$$\begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{3.1.8}$$

(2) We have $F_v = F_v^+ \times F_v^+$.

(3) The extension F_v/F_v^+ is unramified or F_v^+ has residue characteristic $\neq 2$. Moreover, the field F_v^+ is non-Archimedean, there exists a full-rank self-dual $\mathcal{O}_{F_v}$-lattice in V_v, and V_v has even rank.

(4) The field F_v^+ is non-Archimedean, the extension F_v/F_v^+ is unramified, there exists a full-rank self-dual lattice in V_v, and ψ_v is unramified.

Then the Weil index is $\gamma_{\psi_v}(V_v) = 1$.

Proof. We have (3) $\implies$ (1) (see [LL22, Lemma 2.12] for the ramified situation, in which case the even rank assumption is redundant). This implication is false if F_v/F_v^+ is ramified with F_v^+ of residue characteristic 2.

In situations (1) and (2) we may pick a basis $\{1, \alpha\}$ for F_v as an F_v^+ vector space where $\mathrm{tr}_{F_v/F_v^+}(\alpha) = 0$. Applying Lemma 3.1.1 proves the claims.

In situation (4), we may diagonalize the given self-dual lattice, hence reducing to the case where V_v has rank one. In this case, we have $\gamma_{\psi_v}(V_v) = \gamma_{\psi_v}(F_v) = \epsilon(1/2, \eta_v, \psi_v) = 1$. $\square$

3.2. **Weil representation.** Let F_v/F_v^+ and accompanying notation be as in Section 3.1. Assume $F_v/F_v^+ = \mathbb{C}/\mathbb{R}$ if F_v^+ is Archimedean. We also assume F_v^+ has characteristic 0 (because [Kud94] assumes this).

Let V_v be a non-degenerate F_v/F_v^+ Hermitian space of dimension $n \geq 0$. Choose a nontrivial additive character $\psi_v \colon F_v^+ \to \mathbb{C}^\times$, and let $\chi_v \colon F_v^\times \to \mathbb{C}^\times$ be a character such that $\chi_v|_{F_v^{+\times}} = \eta_v^n$.

There is a local *Weil representation* $\omega_v = \omega_{\chi_v, \psi_v}$ of $U(m,m)(F_v^+) \times U(V_v)(F_v^+)$ on the space of Schwartz function $\mathcal{S}(V_v^m)$ (the Schrödinger model [Kud94]), which we normalize as

$$(\omega_v(m(a))\varphi_v)(\underline{x}) = \chi_v(\det a)|\det a|_{F_v}^{n/2}\varphi_v(\underline{x} \cdot a) \qquad\qquad m(a) \in M(F_v^+)$$
$$(\omega_v(n(b))\varphi_v)(\underline{x}) = \psi_v(\operatorname{tr} b(\underline{x},\underline{x}))\varphi_v(\underline{x}) \qquad\qquad n(b) \in N(F_v^+)$$
$$(\omega_v(w)\varphi_v)(\underline{x}) = \gamma_{\psi_v}(V_v)^m \widehat{\varphi}_v(\underline{x}) \qquad\qquad m(a) \in M(F_v^+)$$
$$(\omega_v(h)\varphi_v)(\underline{x}) = \varphi_v(h^{-1} \cdot \underline{x}) \qquad\qquad h \in U(m,m)(F_v^+)$$

for $\varphi_v \in \mathcal{S}(V_v^m)$ and $\underline{x} \in V_v^m$ (viewed as $n \times m$ matrices), where

$$\widehat{\varphi}_v(\underline{x}) = \int_{V_v^m} \varphi_v(\underline{y})\psi_v(\operatorname{tr}_{F_v/F_v^+} \operatorname{tr}(\underline{x},\underline{y}))\, dy \tag{3.2.1}$$

is Fourier transform for the corresponding self-dual Haar measure on V_v^m. The constant $\gamma_{\psi_v}(V_v)$ is the Weil index from Section 3.1

With $s_0 := (n-m)/2$, there is a map $\mathcal{S}(V_v^m) \to I(\chi_v, s_0)$ sending $\varphi_v \in \mathcal{S}(V_v^m)$ to the function $h \mapsto (\omega_v(h)\varphi_v)(0)$. The associated standard section $\Phi_{\varphi_v} \in I(\chi_v, s)$ is the *Siegel–Weil section* for φ_v [GS19, §5.1].

If F_v^+ is non-Archimedean, choose a generator $\mathfrak{d}$ of the different ideal of F_v/F_v^+, and let M_2° be the rank 2 Hermitian $\mathcal{O}_{F_v}$-lattice admitting a basis with Gram matrix

$$\begin{pmatrix} 0 & \mathfrak{d}^{-1} \\ \overline{\mathfrak{d}}^{-1} & 0 \end{pmatrix}. \tag{3.2.2}$$

Note that $M_2^\circ = M_2^{\circ *}$ is self-dual (with respect to the F_v^+-bilinear pairing $\operatorname{tr}_{F_v/F_v^+}(-,-)$).

Lemma 3.2.1. *In the situation above, assume moreover that χ_v and ψ_v are unramified, and that F_v^+ is non-Archimedean. Suppose $\varphi_v = \mathbf{1}_M^{\otimes m}$ where $\mathbf{1}_M$ is the characteristic function of a full rank $\mathcal{O}_{F_v}$-lattice $M \subseteq V_v$ in any of the following situations.*

(1) The lattice M is self-dual. Moreover, the extension F_v/F_v^+ is unramified, or F_v^+ has residue characteristic $\neq 2$.

(2) We have $M \cong (M_2^\circ)^{\oplus d}$ (orthogonal direct sum) for some $d \geq 0$.

Then the associated Siegel-Weil section Φ_{φ_v} is the normalized spherical section Φ_v°, i.e. K_v-fixed with $\Phi_{\varphi_v}(1) = 1$.

Proof. Follows from the explicit formulas above, since w and $P(\mathcal{O}_{F_v^+})$ generate $K_v = U(m,m)(\mathcal{O}_{F_v^+})$ and since the Weil index $\gamma_{\psi_v}(V_v)$ is 1 (Lemma 3.1.3).

If M has even rank, then condition (1) implies condition (2) (the ramified case is [LL22, Lemma 2.12]). $\qquad\square$

Next, consider the case where $F_v/F_v^+ = \mathbb{C}/\mathbb{R}$. Suppose the n-dimensional Hermitian space V_v is positive definite, with Hermitian pairing $(-,-)$. If $\psi_v(x) = e^{2\pi i x}$, the Gaussian

$$\varphi_v(\underline{x}) = e^{-2\pi \operatorname{tr}(\underline{x},\underline{x})} \in \mathcal{S}(V_v^m) \tag{3.2.3}$$

for $\underline{x} = (x_1, \ldots, x_m) \in V_v^m$ (where $\mathrm{tr}(\underline{x}, \underline{x}) = (x_1, x_1) + \cdots + (x_m, x_m)$) has associated Siegel–Weil section

$$\Phi_{\varphi_v} = \Phi_v^{(n)} \qquad\qquad (3.2.4)$$

where $\Phi_v^{(n)}$ is the scalar weight standard section described surrounding (2.2.11), see [GS19, (2.68)].

Remark 3.2.2. Suppose F/F^+ is a CM extension of number fields with associated quadratic character η and accompanying notation as in Section 2.2. With m and n as above, choose any character $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ satisfying $\chi|_{\mathbb{A}^\times} = \eta^n$. Choose nontrivial additive characters $\psi_v \colon F_v^+ \to \mathbb{C}^\times$ for each place v (the ψ_v need not come from a global character). Suppose we are given a collection of local Weil representations ω_{χ_v, ψ_v} on some $\mathcal{S}(V_v^m)$ for each place v of F_v^+ (where the collection $(V_v)_v$ of local Hermitian spaces need not come from a global Hermitian space). Choose $\varphi_v \in \mathcal{S}(V_v^m)$ for each place v, and assume $\varphi_v = \mathbf{1}_{L_v}^m$ for some full-rank self-dual lattice $L_v \subseteq V_v$ for all but finitely many v. Set $\Phi := \bigotimes_v \Phi_{\varphi_v}$.

In this situation, the Eisenstein series variant $\tilde{E}(a, s, \Phi)_n$ (2.2.19) does not depend on the choice of χ. This follows upon inspecting the Weil representation, particularly the action of $m(a)$.

This remark also has a local version, i.e. the Whittaker function variants $\tilde{W}_{T,v}^*(a, s)_n^\circ$ and $\tilde{W}_{T,v}^*(a, s, \Phi_{\varphi_v})_n$ (Sections 4.2 and 4.3) do not depend on the choice of χ_v.

4. LOCAL WHITTAKER FUNCTIONS

Let F_v/F_v^+ and accompanying notation be as in Section 3.2. If F_v^+ has residue characteristic 2, we also assume F_v/F_v^+ is unramified. Let $\chi_v \colon F_v^\times \to \mathbb{C}^\times$ be a character satisfying $\chi_v|_{F_v^{+\times}} = \eta_v^n$ for some integer $n \in \mathbb{Z}$, with n even if F_v/F_v^+ is ramified. Assume χ_v is unramified if F_v^+ is non-Archimedean. Let $\psi_v \colon F_v^+ \to \mathbb{C}^\times$ be an unramified nontrivial additive character. Assume $\psi_v(x) = e^{2\pi i x}$ if $F_v^+ = \mathbb{R}$. These are our default hypotheses, but weaker hypotheses often suffice (as will be indicated below).

Let $\Phi_v^\circ \in I(s, \chi_v)$ be the normalized spherical standard section if F_v^+ is non-Archimedean. Let $\Phi_v^{(n)} \in I(s, \chi_v)$ be the normalized scalar weight standard section from Section 2.2 if $F_v/F_v^+ = \mathbb{C}/\mathbb{R}$.

Given an integer $m \geq 0$ (we do not assume $m \leq n$, unless otherwise specified) and given $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$, we define *normalized local Whittaker functions*

$$W_{T,v}^*(h, s)_n^\circ := \Lambda_{T,v}(s)_n^\circ W_{T,v}(h, s, \Phi_v^\circ) \qquad \text{for } F_v^+ \text{ non-Archimedean} \qquad (4.0.1)$$

$$W_{T,v}^*(h, s)_n^\circ := \Lambda_{T,v}(s)_n^\circ W_{T,v}(h, s, \Phi_v^{(n)}) \qquad \text{for } F_v^+ \text{ Archimedean} \qquad (4.0.2)$$

for certain normalizing factors $\Lambda_{T,v}(s)_n^\circ$ (see (4.3.1) and (4.2.1) below).

The preceding normalization gives $W_{T,v}^*(h, s)_n^\circ$ a clean functional equation (Section 5). Moreover, the normalized function $W_{T,v}^*(h, s)_n^\circ$ (as opposed to the unnormalized versions) seem to correspond more naturally to local information about special cycles (e.g. local contributions to arithmetic degrees) in arithmetic (and non-arithmetic) Siegel–Weil formulas. For example, our main local theorems [Che24a, Section 9] and [Che24b, Section 4] are proved in terms of the derivative of $W_{T,v}^*(1, s)_n^\circ$ and not $W_{T,v}(1, s, \Phi_v^\circ)$ or $W_{T,v}(1, s, \Phi_v^{(n)})$).

The normalizing factors $\Lambda_{T,v}(s)_n^\circ$ also carry geometric information. For example, consider an imaginary quadratic field $F/\mathbb{Q}$ of odd discriminant, suppose $m = n$ is even, and form the product

$2 \prod_v \Lambda_{T,v}(s)_n^\circ$ over all places v of $\mathbb{Q}$. If $n \equiv 0 \pmod 4$, evaluation at $s = 0$ returns the degree of a certain 0-dimensional unitary complex Shimura variety (stack), giving a case of a unitary analogue of the Siegel mass formula. If $n \equiv 2 \pmod 4$, evaluation at $s = 0$ returns the volume of a certain $(n-1)$-dimensional unitary complex Shimura variety (stack). These volume identities will be discussed in Section 8.2 (but are not needed for our main theorems on arithmetic Siegel–Weil).

4.1. Local L-factors.
We use the following (standard) local factors as in [Tat79, §3].

If F_v^+ is a local field (allowing arbitrary characteristic in Section 4.1) and $\xi_v \colon F_v^{+\times} \to \mathbb{C}^\times$ is a quasi-character, we write $L_v(s, \xi_v)$ for the corresponding local L-factor (for the quasi-character $\xi_v | - |_{F_v^+}^s$). Given any nontrivial additive character $\psi_v \colon F_v^+ \to \mathbb{C}^\times$, we write $\epsilon_v(s, \xi_v, \psi_v)$ for the corresponding local epsilon factor (as appeared in Section 3.1) and $\rho_v(s, \xi_v, \psi_v)$ for the local factor from Tate's thesis [Tat67, Theorem 2.4.1], which satisfies

$$\rho_v(s, \xi_v, \psi_v) = \epsilon_v(s, \xi_v, \psi_v)^{-1} L_v(1 - s, \xi_v^{-1})^{-1} L_v(s, \xi_v). \tag{4.1.1}$$

If F^+ is a global field with a quasi-character $\xi \colon F^{+\times} \backslash \mathbb{A}_{F^+} \to \mathbb{C}^\times$ and nontrivial additive character $\psi \colon F^+ \backslash \mathbb{A}_{F^+} \to \mathbb{C}^\times$, we write

$$\Lambda(s, \xi) = \prod_v L_v(s, \xi_v) \qquad L(s, \xi) = \prod_{v < \infty} L_v(s, \xi) \qquad \epsilon(s, \xi) = \prod_v \epsilon_v(s, \xi_v, \psi_v) \tag{4.1.2}$$

and have $\Lambda(s, \xi) = \epsilon(s, \xi)\Lambda(1 - s, \xi^{-1})$. For the reader's convenience, we recall the formulas

$$L_v(s, \xi_v) = \begin{cases} (1 - \xi_v(\varpi_0)|\varpi_0|_{F_v^+}^s)^{-1} & \text{if } \xi_v \text{ is unramified} \\ 1 & \text{if } \xi_v \text{ is ramified} \end{cases} \quad \begin{array}{l} \text{if } F_v^+ \text{ is non-Archimedean} \\ \text{with uniformizer } \varpi_0 \in F_v^+ \end{array}$$

$$L_v(s, \text{sgn}^j) = \begin{cases} \pi^{-s/2}\Gamma(s/2) & \text{if } j \text{ is even} \\ \pi^{-(s+1)/2}\Gamma((s+1)/2) & \text{if } j \text{ is odd} \end{cases} \quad \begin{array}{l} \text{if } F_v^+ = \mathbb{R} \text{ and sgn denotes} \\ \text{the sign character.} \end{array}$$

4.2. Normalized Archimedean Whittaker functions.
With notation as above, assume F_v/F_v^+ is $\mathbb{C}/\mathbb{R}$ and let $\psi_v \colon \mathbb{R} \to \mathbb{C}^\times$ be the standard additive character $x \mapsto e^{2\pi i x}$. The symbol h will denote an element of $U(m, m)(F_v^+)$. Fix integers n, m with $m \geq 0$.

Consider $T \in \text{Herm}_m(F_v^+)$ with $\det T \neq 0$. With $s_0 := (n - m)/2$ as above, we define the normalizing factor

$$\Lambda_{T,v}(s)_n^\circ := \frac{(2\pi)^{m(m-1)/2}}{(-2\pi i)^{nm}} \pi^{m(-s+s_0)} \left(\prod_{j=0}^{m-1} \Gamma(s - s_0 + n - j) \right) |\det T|_{F_v^+}^{-s-s_0} \tag{4.2.1}$$

(compare [GS19, (3.3.14)], also Shimura [Shi82]) where Γ is the usual gamma function.

We define a *normalized Archimedean Whittaker function*

$$W_{T,v}^*(h, s)_n^\circ := \Lambda_{T,v}(s)_n^\circ W_{T,v}(h, s, \Phi_v^{(n)}). \tag{4.2.2}$$

For $a \in \text{GL}_m(F_v)$, we also consider the variant

$$\tilde{W}_{T,v}^*(a, s)_n^\circ := \chi_v(a)^{-1} |\det a|_{F_v}^{-n/2} W_{T,v}^*(m(a), s)_n^\circ \cdot q^{-T} \qquad q^{-T} := e^{-2\pi i \text{tr}(iTy)} \tag{4.2.3}$$

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with $y := a^t\bar{a}$ (temporary notation). This is a (normalized) local analogue of (2.3.6). For any $a \in \mathrm{GL}_m(F_v)$ and $k \in U(m)$, we have the "linear invariance" properties

$$\tilde{W}_{T,v}^*(a,s)_n^\circ = \tilde{W}_{{}^t\bar{a}Ta}^*(1,s)_n^\circ \qquad \tilde{W}_{T,v}^*(1,s)_n^\circ = \tilde{W}_{T,v}^*(k,s)_n^\circ. \qquad (4.2.4)$$

The first expression follows from (2.3.9), and the second expression follows from the scalar weight property of $\Phi_v^{(n)}$. Given $y \in \mathrm{Herm}_m(\mathbb{R})_{>0}$, we also set $W_{T,v}^*(y,s)_n^\circ := \tilde{W}_{T,v}^*(m(a),s)_n^\circ$ for any $a \in \mathrm{GL}_m(F_v)$ satisfying $a^t\bar{a} = y$ (does not depend on the choice of a).[7] We use the shorthand $W_{T,v}^*(s)_n^\circ := \tilde{W}_{T,v}^*(1,s)_n^\circ$.

For all $n \in \mathbb{Z}$, we have the functional equation

$$W_{T,v}^*(h,s)_n^\circ = \eta_v(\det T)^{n-m-1} W_{T,v}^*(h,-s)_n^\circ. \qquad (4.2.5)$$

The case when T is positive definite follows from [Shi82, Theorem 3.1] (via (2.2.13), see also [GS19, (3.54)]). The case of general T (still with $\det T \neq 0$) should follow from [Shi82, Theorem 4.2, (4.34.K)], though we will give an alternative proof (Lemma 5.2.1). Here η_v is the sign character $\mathrm{sgn}(-)$.

Write (r_1, r_2) for the signature of T (temporary notation). If either $n \geq r_1$ or $r_2 = 0$, then the function $W_{T,v}^*(h,s)_n^\circ$ is holomorphic for all $s \in \mathbb{C}$, for fixed h (follows from [Shi82, Theorem 4.2, (4.34.K)]). For any $a \in \mathrm{GL}_m(F_v)$, we also have

$$\tilde{W}_{T,v}^*(a,s_0)_n^\circ = \begin{cases} 1 & \text{if } T \text{ is positive definite} \\ 0 & \text{if } m \leq n \text{ and } T \text{ is not positive definite.} \end{cases} \qquad (4.2.6)$$

For the case when T is positive definite, see [Shi97, (3.15)] (also the proof of [GS19, Proposition 3.2]). The non positive definite case with $m \leq n$ follows from [Shi82, Theorem 4.2, (4.34.K)] (see also [GS19, Proposition 3.3(i)]).

4.3. Normalized non-Archimedean Whittaker functions.
With n, χ_v, ψ_v, η_v, etc. as at the beginning of Section 4, assume F_v^+ is non-Archimedean. For the moment, we only assume F_v^+ has characteristic $\neq 2$, and allow χ_v possibly ramified. We can also allow F_v/F_v^+ to be ramified with F_v^+ of residue characteristic 2 in Section 4.3. The symbol h will denote an element of $U(m,m)(F_v^+)$.

Assume $\psi_v : F_v^+ \to \mathbb{C}^\times$ is a nontrivial unramified additive character. Let ϖ_0 be a uniformizer of F_v^+, and let q_v be the residue cardinality of F_v^+. Consider $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$.

We define the *local normalizing factor*

$$\Lambda_{T,v}(s)_n^\circ := |\Delta|_{F_v^+}^{-m(m-1)/4} \left(\prod_{j=0}^{m-1} L_v(2s+m-j, \eta_v^{j+n}) \right) |(\det T)\Delta^{\lfloor m/2 \rfloor}|_{F_v^+}^{-s-s_0}. \qquad (4.3.1)$$

The local L-factors appearing in $\Lambda_{T,v}(s)_n^\circ$ should be compared with e.g. [HKS96, §6].

Suppose V_v is an n-dimensional non-degenerate F_v/F_v^+ Hermitian space. Consider a full-rank lattice $L_v \subseteq V_v$, and take the Schwartz function $\varphi_v = \mathbf{1}_{L_v}^m \in \mathcal{S}(V_v^m)$. Form the associated Siegel–Weil standard section $\Phi_{\varphi_v} \in I(\chi_v, s)$. Let S be the Gram matrix of any basis for L_v.

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[7]With this notation, there is possible ambiguity for the meaning of $W_{T,v}^*(1,s)_n^\circ$, which could refer to either $W_{T,v}^*(h,s)_n^\circ$ or $W_{T,v}^*(y,s)_n^\circ$ evaluated at $h = 1_{2m}$ or $y = 1_m$. To avoid confusion, we will avoid the symbol $W_{T,v}^*(1,s)_n^\circ$ when v is Archimedean.

We consider the *normalized local Whittaker function* $W_{T,v}^*$ and the variant $\tilde{W}_{T,v}^*$

$$W_{T,v}^*(h, s, \Phi_{\varphi_v})_n := \gamma_{\psi_v}(V_v)^m \mathrm{vol}(L_v)^{-m} \Lambda_{T,v}(s)_n^\circ W_{T,v}(h, s, \Phi_{\varphi_v}) \tag{4.3.2}$$

$$\tilde{W}_{T,v}^*(a, s, \Phi_{\varphi_v})_n := \chi_v(a)^{-1} |\det a|_{F_v}^{-n/2} W_{T,v}^*(m(a), s, \Phi_{\varphi_v}) \tag{4.3.3}$$

for $a \in \mathrm{GL}_m(F_v)$. The volume $\mathrm{vol}(L_v)$ is taken with respect to the self-dual Haar measure with respect to the pairing $x, y \mapsto \psi_v(\mathrm{tr}(x, y))$ on V_v (compare Lemma 4.4.2). The variant $\tilde{W}_{T,v}^*$ is a local analogue of (2.2.19). These will depend on n in general. For any $a \in \mathrm{GL}_m(F_v)$ and $k \in \mathrm{GL}_m(\mathcal{O}_{F_v^+})$, we have the "linear invariance" property

$$\tilde{W}_{T,v}^*(a, s, \Phi_{\varphi_v})_n = \tilde{W}_{{}^t\bar{a}Ta,v}^*(1, s, \Phi_{\varphi_v})_n \qquad W_{T,v}^*(1, s, \Phi_{\varphi_v})_n = \tilde{W}_{T,v}^*(1, s)_n = \tilde{W}_{T,v}^*(k, s, \Phi_{\varphi_v})_n. \tag{4.3.4}$$

The left expression follows from (2.3.9). The right expression follows from the expression $\chi_v(k)^{-1}\omega_v(m(k))\varphi_v = \varphi_v$ for all k, where ω_v is the local Weil representation (Section 3.2).

Now assume χ_v is unramified, and recall the normalized spherical standard section $\Phi_v^\circ \in I(\chi_v, s)$. If L_v is self-dual, we have $\Phi_{\varphi_v} = \Phi_v^\circ$ (Section 3.2), at least outside the case of F_v/F_v^+ ramified with residue characteristic 2. If F_v/F_v^+ is ramified of residue characteristic 2, this still holds if $L_v = (M_2^\circ)^{\oplus d}$ for some $d \geq 0$ (with M_2° the "standard" self-dual lattice from (3.2.2). Note that $\gamma_{\psi_v}(V_v) = 1$ in these cases.

In the situation of the previous paragraph, we set

$$W_{T,v}^*(h, s)_n^\circ := W_{T,v}^*(h, s, \Phi_{\varphi_v})_n \qquad \tilde{W}_{T,v}^*(a, s)_n^\circ := \tilde{W}_{T,v}^*(a, s, \Phi_{\varphi_v})_n$$

for $h \in H(F_v^+)$ and $a \in \mathrm{GL}_m(F_v)$. Note $W_{T,v}^*(h, s)_n^\circ = \Lambda_{T,v}(s)_n^\circ W_{T,v}(h, s, \Phi_v^\circ)$. The alternative normalization

$$W_{T,v}^{(*)}(h, s)_n^\circ := |(\det T)\Delta^{\lfloor m/2 \rfloor}|_{F_v^+}^{s+s_0} W_{T,v}^*(h, s)_n^\circ \tag{4.3.5}$$

will also be useful.

We use the shorthand $W_{T,v}^*(s)_n^\circ := W_{T,v}^*(1, s)_n^\circ$ and $W_{T,v}^{(*)}(s)_n^\circ := W_{T,v}^{(*)}(1, s)_n^\circ$. We further describe these functions in the following sections (e.g. special values and functional equations). We are mostly interested in the spherical local Whittaker function $W_{T,v}^*(h, s)_n^\circ$, and the case of general φ_v plays a very limited role in the present work.

4.4. Local densities.

We relate non-Archimedean Whittaker functions with local densities. This should be essentially known, but we restate the result for clarity (Lemma 4.4.2).[8] In Section 4.4, we do not need to assume χ_v is unramified (but still require $\chi_v|_{F_v^{+\times}} = \eta_v^n$).

Retain notation and assumptions from Section 4.3. In Section 4.4, we now require F_v^+ to have characteristic 0, exclude the case where F_v/F_v^+ is ramified with F_v^+ of residue characteristic 2, and

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[8] The proof is essentially as in [KR14, Proposition 10.1], with a few modifications. In the ramified situation, we should use M_2° (from Section 3.2) instead of $L_{1,1}$ (in the proof of loc. cit.); the proposition statement changes correspondingly, see [Shi22, Proposition 9.7]. Moreover, the quantity $\gamma_p(V)^n$ appearing before [KR14, (10.3)] should be $\gamma_p(V)^{-n}$ for consistency with the Schrödinger model of the Weil representation from [Kud94, Theorem 3.1 §3, §5] (and the same applies to [Shi22, Proposition 9.7]). The interpolation of $W_{T,v}(s, \Phi_{\varphi_v})$ in the two cited references should also be shifted by $s_0 = (n - m)/2$ in the s-variable. The cited results also restrict to the case $F_v^+ = \mathbb{Q}_p$, but the result and (modified) proof hold more generally.

take $n \geq 0$. We write

$$\operatorname{Herm}_m(\mathcal{O}_{F_v^+})^* := \{b \in \operatorname{Herm}_m(F_v^+) : \operatorname{tr}(bc) \in \mathcal{O}_{F_v^+} \text{ for all } c \in \operatorname{Herm}_m(\mathcal{O}_{F_v^+})\} \quad (4.4.1)$$

$$= \{b \in \operatorname{Herm}_m(F_v^+) : b_{i,j} \in \mathcal{O}_{F_v^+} \text{ if } i = j \text{ and } b_{i,j} \in \mathfrak{d}^{-1}\mathcal{O}_{F_v} \text{ if } i \neq j\}.$$

Given nonsingular Hermitian matrices $S \in \operatorname{Herm}_n(F_v^+)$ and $T \in \operatorname{Herm}_m(F_v^+)$, we consider the *local representation density* (or just *local density*)

$$\operatorname{Den}(S, T) := \lim_{k \to \infty} \frac{\operatorname{vol}(\{x \in M_{n,m}(\mathcal{O}_{F_v}) : {}^t\overline{x}Sx - T \in \varpi_0^k \operatorname{Herm}_m(\mathcal{O}_{F_v^+})^*\})}{q_v^{-km^2}} \quad (4.4.2)$$

where $M_{n,m}(\mathcal{O}_{F_v})$ is given the Haar measure of total volume 1. The limit argument stabilizes for $k \gg 0$ (follows from the proof of Lemma 4.4.2). The local density $\operatorname{Den}(S, T)$ depends only on the isomorphism classes of the Hermitian lattices defined by S and T. If $n < m$ then $\operatorname{Den}(S, T) = 0$.

If $S \in \operatorname{Herm}_n(\mathcal{O}_{F_v^+})^*$, we have

$$\operatorname{Den}(S, T) = \lim_{k \to \infty} \frac{\#\{x \in M_{n,m}(\mathcal{O}_{F_v}/\varpi_0^k\mathcal{O}_{F_v}) : {}^t\overline{x}Sx - T \in \varpi_0^k \operatorname{Herm}_m(\mathcal{O}_{F_v^+})^*\}}{q_v^{k \cdot m(2n-m)}}. \quad (4.4.3)$$

If $S \in \operatorname{Herm}_n(\mathcal{O}_{F_v^+})^*$ and $T \notin \operatorname{Herm}_m(\mathcal{O}_{F_v^+})^*$, we have $\operatorname{Den}(S, T) = 0$.

Remark 4.4.1. If $S \in \operatorname{Herm}_n(\mathcal{O}_{F_v^+})^*$ and $T \in \operatorname{Herm}_m(\mathcal{O}_{F_v^+})^*$ with $m \leq n$, the local density $\operatorname{Den}(S, T)$ admits the following equivalent formulation. Suppose M (resp. L) is a Hermitian $\mathcal{O}_{F_v}$-lattice which admits a basis with Gram matrix S (resp. T). Write $\mathfrak{d}$ for any trace-zero generator of the different ideal $\mathfrak{d}$ of F_v/F_v^+, and let M' (resp. L') be the skew-Hermitian lattice with pairing $\mathfrak{d}S$ (resp. $\mathfrak{d}T$). If $\operatorname{Herm}(M', L')$ is the *scheme of skew-Hermitian module homomorphisms* given by

$$\operatorname{Herm}(M', L')(R) := \operatorname{Herm}(M' \otimes R, L' \otimes R) \quad (4.4.4)$$

for $\mathcal{O}_{F_v^+}$-algebras R (where the right-hand side means $\mathcal{O}_{F_v}$-linear homomorphisms preserving the skew-Hermitian pairing), we have

$$\#\operatorname{Herm}(M', L')(\mathcal{O}_{F_v^+}/\varpi_0^k\mathcal{O}_{F_v^+}) = \#\{x \in M_{n,m}(\mathcal{O}_{F_v}/\varpi_0^k\mathcal{O}_{F_v}) : {}^t\overline{x}Sx - T \in \varpi_0^k \operatorname{Herm}_m(\mathcal{O}_{F_v^+})^*\}$$
$$(4.4.5)$$

and also $m(2n - m) = \dim(\operatorname{Herm}(M', L') \times \operatorname{Spec} F_v^+)$ (and the right-hand side is nonempty). This recovers the formulations in [LZ22, §3.1] (inert), [FYZ24, §2.3] (inert and split), and [HLSY23, §5.1] (ramified).

Return to the situation of general S and T (and possibly $m > n$). Fix characters $\chi_v : F_v^\times \to \mathbb{C}^\times$ and $\psi_v : F_v^+ \to \mathbb{C}^\times$ as above, with ψ_v unramified. Let M be a Hermitian $\mathcal{O}_{F_v}$-lattice admitting a basis whose Gram matrix is S. Write $V_v = M \otimes_{\mathcal{O}_{F_v}} F_v$ for the associated F_v/F_v^+ Hermitian space, and let $\varphi_v \in \mathcal{S}(V_v^m)$ be the function $\varphi_v = \mathbf{1}_M^{\otimes m}$, where $\mathbf{1}_M$ is the characteristic function of M. Let $\Phi_{\varphi_v} \in I(s, \chi_v)$ be the associated Siegel–Weil section, and form the local Whittaker function $W_{T,v}(h, s, \Phi_{\varphi_v})$ as in Section 2.3. Set $W_{T,v}(s, \Phi_{\varphi_v}) := W_{T,v}(1, s, \Phi_{\varphi_v})$.

With M_2° being the rank 2 self-dual Hermitian lattice from (3.2.2), let $S_{r,r}$ be the Gram matrix of a basis for $L_{v,r,r} := M \oplus (M_2^\circ)^{\oplus r}$ (orthogonal direct sum). When F_v/F_v^+ is not ramified, we also let S_r be the Gram matrix of a basis for $L_{v,r} := M \oplus \langle 1 \rangle^{\oplus r}$ (orthogonal direct sum), where $\langle 1 \rangle$ is a

rank one self-dual lattice. The notations $L_{v,r,r}$ and $L_{v,r}$ will only be used in the proof of the next lemma.

Lemma 4.4.2. *With notation as above, there exists* $\mathrm{Den}(S,T,X) \in \mathbb{Q}[X]$ *(necessarily unique) such that*

$$W_{T,v}(s_0 + s, \Phi_{\varphi_v}) = \gamma_{\psi_v}(V_v)^{-m} |\det S|^m_{F_v^+} |\Delta|^e_{F_v^+} \mathrm{Den}(S,T,q_v^{-2s}) \qquad \textit{for all } s \in \mathbb{C} \qquad (4.4.6)$$

$$\mathrm{Den}(S_{r,r},T) = \mathrm{Den}(S,T,q_v^{-2r}) \qquad\qquad \textit{for all } r \in \mathbb{Z}_{\geq 0} \qquad (4.4.7)$$

where $\gamma_{\psi_v}(V_v)$ *is the Weil index,* $s_0 = (n-m)/2$, *and* $e = nm/2 + m(m-1)/4$. *For all* $r \in \mathbb{Z}_{\geq 0}$, *we also have*

$$\mathrm{Den}(S_r,T) = \mathrm{Den}(S,T,(-q_v)^{-r}) \qquad\qquad \textit{if } F_v/F_v^+ \textit{ is inert} \qquad (4.4.8)$$

$$\mathrm{Den}(S_r,T) = \mathrm{Den}(S,T,q_v^{-r}) \qquad\qquad \textit{if } F_v/F_v^+ \textit{ is split.} \qquad (4.4.9)$$

Proof. As mentioned above (Footnote 8), this is a restatement of a result which should be essentially known [KR14, Proposition 10.1] [Shi22, Proposition 9.7], up to a few modifications. The modified version stated here may be proved by a similar interpolation argument, as explained below. For any $r \in \mathbb{Z}_{\geq 0}$, set $V_{v,r,r} := L_{v,r,r} \otimes_{\mathcal{O}_{F_v}} F_v$, and let $\varphi_{v,r,r} = \mathbf{1}^m_{L_{v,r,r}}$. Equip $\mathrm{Herm}_m(\mathcal{O}_{F_v^+})$ and $V_{v,r,r}$ with the self-dual Haar measures with respect to $(b,c) \mapsto \psi_v(\mathrm{tr}(bc))$ and $\psi_v(\mathrm{tr}_{F_v/F_v^+}(\mathrm{tr}(-,-)))$ respectively. Using the Weil representation, we compute

$$W_{T,v}(s_0 + r, \Phi_{\varphi_v}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.4.10)$$

$$= \lim_{k\to\infty} \int_{\varpi_0^{-k}\mathrm{Herm}_m(\mathcal{O}_{F_v^+})} \psi_v(-\mathrm{tr}(Tb)) \Phi_{\varphi_v}(w^{-1}n(b), s_0 + r) \, dn(b)$$

$$= \gamma_{\psi_v}(V_v)^{-m} \lim_{k\to\infty} \int_{\varpi_0^{-k}\mathrm{Herm}_m(\mathcal{O}_{F_v^+})} \psi_v(-\mathrm{tr}(Tb)) \int_{V_{v,r,r}^m} \psi_v(\mathrm{tr}(b(x,x))) \varphi_{v,r,r}(x) \, dx \, dn(b)$$

$$= \gamma_{\psi_v}(V_v)^{-m} \lim_{k\to\infty} \mathrm{vol}(\varpi_0^{-k}\mathrm{Herm}_m(\mathcal{O}_{F_v^+})) \int_{\substack{x \in V_{v,r,r}^m \\ (x,x)-T\in\varpi_0^k\mathrm{Herm}_m(\mathcal{O}_{F_v^+})^*}} \varphi_{v,r,r}(x) \, dx$$

$$= \gamma_{\psi_v}(V_v)^{-m} \mathrm{vol}(\mathrm{Herm}_m(\mathcal{O}_{F_v^+})) \mathrm{vol}(L_{v,r,r}^m) \mathrm{Den}(S_{r,r},T)$$

We have the volume identities

$$\mathrm{vol}(\mathrm{Herm}_m(\mathcal{O}_{F_v^+})) = |\Delta|^{m(m-1)/4}_{F_v^+} \qquad \mathrm{vol}(L_{v,r,r}) = |\det S|_{F_v^+} |\Delta|^{n/2}_{F_v^+} \qquad (4.4.11)$$

for the self-dual Haar measures described above. We already know $W_{T,v}(s, \Phi_{\varphi_v}) \in \mathbb{C}[q_v^{-2s}]$ by Lemma 2.3.1. Since $\mathrm{Den}(S_{r,r},T) \in \mathbb{Q}$ for all $r \geq 0$, we conclude $W_{T,v}(s, \Phi_{\varphi_v}) \in \mathbb{Q}[q_v^{-2s}]$. The additional claims involving $\mathrm{Den}(S_r,T)$ in the unramified case may be proved similarly, using $L_{v,r}$ instead of $L_{v,r,r}$. $\qquad\square$

4.5. Local densities and spherical non-Archimedean Whittaker functions. Take F_v/F_v^+, ψ_v, and χ_v as in Section 4.4, and continue to assume $n \geq 0$ for the moment. Set $s_0 = (n-m)/2$. We assume χ_v is unramified.

Let M° be a self-dual Hermitian $\mathcal{O}_{F_v}$-lattice of rank n. This characterizes M° uniquely up to isomorphism, and forces n to be even if F_v/F_v^+ is ramified. We also have $\gamma_{\psi_v}(V_v) = 1$ (Lemma 3.1.3).

Set $V_v = M^\circ \otimes_{\mathcal{O}_{F_v}} F_v$, and let $\varphi_v \in \mathcal{S}(V_v^m)$ be the characteristic function of $M^{\circ m}$. Then the associated Siegel–Weil section $\Phi_{\varphi_v} \in I(s, \chi_v)$ coincides with the normalized spherical Whittaker function Φ_v° (Lemma 3.2.1).

Remark 4.5.1. Even if χ_v is possibly ramified, we still have $W_{T,v}(s, \Phi_{\varphi_v}) = W_{T,v}(s, \Phi_v^\circ)$ for any $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$ (by Lemma 2.3.1(3) or Lemma 4.4.2), where $\Phi_v^\circ \in I(s, \chi_v')$ is the standard normalized spherical section for an unramified χ_v'.

Suppose $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$. If S is the Gram matrix of any basis for M°, Lemma 4.4.2 gives

$$W_{T,v}(s_0 + s, \Phi_v^\circ) = |\Delta|_{F_v^+}^{m(m-1)/4} \mathrm{Den}(S, T, q_v^{-2s}) \tag{4.5.1}$$

for all $s \in \mathbb{C}$.

Suppose $M^{\circ\prime}$ is a rank m Hermitian $\mathcal{O}_{F_v}$-lattice such that

$$\begin{cases} M^{\circ\prime} \text{ is self-dual} & \text{if } F_v/F_v^+ \text{ is unramified or if } m \text{ is even} \\ M^{\circ\prime} \text{ is almost self-dual} & \text{if } F_v/F_v^+ \text{ is ramified and } m \text{ is odd.} \end{cases} \tag{4.5.2}$$

Let $S' \in \mathrm{Herm}(F_v^+)$ be the Gram matrix of a basis for $M^{\circ\prime}$. We have

$$\left(\prod_{j=0}^{m-1} L_v(2(s + s_0) + m - j, \eta_v^{j+n}) \right)^{-1} = \mathrm{Den}(S, S', X)|_{X = q_v^{-2s}}. \tag{4.5.3}$$

See [LZ22, (3.2.0.1)] (inert), [FYZ24, Theorem 2.2] (split and inert), [LL22, Lemma 2.15] (ramified).

There is a (normalized) *local density polynomial* $\mathrm{Den}(X, T)_n \in \mathbb{Z}[1/q_v][X]$ such that

$$W_{T,v}^{(*)}(s + s_0)_n^\circ = \mathrm{Den}(q_v^{-2s}, T)_n \tag{4.5.4}$$

for all $s \in \mathbb{C}$ (with $W_{T,v}^{(*)}$ as in Section 4.3). See the "Cho–Yamauchi formulas" proved in [LZ22, Theorem 3.5.1] (inert), [FYZ24, Theorem 2.2] (split and inert), and [LL22, Lemma 2.15] (ramified). Note that our convention differs slightly from [LL22] in the ramified case, where they consider polynomials in q_v^{-s} instead.

The polynomial $\mathrm{Den}(X, T)_n$ is nonzero if and only if $T \in \mathrm{Herm}(\mathcal{O}_{F_v^+})^*$, in which case $\mathrm{Den}(X, T)_n$ has constant term 1. When $m = n$, we have $\mathrm{Den}(X, T)_n \in \mathbb{Z}[X]$ for any T. More classically, see [Shi97, Theorem 13.6], which implies that $\mathrm{Den}(q_v^n X, T)_n \in \mathbb{Z}[X]$ with constant term 1.

We have

$$\begin{aligned} \mathrm{Den}(X, T)_{n+1} &= \mathrm{Den}(q_v^{-1} X, T)_n & &\text{if } F_v/F_v^+ \text{ is split} \\ \mathrm{Den}(X, T)_{n+1} &= \mathrm{Den}(-q_v^{-1} X, T)_n & &\text{if } F_v/F_v^+ \text{ is inert} \\ \mathrm{Den}(X, T)_{n+2} &= \mathrm{Den}(q_v^{-2} X, T)_n & &\text{if } F_v/F_v^+ \text{ is ramified.} \end{aligned} \tag{4.5.5}$$

For $n < 0$, we define $\mathrm{Den}(X, T)_n$ using (4.5.5). Note that (4.5.4) continues to hold. For the rest of Section 4.5, we allow general $n \in \mathbb{Z}$ (assumed even if F_v/F_v^+ is ramified).

Similarly, there is a (normalized) *local density (Laurent) polynomial* $\mathrm{Den}^*(X,T)_n \in \mathbb{Z}[1/q_v][X, X^{-1/2}]$ such that

$$W^*_{T,v}(s+s_0)^\circ_n = \mathrm{Den}^*(q_v^{-2s}, T)_n \tag{4.5.6}$$

for all $s \in \mathbb{C}$ (with $W^*_{T,v}$ as in Section 4.3).

Remark 4.5.2. On the right-hand side of (4.5.6), we mean evaluating $\mathrm{Den}^*(X,T)_n$ at $X^{1/2} = q_v^{-s}$. We similarly abuse notation elsewhere. For example, $\mathrm{Den}^*(q_v X, T)_n \in \mathbb{Z}[1/q_v^{1/2}][X, X^{-1/2}]$ is obtained from $\mathrm{Den}^*(X,T)_n$ by replacing $X^{1/2}$ with $q_v^{1/2}X^{1/2}$. The notation $\frac{d}{dX} \colon \mathbb{Q}[X, X^{-1/2}] \to \mathbb{Q}[X, X^{-1/2}]$ means the $\mathbb{Q}$-linear map $X^{j/2} \mapsto (j/2)X^{j/2-1}$.

If T defines a self-dual Hermitian lattice when m is even or F_v/F_v^+ is unramified (resp. almost self-dual Hermitian lattice when m is odd and F_v/F_v^+ is ramified), we have

$$W^*_{T,v}(s)^\circ_n = W^{(*)}_{T,v}(s)^\circ_n = 1 \qquad \mathrm{Den}^*(X,T)_n = \mathrm{Den}(X,T)_n = 1 \tag{4.5.7}$$

(follows from (4.5.3)). For such T, an application of Lemma 2.3.1(3) also shows that

$$W_{T,v}(s, \Phi^\circ_v) = |\Delta|^{m(m-1)/4}_{F_v^+} \prod_{j=0}^{m-1} L_v(2s+m-j, \eta_v^j \chi'_v|_{F_v^{+\times}})^{-1} \tag{4.5.8}$$

if $\Phi^\circ_v \in I(s, \chi'_v)$ is the normalized spherical section for any unramified character $\chi'_v \colon F_v^\times \to \mathbb{C}^\times$ (not assuming $\chi'_v|_{F_v^{+\times}} = \eta_v^n$).

If L is a $\mathcal{O}_{F_v}$ Hermitian lattice of rank m, and if L admits a basis with Gram matrix T (allowing arbitrary $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$ again), we write $\mathrm{Den}(X, L)_n := \mathrm{Den}(X, T)_n$ and similarly $\mathrm{Den}^*(X, L)_n := \mathrm{Den}^*(X, T)_n$. We have

$$\mathrm{Den}^*(X, L)_n = (q_v^{2s_0} X^{-1/2})^{\mathrm{val}'(L)} \mathrm{Den}(X, L)_n \tag{4.5.9}$$

$$\mathrm{val}'(L) := \lfloor \mathrm{val}(L) \rfloor = \begin{cases} \mathrm{val}(L) - 1/2 & \text{if } F_v/F_v^+ \text{ is ramified and } m \text{ is odd} \\ \mathrm{val}(L) & \text{else.} \end{cases} \tag{4.5.10}$$

The local densities satisfy a certain cancellation property (which we will use): if L° is a self-dual Hermitian lattice of rank n, then for any non-degenerate Hermitian lattice L and every integer $r \in \mathbb{Z}$ (assume r is even if F_v/F_v^+ is ramified), we have

$$\mathrm{Den}(X, L \oplus L^\circ)_{r+n} = \mathrm{Den}(X, L)_r \qquad \mathrm{Den}^*(X, L \oplus L^\circ)_{r+n} = \mathrm{Den}^*(X, L)_r \tag{4.5.11}$$

where $L \oplus L^\circ$ is the orthogonal direct sum. This follows from the Cho–Yamauchi type formulas cited above and the following linear algebra fact: every lattice $M' \subseteq (L \oplus L^\circ) \otimes_{\mathcal{O}_{F_v}} F_v$ satisfying $L^\circ \subseteq M' \subseteq M'^\vee$ admits an orthogonal direct sum decomposition $M' = L^\circ \oplus M''$ for some sublattice M''.

5. Local functional equations

Let F_v be a degree 2 étale algebra over a local field F_v^+ of characteristic $\neq 2$, with notation $\mathfrak{d}$, Δ, η_v, and $a \mapsto \bar{a}$ as above. If F_v^+ is Archimedean, we also assume F_v/F_v^+ is $\mathbb{C}/\mathbb{R}$. Fix an integer $m \geq 0$.

Consider a character $\chi_v\colon F_v^\times \to \mathbb{C}^\times$ and a nontrivial additive character $\psi_v\colon F^+ \to \mathbb{C}^\times$ (for the moment, we do not require $\chi_v|_{F_v^{+\times}} = \eta_v^n$, and allow χ_v and ψ_v to be ramified).

Set $\check{\chi}_v(a) := \chi_v(\bar{a})^{-1}$. There is a local intertwining operator

$$M(s, \chi_v)\colon I(s, \chi_v) \to I(-s, \check{\chi}_v) \tag{5.0.1}$$

(where $I(s, \chi_v)$ and $I(-s, \check{\chi}_v)$ are degenerate local principal series for $U(m, m)$) defined by the integral

$$M(s, \chi_v)\Phi_v(h) = \int_{N(F_v^+)} \Phi_v(w^{-1} n(b) h, s) \; dn(b) \tag{5.0.2}$$

for $\mathrm{Re}(s) > m/2$, with meromorphic continuation to $\mathbb{C}$ (e.g. see [KS97] in the non-Archimedean case).

Given $T \in \mathrm{Herm}_m(F_v^+)$, we define the quantity

$$\kappa_T(s, \chi_v, \psi_v) = \chi_v(-1)^m \chi_v(\det T)^{-1} |\det T|_{F_v^+}^{-2s} \gamma_{\psi_v}(F_v)^{m(m-1)/2} \eta_v(\det T)^{m-1}$$

$$\cdot \prod_{j=0}^{m-1} \rho_v(2s + j - m + 1, \eta_v^j \cdot \chi_v|_{F_v^{+\times}}, \overline{\psi}_v) \tag{5.0.3}$$

where $\gamma_{\psi_v}(F_v)$ is a Weil index (Section 3.1) and ρ_v is a local factor as in Tate's thesis (Section 4.1). This factor is taken from [KS97, §3][9] (see also [HKS96, Proposition 6.3]).

5.1. **Non-Archimedean.** Suppose F_v^+ is non-Archimedean (with notation as above). For any $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$ and any standard section Φ_v of $I(s, \chi_v)$, there is a functional equation

$$W_{T,v}(h, -s, M(s, \chi_v)\Phi_v) = \kappa_T(s, \chi_v, \psi_v) W_{T,v}(h, s, \Phi_v) \tag{5.1.1}$$

as in [KS97, §3, §7].

We next consider spherical Whittaker functions. Assume ψ_v and χ_v are unramified. We require F_v^+ to be characteristic 0 (because [Shi97, §13] assumes this). With Φ_v° denoting the normalized spherical sections of $I(s, \chi_v)$ and $I(s, \check{\chi}_v)$, we have

$$M(s, \chi_v)\Phi_v^\circ(s) = |\Delta|_{F_v^+}^{m(m-1)/4} \prod_{j=0}^{m-1} \frac{L_v(2s + j - m + 1, \eta_v^j \chi_v|_{F_v^{+\times}})}{L_v(2s + m - j, \eta_v^j \chi_v|_{F_v^{+\times}})} \Phi_v^\circ(-s), \tag{5.1.2}$$

see [Shi97, Theorem 13.6].[10]

Now, we further restrict to the situation where $\chi_v|_{F_v^{+\times}} = \eta_v^n$ for some $n \in \mathbb{Z}$, with n assumed even if F_v/F_v^+ is ramified. Note $\check{\chi}_v = \chi_v$. Combining (5.1.2) with the identities stated above (including

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[9]The factor $\kappa_T(s, \chi_v, \psi_v)$ is given there in the non-Archimedean case, but we will use the same formula in the Archimedean case. For comparing formulas, note the different convention used to define $W_{T,v}$ and $M(s, \chi_v)$ (ψ_v versus $\overline{\psi}_v$ and w vs w^{-1}).

[10]Take $\zeta = 0$ in the notation of loc. cit.. Strictly speaking, the statement there is only for $\chi_v|_{F_v^{+\times}}$ trivial, but the general case follows from this; see (2.1.10) and the proof of Lemma 2.3.1(3).

the relation between Weil indices and epsilon factors in (3.1.3)), a straightforward computation (omitted) yields the functional equations

$$W_{T,v}^{(*)}(h, -s)_n^\circ = |(\det T)\Delta^{\lfloor m/2 \rfloor}|_{F_v^+}^{-2s}\eta_v((-1)^{m(m-1)/2}\det T)^{n-m-1}W_{T,v}^{(*)}(h, s)_n^\circ \tag{5.1.3}$$

$$W_{T,v}^*(h, -s)_n^\circ = \eta_v((-1)^{m(m-1)/2}\det T)^{n-m-1}W_{T,v}^*(h, s)_n^\circ \tag{5.1.4}$$

with $W_{T,v}^{(*)}(h, s)_n^\circ$ and $W_{T,v}^*(h, s)_n^\circ$ as in Section 4.3.

Next, assume that F_v/F_v^+ is unramified or that F_v^+ has residue characteristic $\neq 2$. If L is a Hermitian $\mathcal{O}_{F_v}$-lattice, we thus have

$$\mathrm{Den}(q_v^{2s_0}X^{-1}, L)_n = \varepsilon(L)^{n-m-1}X^{-\mathrm{val}'(L)}\mathrm{Den}(q_v^{2s_0}X, L)_n \tag{5.1.5}$$

$$\mathrm{Den}^*(q_v^{2s_0}X^{-1}, L)_n = \varepsilon(L)^{n-m-1}\mathrm{Den}^*(q_v^{2s_0}X, L)_n \tag{5.1.6}$$

with $\mathrm{val}'(L) := \lfloor \mathrm{val}(L) \rfloor$ as in (4.5.10) (both $\varepsilon(L)$ and $\mathrm{val}(L)$ were defined in [Che24a, Section 2.2]).

In the case where $\chi_v|_{F_v^{+\times}}$ is trivial, these functional equations are essentially [Ike08, Corollary 3.2].

5.2. **Archimedean.** Suppose F_v/F_v^+ is $\mathbb{C}/\mathbb{R}$ (with notation as above). For any $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$ and any standard section Φ_v of $I(s, \chi_v)$, we have

$$W_{T,v}(h, -s, M(s, \chi_v)\Phi_v) = \kappa_T(s, \chi_v, \psi_v)W_{T,v}(h, s, \Phi_v). \tag{5.2.1}$$

This may be deduced, e.g. by combining the non-Archimedean analogue (5.1.1) with the global functional equation (2.2.6).

In the rest of Section 5.2, we require $\chi_v|_{F_v^{+\times}} = \eta_v^n$ for some $n \in \mathbb{Z}$, and let $\psi_v(x) = e^{2\pi i x}$. Recall that we have defined a normalized Archimedean Whittaker function $W_{T,v}^*(h, s)_n^\circ$ (Section 4.2).

Lemma 5.2.1. *For any $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$, we have the functional equation*

$$W_{T,v}^*(h, -s)_n^\circ = \eta_v(\det T)^{n-m-1}W_{T,v}^*(h, s)_n^\circ. \tag{5.2.2}$$

Proof. By (5.2.1), we must have $W_{T,v}^*(h, -s)_n^\circ = \eta_v(\det T)^{n-m-1}f(s)W_{T,v}^*(h, s)_n^\circ$ for some meromorphic factor $f(s)$ which is independent of T. When T is positive definite, we have $f(s) = 1$ (see Section 4.2), so we obtain the claimed functional equation for all $T \in \mathrm{Herm}_m(F_v^+)$ with $\det T \neq 0$. Note that η_v is simply the sign character $\mathrm{sgn}(-)$. $\qquad\square$

Recall that $\Phi_v^{(n)} \in I(s, \chi_v)$ is our notation for a certain scalar weight standard section, as in Section 2.2. For verifying the next lemma, it may be helpful to recall the relation between local epsilon factors $\epsilon_v(-)$ and Weil indices $\gamma_v(-)$ (Section 3.1).

Lemma 5.2.2. *We have*

$$M(s, \chi_v)\Phi_v^{(n)}(s) \tag{5.2.3}$$

$$= \left(\prod_{j=0}^{m-1} \frac{L_v(2s+j-m+1, \eta_v^{n+j})\Gamma(-s-s_0+n-j)}{\epsilon_v(2s+j-m+1, \eta_v^{n+j}, \overline{\psi}_v)L_v(-2s-j+m, \eta_v^{n+j})\Gamma(s-s_0+n-j)} \right)$$

$$\cdot (-1)^{nm}i^{m(m-1)/2}\pi^{2ms}\Phi_v^{(n)}(-s)$$

with $s_0 = (n-m)/2$ as above.

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Proof. A priori, the displayed identity holds up to some meromorphic scale factor. We may compute this scale factor by combining (5.2.1) and Lemma 5.2.1 (take $T = 1_m$). □

Remark 5.2.3. Lemma 5.2.2 should be a reformulation (with alternative proof) of a case of [Shi82, (1.31)] (translating into Shimura's setup via (2.2.13)). Shimura's computation in loc. cit. implies

$$M(s, \chi_v)\Phi_v^{(n)}(s) = \left(\frac{i^{-mn}(2\pi)^{m^2}\pi^{-m(m-1)/2}}{2^{m(m-1)/2+2ms}} \prod_{j=0}^{m-1} \frac{\Gamma(2s-j)}{\Gamma(s-s_0+n-j)\Gamma(s-s_0-j)} \right) \Phi_v^{(n)}(-s).$$

(5.2.4)

Similarly, the functional equation in Lemma 5.2.1 should follow from [Shi82, Theorem 4.2, (4.34.K)] (alternative proof) after some rearranging.

For our later calculations, we prefer to use these results as stated in Lemmas 5.2.1 and 5.2.2.

6. NORMALIZED FOURIER COEFFICIENTS

6.1. **Global normalization.** With notation as in Section 2.2 and Section 2.3, let F/F^+ be a CM extension of number field. For the moment, we allow 2-adic places of F^+ to ramify in F. Write $\mathfrak{d}$ (resp. Δ) for the different ideal (resp. discriminant ideal) of F/F^+. Let $\eta\colon F^{+\times}\backslash \mathbb{A}^\times \to \{\pm 1\}$ be the quadratic character associated with F/F^+.

Assume there exists a nontrivial additive character $\psi\colon F^+\backslash\mathbb{A} \to \mathbb{C}^\times$ such ψ_v is unramified for every non-Archimedean v and $\psi_v(x) = e^{2\pi i x}$ at every Archimedean place. Fix such a ψ. Fix integers m and n with $m \geq 0$, with $s_0 := (n-m)/2$ as above. If any non-Archimedean places of F^+ are ramified in F, we assume n is even. Let $\chi\colon F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ be a character satisfying $\chi|_{\mathbb{A}^\times} = \eta^n$. To simplify, we assume that χ is unramified at every non-Archimedean place (but see also Remark 4.5.1).

Take the standard section

$$\Phi^{(n)\circ} := \left(\bigotimes_{v|\infty} \Phi_v^{(n)} \right) \otimes \left(\bigotimes_{v<\infty} \Phi_v^\circ \right) \in I(s, \chi)$$

(6.1.1)

(scalar weight at Archimedean places and spherical at non-Archimedean places). Form the associated Eisenstein series $E(h, s, \Phi^{(n)\circ})$ and its variants $E(z, s, \Phi^{(n)\circ})_n$ and $\tilde{E}(a, s, \Phi^{(n)\circ})_n$ as in Section 2.2. The Eisenstein series variant $\tilde{E}(a, s, \Phi^{(n)\circ})_n$ does not depend on the choice of χ (Remark 3.2.2).

Define the *global normalizing factor*

$$\Lambda_m(s)_n^\circ := \left(\frac{(2\pi)^{m(m-1)/2}}{(-2\pi i)^{nm}}\pi^{m(-s+s_0)} \right)^{[F^+:\mathbb{Q}]} |N_{F^+/\mathbb{Q}}(\Delta)|^{m(m-1)/4}|N_{F^+/\mathbb{Q}}(\Delta^{\lfloor m/2\rfloor})|^{s+s_0}$$

$$\cdot \left(\prod_{j=0}^{m-1} \Gamma(s-s_0+n-j)^{[F^+:\mathbb{Q}]} \cdot L(2s+m-j, \eta^{j+n}) \right).$$

(6.1.2)

We define the *normalized Eisenstein series* and its variants

$$E^*(h, s)_n^\circ := \Lambda_m(s)_n^\circ E(h, s, \Phi^{(n)\circ})$$

(6.1.3)

$$E^*(z, s)_n^\circ := \Lambda_m(s)_n^\circ E(z, s, \Phi^{(n)\circ})_n \qquad \tilde{E}^*(a, s)_n^\circ := \Lambda_m(s)_n^\circ \tilde{E}(a, s, \Phi^{(n)\circ})_n$$

(6.1.4)

where $h \in U(m, m)(\mathbb{A})$ and $z \in \mathcal{H}_m$ and $a \in \mathrm{GL}_m(\mathbb{A}_F)$. For $T \in \mathrm{Herm}_m(F^+)$, we similarly define

$$E_T^*(h, s)_n^\circ := \Lambda_m(s)_n^\circ E_T(h, s, \Phi^{(n)\circ}) \tag{6.1.5}$$

$$E_T^*(y, s)_n^\circ := \Lambda_m(s)_n^\circ E_T(y, s, \Phi^{(n)\circ})_n \qquad \tilde{E}_T^*(a, s)_n^\circ := \Lambda_m(s)_n^\circ \tilde{E}_T(a, s, \Phi^{(n)\circ})_n \tag{6.1.6}$$

The latter two are *normalized Fourier coefficients*.

Given any $T \in \mathrm{Herm}_m(F^+)$ with $\det T \neq 0$, the local normalizing factors from Sections 4.2 and 4.3 satisfy

$$\Lambda_m(s)_n^\circ := \prod_v \Lambda_{T,v}(s)_n^\circ \tag{6.1.7}$$

where the product (over all places v of F^+) is convergent for $\mathrm{Re}(s) > 0$. For such T, we have factorizations into (normalized) local Whitaker functions

$$E_T^*(h, s)_n^\circ = \prod_v W_{T,v}^*(h_v, s)_n^\circ \qquad \tilde{E}_T^*(a, s)_n^\circ = \prod_v \tilde{W}_{T,v}^*(a_v, s)_n^\circ \tag{6.1.8}$$

where all but finitely many factors are identically equal to 1 (as functions of s) for fixed T, h, and n.

Lemma 6.1.1. *We have*

$$E^*(h, -s)_n^\circ = (-1)^{m(m-1)(n-m-1)[F^+:\mathbb{Q}]/2} E^*(h, s)_n^\circ \tag{6.1.9}$$

Proof. Given $T \in \mathrm{Herm}_m(F^+)$ with $\det T \neq 0$, the local functional equations (Section 5) and the factorization from (6.1.8) imply

$$E_T^*(h, -s)_n^\circ = (-1)^{m(m-1)(n-m-1)[F^+:\mathbb{Q}]/2} E_T^*(h, s)_n^\circ. \tag{6.1.10}$$

The global functional equation (2.2.6) implies that $E^*(h, -s)_n^\circ = f(s) E^*(h, s)_n^\circ$ for some meromorphic function $f(s)$ (temporary notation) independent of T. There exists T with $\det T \neq 0$ and $E_T^*(h, s)_n^\circ$ not identically zero (e.g. $T = 1_m$; this follows from Section 4). So $f(s)$ is identically 1 and (6.1.10) holds for all $T \in \mathrm{Herm}_m(F^+)$. $\square$

6.2. **Singular Fourier coefficients.** Retain notation and assumptions from Section 6.1. In this section, the main result is Corollary 6.2.2 on singular Fourier terms of corank 1.

We use various subscripts to emphasize m-dependence (in the implicit $U(m, m)$). For example, we write $\Phi_{m,v}^\circ$ rather than just Φ_v° for non-Archimedean v (resp. $\Phi_{m,v}^{(n)}$ instead of $\Phi_v^{(n)}$ for Archimedean v), similarly $\Phi_m^{(n)\circ}$ instead of $\Phi^{(n)\circ}$ for the global standard section from Section 6.1, also $M_m(s, \chi)$ instead of $M(s, \chi)$ for the intertwining operator, etc..

Suppose $m \geq 1$ and set $m^\flat = m - 1$. Recall the operators $\mu_{m^\flat}^{m*}(s, \chi)$, $M_m(s, \chi)$, $M_{m^\flat}(s, \chi)$ and $U_{m^\flat}^m(s, \chi)$ as in Section 2.4.

Lemma 6.2.1. *We have*

$$\mu^{m*}_{m^\flat}(s,\chi)\Phi^{(n)\circ}_m(s) = \Phi^{(n)\circ}_{m^\flat}(s+1/2) \tag{6.2.1}$$

$$U^m_{m^\flat}(s,\chi)\Phi^{(n)\circ}_m(s) = (-1)^e \frac{\Lambda_{m^\flat}(s-1/2)^\circ_n\Lambda_m(-s)^\circ_n}{\Lambda_m(s)^\circ_n\Lambda_{m^\flat}(-s+1/2)^\circ_n}\Phi^{(n)\circ}_{m^\flat}(s-1/2) \tag{6.2.2}$$

$$M_m(s,\chi)\Phi^{(n)\circ}_m(s) = |N_{F^+/\mathbb{Q}}(\Delta)|^{-m(m-1)/4}((-1)^{nm}i^{m(m-1)/2}\pi^{-2ms})^{[F^+:\mathbb{Q}]} \tag{6.2.3}$$

$$\cdot\left(\prod_{j=0}^{m-1}\frac{L(2s+j-m+1,\eta^{n+j})}{L(2s+m-j,\eta^{n+j})}\right)$$

$$\cdot\left(\prod_{v|\infty}\prod_{j=0}^{m-1}\frac{L_v(2s+j-m+1,\eta^{n+j}_v)}{\epsilon_v(2s+j-m+1,\eta^{n+j}_v,\overline{\psi}_v)L_v(-2s-j+m,\eta^{n+j}_v)}\right)$$

$$\cdot\left(\prod_{j=0}^{m-1}\frac{\Gamma(-s-s_0+n-j)}{\Gamma(s-s_0+n-j)}\right)^{[F^+:\mathbb{Q}]}$$

$$\cdot\Phi^{(n)\circ}_m(-s),$$

allowing $m = 0$ for in $M_m(s,\chi)$ formula, and where

$$e := (m(m-1)(n-m-1)/2 - m^\flat(m^\flat-1)(n-m^\flat-1)/2)[F^+:\mathbb{Q}]$$

(temporary notation).

Proof. Each identity holds a priori up to a meromorphic scale factor. We may compute this scale factor by evaluating both sides at $1 \in U(m^\flat, m^\flat)$ or $1 \in U(m,m)$ as appropriate.

The identity for $\mu^{m*}_{m^\flat}(s,\chi)$ is then clear. For $M_m(s,\chi)$, the identity follows directly upon combining (5.1.2) and (5.2.3).

Define the temporary notation $\alpha_m(s)_n$ for the meromorphic function (in the lemma statement) satisfying $M_m(s,\chi)\Phi^{(n)\circ}_m(s) = \alpha_m(s)_n\Phi^{(n)\circ}_m(-s)$. By (2.4.6)), proving the claimed identity for $U^m_{m^\flat}(s,\chi)$ is equivalent to showing

$$\frac{\alpha_m(s)_n}{\alpha_{m^\flat}(s-1/2)_n} = (-1)^e\frac{\Lambda_{m^\flat}(s-1/2)^\circ_n\Lambda_m(-s)^\circ_n}{\Lambda_m(s)^\circ_n\Lambda_{m^\flat}(-s+1/2)^\circ_n} \tag{6.2.4}$$

with e as in the lemma statement. This may be computed explicitly as follows. Some rearranging yields

$$\frac{\Lambda_{m^\flat}(s-1/2)^\circ_n\Lambda_m(-s)^\circ_n}{\Lambda_m(s)^\circ_n\Lambda_{m^\flat}(-s+1/2)^\circ_n} = (\pi^{2ms}\pi^{(-2s+1)m^\flat})^{[F^+:\mathbb{Q}]}|N_{F^+/\mathbb{Q}}(\Delta^{\lfloor m/2\rfloor})|^{-2s}|N_{F^+/\mathbb{Q}}(\Delta^{\lfloor m^\flat/2\rfloor})|^{2s-1}$$

$$\cdot\Gamma(s-s_0+n)^{-[F^+:\mathbb{Q}]}\Gamma(-s-s_0+n-m+1)^{[F^+:\mathbb{Q}]}$$

$$\cdot L(2s,\eta^{m+n})L(2s+m,\eta^n)^{-1}L(2s+m-1,\eta^{n+1})^{-1}$$

$$\cdot L(-2s+1,\eta^{m+n+1}).$$

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and

$$\frac{\alpha_m(s)_n}{\alpha_{m^\flat}(s-1/2)_n} = |N_{F^+/\mathbb{Q}}(\Delta)|^{-(m-1)/2}((-1)^n i^{m-1} \pi^{2ms-2m^\flat(s-1/2)})^{[F^+:\mathbb{Q}]}$$

$$\cdot L(2s, \eta^{n+m+1}) \cdot L(2s+m, \eta^n)^{-1} L(2s+m-1, \eta^{n+1})^{-1} L(2s, \eta^{n+m})$$

$$\cdot \left(\prod_{v|\infty} \frac{L_v(2s, \eta_v^{n+m+1})}{\epsilon_v(2s, \eta_v^{n+m+1}, \overline{\psi}_v) L_v(-2s+1, \eta_v^{n+m+1})} \right)$$

$$\cdot \Gamma(-s - s_0 + n - m + 1)^{[F^+:\mathbb{Q}]} \Gamma(s - s_0 + n)^{-[F^+:\mathbb{Q}]}.$$

We then use the global functional equation $\Lambda(s, \eta^{n+m+1}) = \epsilon(s, \eta^{n+m+1})\Lambda(1-s, \eta^{n+m+1})$ (notation as in Section 4.1). Recall the relation between Weil indices and epsilon factors (Section 3.1), the global product formula $\prod_v \gamma_{\overline{\psi}_v}(F_v) = 1$ for Weil indices, and the equality $\gamma_{\psi_v}(\mathbb{C}) = i$. Recall also that we have assumed n even if $\Delta \neq 1$. Combining these facts with some casework (which we omit) on m, n, Δ gives the claim. $\qquad \square$

Corollary 6.2.2. *Consider any* $a = \mathrm{diag}(a^\#, a^\flat) \in \mathrm{GL}_m(\mathbb{A}_F)$ *with* $a^\# \in \mathrm{GL}_1(\mathbb{A}_F)$ *and* $a^\flat \in \mathrm{GL}_{m^\flat}(\mathbb{A}_F)$. *For any* $T \in \mathrm{Herm}_m(F^+)$ *with* $\mathrm{rank}\, T = m-1$ *and* $T = \mathrm{diag}(0, T^\flat)$ *being block diagonal with* $\det T^\flat \neq 0$, *we have*

$$\tilde{E}_T^*(a, s)_n^\circ = |\det a^\#|_F^{s-s_0} \frac{\Lambda_m(s)_n^\circ}{\Lambda_{m^\flat}(s+1/2)_n^\circ} \tilde{E}_{T^\flat}^*(a^\flat, s+1/2)_n^\circ$$

$$+ (-1)^e |\det a^\#|_F^{-s-s_0} \frac{\Lambda_m(-s)_n^\circ}{\Lambda_{m^\flat}(-s+1/2)_n^\circ} \tilde{E}_{T^\flat}^*(a^\flat, s-1/2)_n^\circ$$

with constant e as in Lemma 6.2.1.

Proof. This follows immediately from Lemma 6.2.1, (2.4.3), and the definition of the normalized Fourier coefficients $\tilde{E}_T^*(a, s)_n^\circ$ and $\tilde{E}_{T^\flat}^*(a^\flat, s)$ (Section 6.1). $\qquad \square$

Remark 6.2.3. In the situation of Corollary 6.2.2, the functional equation

$$\tilde{E}_T^*(a, s)_n^\circ = (-1)^{m(m-1)(n-m-1)[F^+:\mathbb{Q}]/2} \tilde{E}_T^*(a, -s)_n^\circ \tag{6.2.5}$$

is a visible consequence of the identity $\tilde{E}_{T^\flat}^*(a^\flat, s)_n^\circ = (-1)^{m^\flat(m^\flat-1)(n-m^\flat-1)[F^+:\mathbb{Q}]/2} \tilde{E}_{T^\flat}^*(a^\flat, -s)_n^\circ$.

Part 2. **Siegel–Weil**

Our main results (arithmetic Siegel–Weil) are in Section 9. We also give some explicit formulas for special values (local Siegel–Weil and geometric Siegel–Weil) in Sections 7 and 8. These special value formulas will be needed as ingredients in the proofs of our arithmetic Siegel–Weil results.

7. Local Siegel–Weil

7.1. **Volume forms.** Given a scheme X which is smooth and equidimensional over a field A, a *volume form* (or *gauge form*) on X will mean a nowhere vanishing (algebraic) differential form of top degree on X. When X is also affine and A is a local field, the set $X(A)$ has the natural structure of an A-analytic manifold (in the sense of [Ser06, Part II, Chapter III]). In this case, a volume form on X defines a Borel measure on $X(A)$ in a standard way (see [Wei82, §2.2]).

We use volume forms to normalize various Haar measures. Let B be a degree 2 étale algebra over a field A of characteristic $\neq 2$, and write $b \mapsto \bar{b}$ for the nontrivial involution on B. Let V be a B/A Hermitian space which is free of rank n, and set $G = U(V)$. Fix a nonnegative integer $m \leq n$, and choose translation invariant volume forms α and β on V^m and Herm_m respectively (viewed as group schemes over A). The forms α and β have degrees $2nm$ and m^2 respectively.

Consider the moment map

$$V^m \xrightarrow{\;\mathcal{T}\;} \mathrm{Herm}_m \tag{7.1.1}$$
$$\underline{x} \longmapsto (\underline{x}, \underline{x}).$$

We assume $n \geq m$, and write $V^m_{\mathrm{reg}} \subseteq V^m$ for the open subscheme where $\det \mathcal{T}$ is invertible. A tangent space calculation shows that $\mathcal{T}$ is smooth when restricted to V^m_{reg}.

Given $T \in \mathrm{Herm}_m(A)$, we write $\Omega_T \subseteq V^m$ for the fiber of the moment map over T. If $\underline{x} \in V^m(A)$ has Gram matrix $T = (\underline{x}, \underline{x})$, then $g \mapsto g^{-1} \cdot \underline{x}$ defines a morphism $\iota_{\underline{x}} \colon G \to \Omega_T$. If $\det T$ is invertible, then a dimension count and tangent space calculation shows that $\iota_{\underline{x}}$ is smooth. If $\det T$ is invertible, if A is a local field, and if $G_{\underline{x}} \subseteq G$ denotes the stabilizer of $\underline{x}$, then the induced map $G_{\underline{x}}(A) \backslash G(A) \to \Omega_T(A)$ is a homeomorphism (surjectivity is from Witt's theorem, and openness is from the submersivity of $G(A) \to \Omega_T(A)$, which in turn comes from smoothness of $\iota_{\underline{x}}$).

Lemma 7.1.1. *There exists an algebraic differential form ν on V^m_{reg} of degree $m(2n - m)$ satisfying the following conditions.*

(1) *We have $\alpha = \mathcal{T}^*(\beta) \wedge \nu$.*

(2) *For the $G \times \mathrm{Res}_{B/A} \mathrm{GL}_m$ action on V^m_{reg} given by $x \mapsto gxh^{-1}$ for $(g, h) \in G \times \mathrm{Res}_{F/F^+} \mathrm{GL}_m$, we have $(g, h)^* \nu = \det({}^t\bar{h}h)^{m-n} \nu$.*

(3) *For each $x \in V^m_{\mathrm{reg}}$, the restriction of $\nu \colon T_x(V^m) \to \mathbb{G}_a$ to $\ker d\mathcal{T}_x$ is nonzero.*

(4) *For any fixed non-degenerate subspace $V^{\flat} \subseteq V$ which is free of rank m, and for $\underline{x} \in V^{\flat m}_{\mathrm{reg}}(A)$, the differential form*

$$\det(\underline{x}, \underline{x})^{m-n} \iota_{\underline{x}}^* \nu \tag{7.1.2}$$

on G is independent of the choice of $\underline{x}$. This form is right G-invariant.

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Proof. The case $m = n$ is stated in [KR14, §10]. The analogue of that case for orthogonal groups is discussed in [KRY06, Lemmas 5.3.1, 5.3.2] (there stated and proved for three dimensional quadratic spaces). The present lemma may be proved by a similar computation.

Part (4) follows from part (2) (where "non-degenerate subspace" means that the restriction of the Hermitian pairing is non-degenerate). In part (3), $x \in V_{\mathrm{reg}}^m$ means $x \in V_{\mathrm{reg}}^m(S)$ for some suppressed A-scheme S, and we similarly abused notation in part (2). In part (3), the symbol $\mathbb{G}_a$ denotes the additive group scheme. $\square$

7.2. Special value formula.

We retain notation from Section 7.1, and specialize to the case where B/A is the extension F_v/F_v^+ where F_v^+ is a local field of characteristic $\neq 2$. If F_v^+ is Archimedean, we assume F_v/F_v^+ is $\mathbb{C}/\mathbb{R}$. We often use subscripts v to emphasize F_v^+ being a local field, e.g. we write $\underline{x}_v$ for elements of $V_{\mathrm{reg}}^m(F_v^+)$.

Fix a nontrivial additive character $\psi_v \colon F_v^+ \to \mathbb{C}^\times$. We write db_v for the self-dual Haar measure on $\mathrm{Herm}_m(F_v^+)$ with respect to the trace pairing $(b, c) \mapsto \psi_v(\mathrm{tr}(bc))$. We also write $d\underline{x}_v$ for the self-dual Haar measure on $V^m(F_v^+)$ with respect to the pairing $\psi_v(\mathrm{tr}_{F_v/F_v^+}(\mathrm{tr}(-, -)))$.

Fix translation-invariant volume forms α and β as in Section 7.1. These determine Haar measures $d_\beta b_v$ and $d_\alpha \underline{x}_v$ on $\mathrm{Herm}_m(F_v^+)$ and $V^m(F_v^+)$ respectively. Define positive real constants $c_v(\alpha, \psi_v)$ and $c_v(\beta, \psi_v)$ such that

$$d_\alpha \underline{x}_v = c_v(\alpha, \psi_v) d\underline{x}_v \qquad d_\beta b_v = c_v(\beta, \psi_v) db_v. \tag{7.2.1}$$

Suppose $T \in \mathrm{Herm}_m(F_v^+)$ is a matrix with $\det T \neq 0$. For the rest of Section 7.2, fix a differential form ν as in Lemma 7.1.1. The restriction of $\det(T)^{m-n}\nu$ to Ω_T is a G-invariant volume form on Ω_T, and we write $d_{T,\nu}\underline{x}_v$ for the resulting measure on $\Omega_T(F_v^+)$.

It is known that there exists a constant $c_{T,v}$ (depending on T, the measure $d_{T,\nu}\underline{x}_v$, and the character ψ_v) such that

$$W_{T,v}(s_0, \Phi_{\varphi_v}) = c_{T,v} \int_{\Omega_T(F_v^+)} \varphi_v(\underline{x}_v) \, d_{T,\nu}\underline{x}_v \tag{7.2.2}$$

holds for any Schwartz function $\varphi_v \in \mathcal{S}(V^m(F_v^+))$ (see [Ich04, Lemma 5.1, Lemma 5.2]). Here we set $s_0 := (n - m)/2$ as usual, and Φ_{φ_v} is the Siegel–Weil section associated with φ_v (Section 3.2). If $\Omega_T(F_v^+) = \emptyset$, we thus have $W_{T,v}(s_0, \Phi_{\varphi_v}) = 0$ for all φ_v.

We may compute the constant $c_{T,v}$ by evaluating (7.2.2) on any nonzero nonnegative Schwartz function φ_v. We may take φ_v to have support which is compact and contained in $V_{\mathrm{reg}}^m(F_v^+)$. The relation $\alpha = \mathcal{T}^*(\beta) \wedge \nu$ and an "integrate along the fibers of $\mathcal{T}$" computation (similar to the proof of [KRY06, Proposition 5.3.3]) gives

$$c_{T,v} = \frac{\gamma_{\psi_v}(V)^{-m} c_v(\beta, \psi_v)}{c_v(\alpha, \psi_v)} |\det T|_{F_v^+}^{n-m}. \tag{7.2.3}$$

Here $\gamma_{\psi_v}(V)$ is the Weil index, as appearing in the Weil representation (Section 3.2).

Lemma 7.2.1 (Local Siegel–Weil). *Let V be a F_v/F_v^+ Hermitian space of rank n, and let $\psi_v \colon F_v^+ \to \mathbb{C}^\times$ be a nontrivial additive character. Fix a non-degenerate subspace $V^\flat \subseteq V$ which is free of rank m, and fix a Haar measure on $U(V^{\flat\perp})(F_v^+)$.*

There exists a unique Haar measure dg_v on $G(F_v^+)$ such that, for any basis $\underline{x}_v \in V^{\flat m}$ of $V^\flat$ and any Schwartz function $\varphi_v \in \mathcal{S}(V^m(F_v^+))$, we have

$$W_{T,v}(s_0, \Phi_{\varphi_v}) = \gamma_{\psi_v}(V)^{-m} |\det T|_{F_v^+}^{n-m} \int_{G_{\underline{x}_v}(F_v^+) \backslash G(F_v^+)} \varphi_v(g_v^{-1} \underline{x}_v) \, dg_v \tag{7.2.4}$$

for the corresponding quotient measure, where $T = (\underline{x}_v, \underline{x}_v)$ is the Gram matrix of $\underline{x}_v$ (and where the Haar measure on $G_{\underline{x}_v}(F_v^+)$ is induced by the canonical identification $G_{\underline{x}_v} \cong U(V^{\flat \perp})$).

Proof. Select any basis $\underline{x}_v$ of $V^\flat$. Set $\omega_1 = \det(\underline{x}_v, \underline{x}_v)^{m-n} \iota_{\underline{x}}^* \nu$ (temporary notation). We know ω_1 does not depend on the choice of $\underline{x}_v$, by Lemma 7.1.1(4). Let ω_2 be a right G-invariant differential form of degree $(n-m)^2$ on G such what $\omega_1 \wedge \omega_2$ is a nowhere vanishing differential form of top degree n^2 (also right G-invariant). The volume form $\omega_1 \wedge \omega_2$ on G defines a Haar measure on $G(F_v^+)$. The restriction $\omega_2|_{G_{\underline{x}}}$ is a volume form on $G_{\underline{x}}$ (by smoothness of $\iota_{\underline{x}}$), and defines a Haar measure on $G_{\underline{x}_v}(F_v^+)$. The resulting quotient measure on $G_{\underline{x}}(F_v^+) \backslash G(F_v^+) \cong \Omega_T(F_v^+)$ is precisely the measure for the volume form $(\det T)^{m-n} \nu|_{\Omega_T}$ on Ω_T.

The lemma then follows from (7.2.2) and the constant calculated in (7.2.3). $\qquad\square$

Remark 7.2.2. Consider the situation of Lemma 7.2.1, and suppose $V^{\flat\prime} \subseteq V$ is a subspace which is isomorphic to $V^\flat$ as a Hermitian space. Suppose $f_v \in U(V)(F_v^+)$ satisfies $f_v(V^\flat) = V^{\flat\prime}$, and equip $U(V^{\flat\prime\perp})(F_v^+)$ with the Haar measure induced from $U(V^{\flat\perp})(F_v^+)$ via f_v. If dg_v and dg_v' are the induced Haar measures on $G(F_v^+)$ corresponding to $V^\flat$ and $V^{\flat\prime}$ respectively (Lemma 7.2.1), a change of variables shows $dg_v = dg_v'$.

7.3. **Explicit Haar measures.** For our application to uniformization of special cycles (Section 7.4), we need to explicitly compute the Haar measures from Lemma 7.2.1 in a few cases. The main result of this subsection is Lemma 7.3.5, and the other lemmas are auxiliary.

We retain notation from Section 7.2. In addition, we assume that F_v^+ is non-Archimedean and that ψ_v is unramified. Let ϖ_0 be a uniformizer of F_v^+. If F_v/F_v^+ is ramified, let ϖ be a uniformizer of F_v. Throughout Section 7.3, we assume that F_v/F_v^+ is unramified if F_v^+ has residue characteristic 2.

Let M_2° be the rank 2 self-dual lattice described in Section 3.2, and write $U(M_2^\circ)$ for the group of (unitary) automorphisms of M_2°. Let q_v be the residue cardinality of F_v^+.

The next lemma should be compared with Witt's theorem for lattices with quadratic forms, as in [Mor79].

Lemma 7.3.1. *For any given $c \in \mathcal{O}_{F_v^+}^\times$, the group $U(M_2^\circ)$ acts transitively on the set*

$$\{x \in M_2^\circ : (x,x) = c\}. \tag{7.3.1}$$

If F_v/F_v^+ is inert, the same holds for any $c \in \varpi_0 \mathcal{O}_{F_v^+}^\times$.

Proof. Given $y \in M_2^\circ$, we write $\langle y \rangle \subseteq M_2^\circ$ for the submodule generated by y. If F_v/F_v^+ is ramified, we view ϖ as a generator of the different ideal $\mathfrak{d}$, and we otherwise view 1 as a generator of $\mathfrak{d}$. Choose a basis e_1, e_2 of M_2° with Gram matrix given by (3.2.2). In this basis, we also consider the

elements

$$w' = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \qquad m(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} \qquad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \epsilon = \begin{cases} -1 & \text{if } F_v/F_v^+ \text{ is ramified} \\ 1 & \text{else} \end{cases}$$

$$(7.3.2)$$

of $U(M_2^\circ)$ (acting on column vectors), where $a \in \mathcal{O}_{F_v}^\times$ and $b \in \mathcal{O}_{F_v}$ satisfies $\bar{b} = -\epsilon b$.

Case 1. Assume F_v/F_v^+ is unramified and $c \in \mathcal{O}_{F_v^+}^\times$. Given any $x \in M_2^\circ$ with $(x, x) = c$, there exists an orthogonal direct sum decomposition $M_2^\circ = \langle x \rangle \oplus \langle y \rangle$ for some $y \in M_2^\circ$ with $(y, y) = 1$ (by self-dualness). Via this decomposition, the lemma is clear in this case.

Case 2. Assume F_v/F_v^+ is ramified and $c \in \mathcal{O}_{F_v^+}^\times$. Suppose $x = a_1 e_1 + a_2 e_2 \in M_2^\circ$ with $(x, x) = c$. Without loss of generality, we may assume $a_2 \in \mathcal{O}_{F_v}^\times$ (replace x with $w'x$ if necessary), and we may further assume $a_2 = 1$ (replace x with $m(\bar{a}_2)x$). We then have $\text{tr}_{F_v/F_v^+}(\varpi^{-1}a_1) = -c$. Given another $x' = a_1'e_1 + e_2 \in M_2^\circ$ with $(x', x') = c$, we take $b = a_1' - a_1$ and have $n(b)x = x'$.

Case 3. Assume F_v/F_v^+ is inert and $c \in \varpi_0\mathcal{O}_{F_v^+}^\times$. Suppose $x = a_1 e_1 + a_2 e_2 \in M_2^\circ$ with $(x, x) = c$. Without loss of generality, we may assume $a_2 = 1$ and $\text{tr}_{F_v/F_v^+}(a_1) = c$ (argue as in Case 2). Given another $x' = a_1'e_1 + e_2 \in M_2^\circ$ with $(x', x') = c$, we take $b = a' - a$ and have $n(b)x = x'$. $\square$

Lemma 7.3.2. *Let L be a self-dual hermitian $\mathcal{O}_{F_v}$-lattice of rank n. Any isomorphism between self-dual sublattices of L extends to a (unitary) automorphism of L. The same holds for almost self-dual lattices of rank $n - 1$.*

Proof. Any self-dual lattice $L^\flat \subseteq L$ admits a (unique) orthogonal direct sum decomposition $L = L^\flat \oplus L^\#$ where $L^\#$ is also self-dual. This immediately implies the claim for self-dual sublattices of L, as self-dual lattices are unique up to isomorphism (for a fixed rank).

Next, assume that F_v/F_v^+ is nonsplit and that $L^\flat \subseteq L$ is almost self-dual of rank $n - 1$. There is an orthogonal direct sum decomposition $L^\flat = L^{\flat\flat} \oplus L^{\flat\#}$, where $L^{\flat\flat}$ is self-dual of rank $n - 2$ and $L^{\flat\#}$ is almost self-dual of rank 1. We also have an orthogonal direct sum decomposition $L = L^{\flat\flat} \oplus L^\#$ where $L^\#$ is self-dual of rank 2.

Suppose $L^{\flat\prime} \subseteq L$ is another almost self-dual lattice of rank $n - 1$, equipped with an isomorphism $L^\flat \to L^{\flat\prime}$. Applying the result just proved above (in the case of rank $n - 2$ self-dual sublattices), we may assume there is an orthogonal decomposition $L^{\flat\prime} = L^{\flat\flat} \oplus L^{\flat\prime\#}$ where $L^{\flat\#} \cong L^{\flat\prime\#}$. We thus reduce to the case $n = 2$ (the claim for $L^\#$), which was proved in Lemma 7.3.1. $\square$

Lemma 7.3.3. *Assume F_v/F_v^+ is nonsplit, and let V be a F_v/F_v^+ Hermitian space of rank n, and assume that V contains a full-rank self-dual lattice. Suppose $L^\flat \subseteq V$ is a non-degenerate lattice of rank $n - 1$ satisfying $L^\flat \subseteq L^{\flat*}$ and $t(L^\flat) \leq 1$. Then $L^\flat$ is contained in a self-dual lattice of rank n.*

Proof. Recall that $t(L^\flat) := \dim_k((L^{\flat*}/L^\flat) \otimes k)$ where k is the residue field of $\mathcal{O}_{F_v}$.

Let $L^\flat \subseteq V$ be as in the lemma statement. The existence of such $L^\flat$ implies $n \geq 2$. There exists an orthogonal decomposition $L^\flat = L^{\flat\flat} \oplus L^{\flat\#}$ where $L^{\flat\flat}$ is self-dual of rank $n - 2$. Replacing V with the orthogonal complement of $L^{\flat\flat}$, we reduce immediately to the case $n = 2$, which we now assume.

Let ϖ be a uniformizer for F_v (take $\varpi = \varpi_0$ if F_v/F_v^+ is inert). We may take $V = M_2^\circ \otimes F_v$, where M_2° is as in Lemma 7.3.1. We also choose a standard basis e_1, e_2 for M_2° and consider the

elements $w', m(a), n(b) \in U(V)$ as in the proof of that lemma (now allowing $a \in F_v^\times$ and allowing $b \in F_v$ satisfying $\bar{b} = -\epsilon b$).

The rank one lattice $L^\flat$ is generated by an element $x = a_1 e_1 + a_2 e_2$ for some $a_1, a_2 \in F_v$ (such that (x, x) is nonzero and lies in $\mathcal{O}_{F_v^+}$). It is enough to check chat the orbit $U(V) \cdot x$ intersects M_2°. Acting on x by $m(a) \in U(V)$ for suitable a, we see that it is enough to check the case where $a_2 = 1$ and $a_1 \in F_v^\times$.

If F_v/F_v^+ is inert, there exists $a' \in \mathcal{O}_{F_v}$ such that $\mathrm{tr}_{F_v/F_v^+}(a') = (x, x)$ since $\mathcal{O}_{F_v}$ is self-dual with respect to the trace pairing. If F_v/F_v^+ is ramified, there exists $a' \in \mathcal{O}_{F_v}$ such that $\mathrm{tr}_{F_v/F_v^+}(-\varpi^{-1}a') = (x, x)$ since $\mathcal{O}_{F_v}$ and $\varpi^{-1}\mathcal{O}_{F_v}$ are dual. In either case, we can take $b = a' - a_1$, and have $n(b)x \in M_2^\circ$. $\qquad\square$

Lemma 7.3.4. *In the situations of Lemma 7.3.1, choose $x \in M_2^\circ$ with $(x, x) = c$ and form the orthogonal complement lattice $x^\perp \subseteq M_2^\circ$ (of rank one). We view both $U(M_2^\circ)$ and $U(x^\perp)$ as subgroups of $U(M_2^\circ \otimes F_v)$.*

Viewing $U(x^\perp)$ as the norm-one subgroup of $\mathcal{O}_{F_v}^\times$, we have

$$U(M_2^\circ) \cap U(x^\perp) = \{\alpha \in \mathcal{O}_{F_v}^\times : \alpha\bar{\alpha} = 1, \text{ and } \alpha \equiv 1 \pmod{c\eth\mathcal{O}_{F_v}}\} \subseteq U(x^\perp). \tag{7.3.3}$$

The subgroup $U(M_2^\circ) \cap U(x^\perp) \subseteq U(x^\perp)$ has index

$$\begin{cases} 1 & \text{if } c \in \mathcal{O}_{F_v^+}^\times \text{ and } F_v/F_v^+ \text{ is unramified} \\ 2 & \text{if } c \in \mathcal{O}_{F_v^+}^\times \text{ and } F_v/F_v^+ \text{ is ramified} \\ q_v + 1 & \text{if } c \in \varpi_0 \mathcal{O}_{F_v^+}^\times \text{ and } F_v/F_v^+ \text{ is inert.} \end{cases} \tag{7.3.4}$$

Proof. We express elements of $U(M_2^\circ \otimes F_v)$ in a standard basis e_1, e_2 of M_2°, as in the proof of Lemma 7.3.1.

Case 1. Assume F_v/F_v^+ is unramified and $c \in \mathcal{O}_{F_v^+}^\times$. We then have $U(M_2^\circ) \cap U(x^\perp) = U(x^\perp)$, as follows immediately from an orthogonal direct sum decomposition $M_2^\circ = \langle x \rangle \oplus \langle y \rangle$ as in the proof of Lemma 7.3.1 Case 1.

Case 2. Assume F_v/F_v^+ is ramified and $c \in \mathcal{O}_{F_v^+}^\times$. By the proof of Lemma 7.3.1 Case 2, we may assume (after conjugating $U(M_2^\circ \otimes F_v)$ by an appropriate element of $U(M_2^\circ)$) that $x = a_1 e_1 + e_2$ for some $a_1 \in \mathcal{O}_{F_v}$, where $a_1 - \bar{a}_1 = -\varpi c$. Then $\bar{a}_1 e_1 + e_2$ is orthogonal to x. For every $\alpha \in \mathcal{O}_{F_v}^\times$, the matrix

$$\begin{pmatrix} a_1 & \bar{a}_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a_1 & \bar{a}_1 \\ 1 & 1 \end{pmatrix}^{-1} = (-\varpi c)^{-1} \begin{pmatrix} a_1 - \bar{a}_1\alpha & (-1+\alpha)a_1\bar{a}_1 \\ 1 - \alpha & -\bar{a}_1 + a_1\alpha \end{pmatrix} \tag{7.3.5}$$

lies in $U(M_2^\circ)$ if and only if $\alpha \equiv 1 \pmod{\varpi \mathcal{O}_{F_v}}$. The claim about index follows from surjectivity of the reduction modulo ϖ map

$$\{\alpha \in \mathcal{O}_{F_v}^\times : \alpha\bar{\alpha} = 1\} \to \{\alpha \in \mathbb{F}_{q_v}^\times : \alpha^2 = 1\} \tag{7.3.6}$$

(surjectivity is by smoothness of the corresponding unitary group over $\mathrm{Spec}\,\mathcal{O}_{F_v^+}$).

Case 3. Assume F_v/F_v^+ is inert and $c \in \varpi_0 \mathcal{O}_{F_v^+}^\times$. By the proof of Lemma 7.3.1 Case 3, we may assume (after conjugating $U(M_2^\circ \otimes F_v)$ by an appropriate element of $U(M_2^\circ)$) that $x = a_1 e_1 + e_2$

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for some $a_1 \in \mathcal{O}_{F_v}$, where $a_1 + \bar{a}_1 = c$. Then $-\bar{a}_1 e_1 + e_2$ is orthogonal to x. For every $\alpha \in \mathcal{O}_{F_v}^\times$, the matrix

$$\begin{pmatrix} a_1 & -\bar{a}_1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a_1 & -\bar{a}_1 \\ 1 & 1 \end{pmatrix}^{-1} = c^{-1} \begin{pmatrix} a_1 + \bar{a}_1 \alpha & (1-\alpha)a_1 \bar{a}_1 \\ 1 - \alpha & \bar{a}_1 + a_1 \alpha \end{pmatrix} \tag{7.3.7}$$

lies in $U(M_2^\circ)$ if and only if $\alpha \equiv 1 \pmod{\varpi_0 \mathcal{O}_{F_v}}$. The claim about index follows from surjectivity of the reduction modulo ϖ_0 map

$$\{\alpha \in \mathcal{O}_{F_v}^\times : \alpha\bar{\alpha} = 1\} \to \{\alpha \in \mathbb{F}_{q_v^2}^\times : \alpha\bar{\alpha} = 1\} \tag{7.3.8}$$

(surjectivity is by smoothness of the corresponding unitary group over $\operatorname{Spec} \mathcal{O}_{F_v^+}$). $\qquad\square$

We take a particular choice of Schwartz function φ_v in the next lemma, which immediately determines the Haar measure for other choices of φ_v in Lemma 7.2.1. If mn is odd and F_v/F_v^+ is inert with F_v^+ of residue characteristic 2, we also require $F_v^+ = \mathbb{Q}_2$ (because of Lemma 3.2.1).

Lemma 7.3.5. *Take $m = n - 1$ or $m = n$ and $s_0 := (n - m)/2$. Assume the rank n Hermitian space V contains a full-rank self-dual lattice L of full rank. Let $K_v \subseteq G = U(V)$ be the stabilizer of such a lattice L.*

Consider any $\underline{x}_v \in V^m(F_v^+)$ with nonsingular Gram matrix $T = (\underline{x}_v, \underline{x}_v) \in \operatorname{Herm}_m(F_v^+)$. Let $\mathbf{1}_L$ be the characteristic function of L, and set $\varphi_v = \mathbf{1}_L^{\otimes m} \in \mathcal{S}(V^m(F_v^+))$.

Give $G(F_v^+)$ the Haar measure which assigns volume 1 to K_v. Give $G_{\underline{x}_v}(F_v^+)$ the Haar measure which assigns volume 1 to the (unique) maximal open compact subgroup. We have

$$W_{T,v}^*(s_0)_n^\circ = \frac{1}{e} \int_{G_{\underline{x}_v}(F_v^+) \backslash G(F_v^+)} \varphi_v(g_v^{-1} \underline{x}_v) \, dg_v \qquad e := \begin{cases} 2 & \text{if } F_v/F_v^+ \text{ is ramified and } m = n - 1 \\ 1 & \text{else} \end{cases}$$
$$\tag{7.3.9}$$

with respect to the associated quotient measure.

Proof. Recall that $W_{T,v}^*(s)_n^\circ$ is our notation for a certain normalized spherical Whittaker function (Section 4.3), which is a rescaled version of $W_{T,v}(s, \Phi_{\varphi_v})$.

In the lemma statement, the stabilizer in $G(F_v^+)$ of any full-rank self-dual lattice in V has volume 1 (because any such stabilizer is conjugate to K_v). To verify (7.3.9), we can (and will) replace L by any full-rank self-dual lattice in V (by Lemma 7.2.1 again).

Let $V^\flat \subseteq V$ be the rank m subspace spanned by $\underline{x}_v$. Then $V^\flat$ is free of rank m. By Lemma 7.2.1, it is enough to show (7.3.9) holds for one choice of basis $\underline{x}_v$ for $V^\flat$. We choose $\underline{x}_v$ to be a basis for a full-rank lattice $L^\flat \subseteq V^\flat$ which is

$$\begin{cases} \text{self-dual} & \text{if } V^\flat \text{ contains a full-rank self-dual lattice} \\ \text{almost self-dual} & \text{else.} \end{cases} \tag{7.3.10}$$

Note that $V^\flat$ always contains a full-rank self-dual lattice if F_v/F_v^+ is split.

Case 1. Assume $L^\flat$ is self-dual. There exists a rank $n - m$ self-dual lattice $L^\# \subseteq V$ which is orthogonal to $L^\flat$. Form the rank n self-dual lattice $L = L^\flat \oplus L^\#$. Any isomorphism between self-dual sublattices of L lifts to an element of $K_v = U(L)$ (Lemma 7.3.2). This implies that $g_v \mapsto \varphi_v(g_v^{-1} \underline{x}_v)$ is the characteristic function of $G_{\underline{x}_v}(F_v^+) \backslash (G_{\underline{x}_v}(F_v^+) K_v)$.

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We know that $K_v \cap G_{\underline{x}_v}(F_v^+)$ is the unique maximal open compact subgroup in $G_{\underline{x}_v}(F_v^+)$ (i.e. $U(L^\#)$). We compute

$$\int_{G_{\underline{x}_v}(F_v^+)\backslash G(F_v^+)} \varphi_v(g_v^{-1}\underline{x}_v)\, dg_v = \mathrm{vol}(G_{\underline{x}_v}(F_v^+)\backslash(G_{\underline{x}_v}(F_v^+)K_v)) = \frac{\mathrm{vol}(K_v)}{\mathrm{vol}(K_v \cap G_{\underline{x}_v}(F_v^+))} = 1.$$
(7.3.11)

Since $T = (\underline{x}_v, \underline{x}_v)$ and $\underline{x}_v$ is a basis for the self-dual lattice $L^\flat$, we also know $W_{T,v}^*(s_0)_n = 1$ (see (4.5.7); note that $V^\flat$ containing self-dual lattice means that F_v/F_v^+ is unramified if m is odd).

Case 2 Assume that $L^\flat$ is almost self-dual and that F_v/F_v^+ is ramified. Then $n \geq 2$ and $m = n - 1$. There is an orthogonal direct sum decomposition $L^\flat = L^{\flat\flat} \oplus L^{\flat\#}$, where $L^{\flat\flat}$ is self-dual of rank $m - 1$ and $L^{\flat\#}$ is almost self-dual of rank 1. There exists a rank 2 self-dual lattice $L^\# \subseteq V$ which is orthogonal to $L^{\flat\flat}$. We can assume $L^{\flat\#} \subseteq L^\#$ (Lemma 7.3.3). Form the rank n self-dual lattice $L = L^{\flat\flat} \oplus L^\#$. Any isomorphism between rank $n - 1$ almost self-dual sublattices in L lifts to an element of $K_v = U(L)$ (Lemma 7.3.2). This implies that $g_v \mapsto \varphi_v(g_v^{-1}\underline{x}_v)$ is the characteristic function of $G_{\underline{x}_v}(F_v^+)\backslash(G_{\underline{x}_v}(F_v^+)K_v)$.

We know that $K_v \cap G_{\underline{x}_v}(F_v^+) = U(L^\#) \cap G_{\underline{x}_v}(F_v^+)$ has index 2 inside the unique maximal open compact subgroup of $G_{\underline{x}_v}(F_v^+)$ (reduces immediately to the case $n = 2$, which is Lemma 7.3.4). We compute

$$\int_{G_{\underline{x}_v}(F_v^+)\backslash G(F_v^+)} \varphi_v(g_v^{-1}\underline{x}_v)\, dg_v = \mathrm{vol}(G_{\underline{x}_v}(F_v^+)\backslash(G_{\underline{x}_v}(F_v^+)K_v)) = \frac{\mathrm{vol}(K_v)}{\mathrm{vol}(K_v \cap G_{\underline{x}_v}(F_v^+))} = 2.$$
(7.3.12)

Since $T = (\underline{x}_v, \underline{x}_v)$ and since $\underline{x}_v$ is a basis for the almost self-dual lattice $L^\flat$, we also know $W_{T,v}^*(s_0)_n^\circ = 1$ (4.5.7).

Case 3 Assume that $L^\flat$ is almost self-dual and that F_v/F_v^+ is inert. This implies $n \geq 2$ and $m = n - 1$. Arguing as in Case 2 (use the same notation; the first paragraph applies verbatim), again apply Lemma 7.3.2 and Lemma 7.3.4 to compute

$$\int_{G_{\underline{x}_v}(F_v^+)\backslash G(F_v^+)} \varphi_v(g_v^{-1}\underline{x}_v)\, dg_v = \mathrm{vol}(G_{\underline{x}_v}(F_v^+)\backslash(G_{\underline{x}_v}(F_v^+)K_v)) = \frac{\mathrm{vol}(K_v)}{\mathrm{vol}(K_v \cap G_{\underline{x}_v}(F_v^+))} = q_v + 1.$$
(7.3.13)

When $n = 2$, we have $\mathrm{Den}^*(X, L^\flat)_n = q_v X^{-1/2} + X^{1/2}$ (follows from the relevant Cho–Yamauchi type formula; see [LZ22, Example 3.5.2] [FYZ24, Theorem 2.2]). The "cancellation" property for local densities and self-dual lattices (4.5.11) implies $\mathrm{Den}^*(X, L^\flat)_n = q_v X^{-1/2} + X^{1/2}$ for $n \geq 2$. We thus have $W_{T,v}^*(s_0)_n^\circ = \mathrm{Den}^*(1, L^\flat)_n = q_v + 1$. $\qquad\square$

7.4. Uniformization degrees for special cycles.
The purpose of this section is to express the groupoid cardinality of (7.4.4) in terms of special values of local Whittaker functions, with explicit constants (Lemma 7.4.1). This groupoid has already appeared as a "uniformization degree" for special cycles (see [Che24c, (4.5.12)], also [Che24c, Sections 4.8 and 4.9] and [Che24c, Section 5.4]). This calculation will be needed for our main arithmetic Siegel–Weil results (Section 9.1).

Let F/F^+ be a CM extension of number fields, with respective adèle rings $\mathbb{A}_F$ and $\mathbb{A}$ and finite adèle rings $\mathbb{A}_{F,f}$ and $\mathbb{A}_f$, etc.. As in Sections 2 to 6, we write v for places of F^+ with completions F_v^+, and set $F_v := F \otimes_{F^+} F_v^+$.

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Let $T \in \mathrm{Herm}_m(F^+)$ be a Hermitian matrix (with F-coefficients) for any integer $m \geq 0$. Set $m^\flat := \mathrm{rank}(T)$. For each place v, select any $a_v \in \mathrm{GL}_m(F_v)$ such that ${}^t\overline{a}_v^{-1} T a_v^{-1} = \mathrm{diag}(0, T_v^\flat)$ for some $T_v^\flat \in \mathrm{Herm}_{m^\flat}(F_v^+)$ with $\det T_v^\flat \neq 0$. For each v, choose any decomposition (Iwasawa decomposition)

$$a_v = \begin{pmatrix} 1_{m-m^\flat} & * \\ 0 & 1_{m^\flat} \end{pmatrix} \begin{pmatrix} a_v^\# & 0 \\ 0 & a_v^\flat \end{pmatrix} k_v \qquad k_v \in \begin{cases} \mathrm{GL}_m(\mathcal{O}_{F_v}) & \text{if } v \text{ is non-Archimedean} \\ U(m) & \text{if } v \text{ is Archimedean,} \end{cases} \tag{7.4.1}$$

where $a_v^\# \in \mathrm{GL}_{m-m^\flat}(F_v)$, $a_v^\flat \in \mathrm{GL}_{m^\flat}(F_v)$, and $U(m) \subseteq \mathrm{GL}_m(\mathbb{C})$ is the unitary group for the standard diagonal positive definite Hermitian pairing.

Let L be a non-degenerate Hermitian $\mathcal{O}_F$-lattice of any rank n, set $V := L \otimes_{\mathcal{O}_F} F$, and let $G = U(V)$ be the associated unitary group. Set $s_0^\flat := (n - m^\flat)/2$. For any place v of F_v^+, we set $V_v := V \otimes_{F^+} F_v^+$. Let $K_{L,f} = \prod K_{L,v} \subseteq U(V)(\mathbb{A}_f)$ be the adèlic stabilizer of L (i.e. $K_{L,v}$ is the stabilizer of $L_v := L \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_{F_v^+}$ for every place $v < \infty$ of F_v^+). Fix a place v_0 of F_v^+ (Archimedean or non-Archimedean). Assume V_v is positive definite for every Archimedean $v \neq v_0$.

Given $\underline{x}_f^{v_0} \in (V \otimes_{F^+} \mathbb{A}_f^{v_0})^m$, we define the "away from v_0 special cycle" (compare Sections [Che24c, Section 4.2] and [Che24c, Section 5.1])

$$\mathcal{Z}(\underline{x}_f^{v_0}) := \{g_f \in G(\mathbb{A}_f^{v_0})/K_{L,f}^{v_0} : g_{f,v}^{-1}\underline{x}_v \in L_v \text{ for all non-Archimedean } v \neq v_0\} \tag{7.4.2}$$

where $\underline{x}_v \in V_v^m$ is the v-component of $\underline{x}_f^{v_0}$.

Fix a nontrivial additive character ψ_v for each place v. Assume ψ_v is unramified if $v < \infty$, and assume $\psi_v(x) = e^{2\pi i x}$ if $F_v^+ = \mathbb{R}$. For every non-Archimedean place $v \neq v_0$, set $\varphi_v := \mathbf{1}_{L_v}^m$ (characteristic function of $L_v^m \subseteq V_v^m$) and set

$$\varphi_f^{v_0} = \otimes_{\substack{v < \infty \\ v \neq v_0}} \varphi_v \in \mathcal{S}(V(\mathbb{A}_f^{v_0})^m). \tag{7.4.3}$$

Similarly set $\varphi_v^\flat := \mathbf{1}_{L_v}^{m^\flat} \in \mathcal{S}(V(F_v^+)^{m^\flat})$ for such v.

For every place v of F^+, let $\eta_v : F_v^{+\times} \to \{\pm 1\}$ be the quadratic character associated to F_v/F_v^+. Let $\chi_v : F_v^\times \to \mathbb{C}^\times$ be any character satisfying $\chi_v|_{F_v^{+\times}} = \eta_v^n$. Form the associated Siegel–Weil standard section $\Phi_{\varphi_v} \in I(s, \chi_v)$ (Section 3.2) for every place $v < \infty$ with $v \neq v_0$. To simplify slightly, we assume that 2-adic places of F^+ are unramified in F for the rest of Section 7.4.

For $v < \infty$ with $v \neq v_0$, the local Whittaker function variant $\tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s, \Phi_{\varphi_v})_n$ does not depend on the choice of a_v or $a_v^\flat$. Indeed, the $\mathrm{GL}_m(\mathcal{O}_{F_v})$-equivalence class of the Hermitian matrix ${}^t\overline{a}_v^\flat T_v^\flat a_v^\flat$ does not depend on the choice of a_v (follows from the invariance properties in (4.3.4)). For $v \mid \infty$ with $v \neq v_0$, the local Whittaker function variant $\tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s)_n^\circ$ similarly does not depend on the choice of a_v or $a_v^\flat$, as the $U(m)$-equivalence class of ${}^t\overline{a}_v^\flat T_v^\flat a_v^\flat$ is well-defined (then apply (4.2.4)).

Given any tuple $\underline{x} \in V^m$ which spans a non-degenerate Hermitian space, we write $G_{\underline{x}} \subseteq G$ for the stabilizer of $\underline{x}$ (i.e. the unitary group of the orthogonal complement $\mathrm{span}(\underline{x})^\perp \subseteq V$). We write $\underline{x}_f^{v_0}$ for the image of $\underline{x}$ in $(V \otimes_{F^+} \mathbb{A}_f^{v_0})^m$.

Suppose there exists $\underline{x} \in V^m$ with Gram matrix $(\underline{x}, \underline{x})$. Fix such an $\underline{x}$, and assume $\mathrm{span}(\underline{x})^\perp$ is positive definite at every Archimedean place. Let $K_{\underline{x},v_0} \subseteq G_{\underline{x}}(F_{v_0}^+)$ be any open compact subgroup, and assume $K_{\underline{x},v_0}(F_{v_0}^+) = G_{\underline{x}}(F_{v_0}^+)$ if v_0 is Archimedean.

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We are mostly interested in applying Lemma 7.4.1 below when $m^\flat \geq n - 1$ and L_v is self-dual for all $v < \infty$ with $v \neq v_0$. The result and proof is simpler in that case, and the lemma may not be optimal otherwise.

Lemma 7.4.1. *Consider the groupoid quotient*

$$\left[G_{\underline{x}}(F^+) \backslash \left(G_{\underline{x}}(F_{v_0}^+)/K_{\underline{x},v_0} \times \mathcal{Z}(\underline{x}_f^{v_0}) \right) \right]. \tag{7.4.4}$$

The displayed groupoid has finite automorphism groups and finitely many isomorphism classes. Its groupoid cardinality is

$$C \cdot \prod_{\substack{v|\infty \\ v\neq v_0}} \tilde{W}^*_{T_v^\flat, v}(a_v^\flat, s_0^\flat)_n^\circ \prod_{\substack{v<\infty \\ v\neq v_0}} \tilde{W}^*_{T_v^\flat, v}(a_v^\flat, s_0^\flat, \Phi_{\varphi_v^\flat})_n. \tag{7.4.5}$$

for some volume constant $C \in \mathbb{Q}_{>0}$ which we describe in the following three situations.

(1) *Suppose v_0 is Archimedean. Assume the local characters $(\psi_v)_v$ and $(\chi_v)_v$ arise from global characters $\psi \colon F^+\backslash \mathbb{A} \to \mathbb{C}^\times$ and $F^\times\backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. The constant C may depend on V, n, $m^\flat$, F, and the isomorphism classes of the local Hermitian lattices $\{L_v\}_{v<\infty}$. The constant C does not otherwise depend on T or $V^\flat$ or $\underline{x}$.*

(2) *Suppose $m^\flat = n$ (with v_0 not necessarily Archimedean). Then*

$$C = \prod_{\substack{v<\infty \\ v\neq v_0}} c_v \tag{7.4.6}$$

for some constants $c_v \in \mathbb{Q}_{>0}$, all but finitely many of which are 1. For any given $v < \infty$ with $v \neq v_0$, the constant c_v may depend on the local Hermitian lattice L_v and the quadratic extension F_v/F_v^+, but otherwise does not depend on T or V or $\underline{x}$ or v_0 or F/F^+.

If L_v is self-dual, then $c_v = 1$.

(3) *Suppose $m^\flat = n - 1$ (with v_0 not necessarily Archimedean). Assume $K_{\underline{x},v_0} \subseteq G_{\underline{x}}(F_v^+)$ is the unique maximal open compact subgroup. Then there are constants $c_v' \in \mathbb{Q}_{>0}$ such that*

$$C = \frac{2^{1-o(\Delta)} h_F}{w_F h_{F^+} \cdot \#(\mathcal{O}_F^\times/(W\mathcal{O}_{F^+}^\times))} \prod_{\substack{v<\infty \\ v\neq v_0}} c_v' \tag{7.4.7}$$

where $o(\Delta)$ is the number of prime ideals of $\mathcal{O}_{F^+}$ which ramify in $\mathcal{O}_F$, where h_F (resp. h_{F^+}) is the class number of F (resp. F^+), where w_F (resp. W) is the number of (resp. group of) roots of unity in F. All but finitely many c_v' are equal to 1.

For each $v < \infty$ with $v \neq v_0$, the constant c_v' may depend on the local Hermitian lattice L_v, the quadratic extension F_v/F_v^+, and the local invariant $\varepsilon(V_v^\flat) \in \{\pm 1\}$. The constant c_v' does not otherwise depend on T or V or $V^\flat$ or $\underline{x}$ or v_0 or F/F^+.

If L_v is self-dual, then $c_v' = 1$ if F_v/F_v^+ is unramified (resp. $c_v' = 2$ if F_v/F_v^+ is ramified).

Proof. For the moment, we allow v_0 Archimedean or not. The groupoid in the lemma statement indeed has finite stabilizer groups, by discreteness of $G_{\underline{x}}(F^+)$. Take any factorizable open compact subgroup $K_{\underline{x}} = \prod_v K_{\underline{x},v} \subseteq G_{\underline{x}}(\mathbb{A})$. Assume $K_{\underline{x},v} = G_{\underline{x},v}(F_v^+)$ for every Archimedean v, and assume $K_{\underline{x},v} = K_{\underline{x},v_0}$ is the open compact subgroup fixed in the lemma statement when $v = v_0$. For each

v, define $\underline{x}_v^\flat = [x_{1,v}^\flat, \ldots, x_{m^\flat,v}^\flat] \in V_v^{m^\flat}$ to be the tuple satisfying $\underline{x} \cdot a_v^{-1} = [0, \ldots, 0, x_{1,v}^\flat, \ldots, x_{m^\flat,v}^\flat]$ (so $T_v^\flat = (\underline{x}_v^\flat, \underline{x}_v^\flat)$).

We have $\tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s_0)_n^\circ = 1$ for all Archimedean $v \neq v_0$ by positive definite-ness of $T_v^\flat$ (Section 4.2). For all but finitely many v, the Hermitian matrix ${}^t\overline{a}_v^\flat T_v^\flat a_v^\flat$ defines a self-dual $\mathcal{O}_{F_v}$-lattice (first check the case where the collection $(a_v)_v$ comes from a single element $a \in \mathrm{GL}_m(F)$; then recall that $\tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s, \Phi_{\varphi_v})_n$ does not depend on the choice of a_v or $a_v^\flat$). For such non-Archimedean $v \neq v_0$, we have $\tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s, \Phi_{\varphi_v^\flat})_n = \tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s)_n^\circ = 1$ if L_v is self-dual (see (4.5.7), Remark 4.5.1, and the invariance property in (4.3.4)). Hence $\tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s, \Phi_{\varphi_v^\flat})_n = 1$ for all but finitely many v.

Choose Haar measures $dg_{x,v}$ on $G_{\underline{x}}(F_v^+)$ for each v. Assume that $\mathrm{vol}_{dg_{x,v}}(K_{\underline{x},v}) \in \mathbb{Q}$ for all v, that $\mathrm{vol}_{dg_{x,v}}(K_{\underline{x},v}) = 1$ for all but finitely many v, and that $\mathrm{vol}_{dg_x,v}(K_{\underline{x},v}) = 1$ if $v = v_0$ or if $v \mid \infty$.

For $v < \infty$ with $v \neq v_0$, we give $G(F_v^+)$ the unique Haar measure dg_v such that

$$W_{T_v^{\flat\prime},v}^*(1, s_0^\flat, \Phi_{\varphi_v^\flat})_n = \int_{G_{\underline{x}_v}(F_v^+)\backslash G(F_v^+)} \varphi_v^\flat(g_v^{-1}\underline{x}_v') \, dg_v \tag{7.4.8}$$

for any tuple $\underline{x}_v' \in V_v^m$ (temporary notation) with nonsingular Gram matrix $T_v^{\flat\prime} := (\underline{x}_v', \underline{x}_v')$ (Lemma 7.2.1). The integral is taken with respect to the quotient measure induced by $dg_{x,v}$. This measure dg_v on $G(F_v^+)$ may depend on n, $m^\flat$, the isomorphism class of L_v (as the normalization defining $\tilde{W}_{T^\flat,v}^*$ depended on L_v) as well as the local invariant $\varepsilon(V_v^\flat)$ (Remark 7.2.2). The measure dg_v does not otherwise depend on $T_v^\flat$. Note $\mathrm{vol}_{dg_v}(K_{L,v}) \in \mathbb{Q}_{>0}$ for any $v < \infty$ with $v \neq v_0$, since the left-hand side of (7.4.8) lies in $\mathbb{Q}$ by Lemma 4.4.2. We have $\mathrm{vol}_{dg_v}(K_{L,v}) = 1$ for all but finitely many v (cf. the proof of Lemma 7.3.5; we have $W_{T_v^\flat,v}^*(s_0^\flat)_n^\circ = 1$ for all but finitely many v). We equip $G(\mathbb{A}_f^{v_0})$ with the product measure $dg = \prod_{\substack{v<\infty \\ v\neq v_0}} dg_v$.

Using the Haar measures specified above, we may unfold the groupoid cardinality as

$$\deg\left[G_{\underline{x}}(F^+)\backslash\left(G_{\underline{x}}(F_{v_0}^+)/K_{\underline{x},v_0} \times \mathcal{Z}(\underline{x}_f^{v_0})\right)\right] \tag{7.4.9}$$

$$= \mathrm{vol}_{dg}(K_{L,f}^{v_0})^{-1} \int_{G_{\underline{x}}(F^+)\backslash((\prod_{\substack{v=v_0 \\ \text{or } v\mid\infty}} G_{\underline{x}}(F_v^+))\times G(\mathbb{A}_f^{v_0}))} \varphi_f^{v_0}(g^{-1}\underline{x}) \, dg \tag{7.4.10}$$

$$= \mathrm{vol}(G_{\underline{x}}(F^+)\backslash G_{\underline{x}}(\mathbb{A}))\mathrm{vol}_{dg}(K_{L,f}^{v_0})^{-1}\left(\int_{G_{\underline{x}}(\mathbb{A}_f^{v_0})\backslash G(\mathbb{A}_f^{v_0})} \varphi_f^{v_0}(g^{-1}\underline{x}) \, dg\right) \tag{7.4.11}$$

$$= \mathrm{vol}(G_{\underline{x}}(F^+)\backslash G_{\underline{x}}(\mathbb{A}))\mathrm{vol}_{dg}(K_{L,f}^{v_0})^{-1} \prod_{\substack{v<\infty \\ v\neq v_0}} \int_{G_{\underline{x}}(F_v^+)\backslash G(F_v^+)} \varphi_v^\flat(g_v^{-1}\underline{x}_v^\flat a_v^\flat) \, dg_v \tag{7.4.12}$$

$$= C \prod_{\substack{v<\infty \\ v\neq v_0}} \tilde{W}_{T_v^\flat,v}^*(a_v^\flat, s_0^\flat, \Phi_{\varphi_v^\flat})_n \tag{7.4.13}$$

with

$$C := \mathrm{vol}(G_{\underline{x}}(F^+)\backslash G_{\underline{x}}(\mathbb{A})) \prod_{\substack{v<\infty \\ v\neq v_0}} \mathrm{vol}_{dg_v}(K_{L,v})^{-1}. \tag{7.4.14}$$

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Note that the integrals are absolutely convergent, since the integrands are continuous and compactly supported. This unfolding also shows that the groupoid in (7.4.4) has finitely many isomorphism classes.

(1) Suppose v_0 is Archimedean. Recall that the Tamagawa number of any nontrivial unitary group is 2 [Ich04, Section 4]. After scaling one of the non-Archimedean local measures $dg_{x,v}$ by an element of $\mathbb{Q}_{>0}$, we may assume $\prod_v dg_{x,v}$ is the Tamagawa measure on $G_{\underline{x}}(\mathbb{A})$. If $v \mid \infty$, let dg_v be the Haar measure on $G(F_v^+)$ given by Lemma 7.2.1 (induced by $dg_{x,v}$). For $v \mid \infty$, the local invariant $\varepsilon(V_v^\flat)$ is already determined by V and the requirement that $V_v^{\flat\perp}$ is definite. Hence the measures dg_v for $v \mid \infty$ do not depend on $V^\flat$ (apply Remark 7.2.2).

By construction of the measures in Lemma 7.2.1 (via invariant differentials), we find that $\prod_v dg_v$ equals the Tamagawa measure on $G(\mathbb{A})$ up to scaling by a constant which may depend on the lattices $\{L_v\}_{v<\infty}$ as well as n and $m^\flat$ (coming from our normalization of local Whittaker functions $\tilde{W}^*_{T^\flat,v}$, Section 4.3). We conclude that the measure dg on $G(\mathbb{A}_f)$ may depend on V, n, $m^\flat$, F, and the lattices $\{L_v\}_{v<\infty}$, but it does not otherwise depend on T or $V^\flat$ or $\underline{x}$.

(2) Suppose $m^\flat = n$. Then $G_{\underline{x}}$ is the trivial group. Take $\mathrm{vol}_{dg_{x,v}}(K_{\underline{x},v}) = 1$ for all v. Consider $v < \infty$ with $v \neq v_0$ and set $c_v = \mathrm{vol}_{dg_v}(K_{L,v})^{-1}$. If L_v is self-dual, then $c_v = 1$ by Lemma 7.3.5. In general, dg_v may depend on L_v (but not on T or $T_v^\flat$).

(3) Suppose $m^\flat = n - 1$. Then $G_{\underline{x}}$ is isomorphic to the norm-one torus inside $\mathrm{Res}_{F/F^+} \mathbb{G}_m$. Assume $K_{\underline{x},v} \subseteq G_{\underline{x}}(F_v^+)$ is the unique maximal open compact subgroup for every v. Take $\mathrm{vol}_{dg_{x,v}}(K_{\underline{x},v}) = 1$ for all v. Consider $v < \infty$ with $v \neq v_0$ and set $c_v' = \mathrm{vol}_{dg_v}(K_{L,v})^{-1}$. If L_v is self-dual, then $c_v' = 1$ if F_v/F_v^+ is unramified (resp. $c_v' = 2$ if F_v/F_v^+ is ramified) by Lemma 7.3.5. In general, dg_v may depend on L_v, $m^\flat$ and the local invariant $\varepsilon(V_v^\flat)$ (but not on T or $T_v^\flat$).

We have

$$\mathrm{vol}(G_{\underline{x}}(F^+)\backslash G_{\underline{x}}(\mathbb{A})) = \deg[G_{\underline{x}}(F^+)\backslash(G_{\underline{x}}(\mathbb{A})/K_{\underline{x}})] = \frac{\deg(G_{\underline{x}}(F^+)\backslash G_{\underline{x}}(\mathbb{A})/K_{\underline{x}})}{w_F}$$

where $\deg[-]$ denotes groupoid cardinality and $\deg(-)$ denotes set cardinality. We have

$$\deg(G_{\underline{x}}(F^+)\backslash G_{\underline{x}}(\mathbb{A})/K_{\underline{x}}) = 2^{u-t}h_F h_{F^+}^{-1}, \tag{7.4.15}$$

where t is the number of prime ideals of F^+ which ramify in F, and where $u \in \mathbb{Z}$ is such that $H^1(\mathrm{Gal}(F/F^+), \mathcal{O}_F^\times) \cong (\mathbb{Z}/2\mathbb{Z})^u$ [Ono85, (9)]. A group cohomology computation (omitted) shows that $2^{-u} = \#(\mathcal{O}_F^\times/(W\mathcal{O}_{F^+}^\times))/2$ (where $\#$ also means cardinality). $\qquad\square$

8. Geometric Siegel–Weil

8.1. Degrees of 0-cycles. Let L be any non-degenerate Hermitian $\mathcal{O}_F$-lattice of signature $(n-1, 1)$ (not assuming n is even). Let $\mathcal{M} \to \mathrm{Spec}\,\mathcal{O}_F[1/d_L]$ be the associated moduli stack ([Che24c, Section 2.1]). Recall that $d_L \in \mathbb{Z}$ is a certain integer associated to L, with $d_L = 1$ if L is self-dual when $2 \nmid \Delta$. Let $V := L \otimes_{\mathcal{O}_F} F$ be the associated $F/\mathbb{Q}$ Hermitian space.

Consider an integer m with $m = n$ or $m = n - 1$. Pick any embedding $F \to \mathbb{C}$, and set $\mathcal{M}_{\mathbb{C}} := \mathcal{M} \times_{\operatorname{Spec} \mathcal{O}_F} \operatorname{Spec} \mathbb{C}$, etc.. Given $T \in \operatorname{Herm}_m(\mathbb{Q})$ with $\operatorname{rank} T = n - 1$, recall that there is an associated Kudla–Rapoport special cycle $\mathcal{Z}(T) \to \mathcal{M}$ [Che24c, Definition 2.1.4]. The base change $\mathcal{Z}(T)_{\mathbb{C}}$ is smooth, proper, and quasi-finite (and of dimension zero) over $\operatorname{Spec} \mathbb{C}$ [Che24c, Lemmas 2.3.5 and 3.3.4].

For each place v of $\mathbb{Q}$, select any $a_v \in \operatorname{GL}_m(F_v)$ such that ${}^t\overline{a}_v^{-1} T a_v^{-1} = \operatorname{diag}(0, T_v^{\flat})$ for some $T_v^{\flat} \in \operatorname{Herm}_{n-1}(F_v^+)$ with $\det T_v^{\flat} \neq 0$. Choose any $a_v^{\flat} \in \operatorname{GL}_{n-1}(F_v)$ associated to a_v via the Iwasawa decomposition, as in (7.4.1) (if $m = n - 1$, we can just take $a_v^{\flat} = a_v$).

For formation of local Whittaker functions, we use the standard additive character $\psi \colon \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^{\times}$ with $\psi_{\infty}(x) = e^{2\pi i x}$. Suppose $\chi := F^{\times}\backslash\mathbb{A}_F^{\times} \to \mathbb{C}^{\times}$ is a character satisfying $\chi|_{\mathbb{A}^{\times}} = \eta^n$, where η is the quadratic character associated to $F/\mathbb{Q}$. For each prime p, we let $\varphi_v^{\flat} = \mathbf{1}_{L_p}^{n-1} \in \mathcal{S}(V(\mathbb{Q}_p)^{n-1})$ where $\mathbf{1}_{L_p}$ is the characteristic function of the lattice $L_p \subseteq V(\mathbb{Q}_p)$.

Proposition 8.1.1. *Let $C \in \mathbb{Q}_{>0}$ be the volume constant from Lemma 7.4.1(3), for the Hermitian space V and with $v_0 = \infty$ in the notation of loc. cit.. In the situation above, we have*

$$\deg \mathcal{Z}(T)_{\mathbb{C}} = \frac{h_F}{w_F} C \cdot \tilde{W}^*_{T_{\infty}^{\flat}, \infty}(a_{\infty}^{\flat}, 1/2)_n^{\circ} \prod_p \tilde{W}^*_{T_p^{\flat}, p}(a_p^{\flat}, 1/2, \Phi_{\varphi_v^{\flat}})_n. \tag{8.1.1}$$

Proof. As in Section 7.1, we write $\Omega_T(R) := \{\underline{x} \in (V \otimes_{\mathbb{Q}} R)^m : (\underline{x}, \underline{x}) = T\}$ for $\mathbb{Q}$-algebras R. Here $\deg \mathcal{Z}(T)_{\mathbb{C}}$ denotes the (stacky) degree of $\mathcal{Z}(T)_{\mathbb{C}}$ over $\operatorname{Spec} \mathbb{C}$, as explained at the end of [Che24a, Appendix A.1].

Suppose there is no tuple $\underline{x} \in V^m$ such that $(\underline{x}, \underline{x}) = T$. By the Hasse principle, we conclude $\Omega_T(\mathbb{Q}_{v_0}) = \emptyset$ for some place v_0 of $\mathbb{Q}$. Since $\operatorname{rank}(T) < n$, we must have $v_0 = \infty$ (i.e. for $F_v < \infty$, any non-degenerate hermitian F_v vector space of rank $n - 1$ embeds into any non-degenerate Hermitian F_v vector space of rank n). We conclude that $T_{\infty}^{\flat}$ (and ${}^t\overline{a}_{\infty}^{\flat} T_{\infty}^{\flat} a_{\infty}^{\flat}$) has signature $(n - 1 - r, r)$ for some $r \geq 2$. The proposition holds in this case because $\tilde{W}^*_{T_{\infty}^{\flat}, \infty}(a_{\infty}^{\flat}, 1/2)_n^{\circ} = 0$ (by (4.2.6) or (7.2.2)).

Suppose there exists $\underline{x} \in V^m$ such that $(\underline{x}, \underline{x}) = T$. For such $\underline{x}$, write $\underline{x}_{\infty} \in V_{\mathbb{R}}^m$ and $\underline{x}_f \in (V \otimes_{\mathbb{Q}} \mathbb{A}_f)^m$ for the respective images. By complex uniformization of special cycles [Che24c, Section 4.5], we have

$$\deg \mathcal{Z}(T)_{\mathbb{C}} = \frac{h_F}{w_F} \cdot \deg \mathcal{D}(\underline{x}_{\infty}) \cdot \deg \left[U(V)(\mathbb{Q}) \backslash \coprod_{\substack{\underline{x} \in V^m \\ (\underline{x}, \underline{x}) = T}} \mathcal{D}(\underline{x}_f) \right]. \tag{8.1.2}$$

Here $\deg \mathcal{D}(\underline{x}_{\infty})$ is the degree of the Archimedean local special cycle $\mathcal{D}(\underline{x}_{\infty}) \subseteq \mathcal{D}$ [Che24b, Section 2.2] for any $\underline{x} \in V^m$ with $(\underline{x}, \underline{x}) = T$. We know $\mathcal{D}(\underline{x}_{\infty})$ is a single point if T is positive semidefinite, and empty otherwise. Hence $\deg \mathcal{D}(\underline{x}_{\infty}) = \tilde{W}^*_{T_{\infty}^{\flat}, \infty}(a_{\infty}^{\flat}, 1/2)_n^{\circ}$ (by (4.2.6), the right-hand side is 1 if $T_{\infty}^{\flat}$ is positive definite and 0 otherwise).

We then use Lemma 7.4.1 to evaluate the groupoid cardinality in (8.1.2). $\qquad \square$

Remark 8.1.2. Suppose $2 \nmid \Delta$ and that L is self-dual (for the trace pairing, as is our running convention). We then have $C = 2h_F/w_F$ in Proposition 8.1.1. Take any $a \in \operatorname{GL}_m(F)$ such that ${}^t\overline{a}^{-1} T a^{-1} = \operatorname{diag}(0, T^{\flat})$ where $T^{\flat} \in \operatorname{Herm}_{n-1}(\mathbb{Q})$ with $\det T^{\flat} \neq 0$. For each place v of $\mathbb{Q}$, let

$a_v := a \in \mathrm{GL}_m(F_v)$. Set $a^\flat = (a_v^\flat)_v \in \mathrm{GL}_m(\mathbb{A}_F)$ (running over places v of $\mathbb{Q}$) in the notation above. The proposition then states

$$\deg \mathcal{Z}(T)_{\mathbb{C}} = 2\frac{h_F^2}{w_F^2} \cdot \tilde{E}_{T^\flat}^*(a^\flat, 1/2)_n^\circ. \tag{8.1.3}$$

Remark 8.1.3. As observed by Li and Zhang [LZ22, Remark 4.6.2], Proposition 8.1.1 may be proved using Rapoport–Zink non-Archimedean uniformization in essentially the same way. Indeed, the horizontal local special cycle $\mathcal{Z}(T)_{\mathscr{H}} \to \mathrm{Spec}\,\mathcal{O}_F[1/d_L]$ is proper, quasi-finite, and flat [Che24c, Lemma 3.3.4], so we may calculate its degree in the fiber over any geometric point of $\mathrm{Spec}\,\mathcal{O}_F[1/d_L]$. Fix a geometric point in characteristic $p > 0$. Assume $p \neq 2$ if 2 is nonsplit in $\mathcal{O}_F$, assume L_p is self-dual, and assume either $p \nmid \Delta$ or that L is self-dual and $2 \nmid \Delta$. Consider the n-dimensional positive definite non-degenerate Hermitian space $\mathbf{V}$ with $\varepsilon(\mathbf{V}_p) = -1$ and $\varepsilon(\mathbf{V}_\ell) = \varepsilon(V_\ell)$ for any $\ell \neq p$.

Using non-Archimedean uniformization, we may then argue as in the proof of Proposition 8.1.1 (see [Che24c, (4.9.6)]), using the special value formula for degrees of local special cycles [Che24a, Lemma 9.1.3], and the formula for uniformization degrees (Lemma 7.4.1) for $\mathbf{V}$ and $v_0 = p$.

8.2. **Complex volumes.** Assume 2 is unramified in $\mathcal{O}_F$. For even integers $n \in \mathbb{Z}_{>0}$, we show that the global normalizing factors $\Lambda_n(s)_n^\circ$ (Section 6.1) encode complex volumes of certain unitary Shimura varieties (Propositions 8.2.1 and 8.2.3).

First consider $n \equiv 0 \pmod 4$. Let V be the unique $F/\mathbb{Q}$ Hermitian space of signature $(n, 0)$ which satisfies $\varepsilon(V_p) = 1$ for all primes p (with ε as in [Che24a, Section 2.2], i.e. V contains a full-rank self-dual $\mathcal{O}_F$-lattice). Set $G := U(V)$, let $L \subseteq V$ be a full-rank self-dual lattice, and write $K_{L,f} \subseteq G(\mathbb{A}_f)$ for the adèlic stabilizer of L. The following proposition should be a special case of a unitary analogue of the classical Siegel mass formula. It is included for comparison with the analogous volume identity for a signature $(n-1, 1)$ unitary complex Shimura variety. The left-hand side counts self-dual positive definite $\mathcal{O}_F$-lattices of rank n, weighted by the inverses of the sizes of their automorphism groups.

Proposition 8.2.1. *We have*

$$\#[G(\mathbb{Q}) \backslash (G(\mathbb{A}_f)/K_{L,f})] = 2\Lambda_n(0)_n^\circ \tag{8.2.1}$$

where the left-hand side denotes groupoid cardinality.

Proof. Let $\psi \colon \mathbb{Q} \backslash \mathbb{A} \to \mathbb{C}^\times$ be the standard additive character with $\psi_\infty(s) = e^{2\pi i x}$. Let $\chi \colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the trivial character.

For $v = \infty$, let $\varphi_v(\underline{x}) = e^{2\pi i \mathrm{tr}(\underline{x}, \underline{x})} \in \mathcal{S}(V(\mathbb{R})^n)$ and let $T \in \mathrm{Herm}_n(\mathbb{R})$ be an arbitrary positive definite matrix. For $v < \infty$ corresponding to a prime p, let $\varphi_v = \mathbf{1}_{L_v}^n \in \mathcal{S}(V(\mathbb{Q}_p)^n)$ and let T be the Gram matrix for any basis of L_v. For such T, we have $W_{T,v}^*(s_0)_n^\circ = 1$ for all v (Sections 4.2 and 4.5.7). Recall $W_{T,v}^*(s)_n^\circ = \Lambda_{T,v}(s)_n^\circ W_{T,v}(s, \Phi_{\varphi_v})$ if $v < \infty$ (resp. $W_{T,v}^*(s)_n^\circ e^{-2\pi \mathrm{tr}(T)} = \Lambda_{T,v}(s)_n^\circ W_{T,v}(s, \Phi_{\varphi_v})$ if $v = \infty$); see Section 3.2.

Using these data, the local Siegel–Weil formula (Lemma 7.2.1) for each place v of $\mathbb{Q}$ shows that $\mathrm{vol}(G(\mathbb{R}) \times K_{L,f})^{-1} = \Lambda_n(0)_n^\circ$ for the Tamagawa measure on $G(\mathbb{A})$. Since G has Tamagawa number 2 [Ich04, §4], the proposition follows. $\qquad\square$

Next, consider $n \equiv 2 \pmod 4$. Let V be the unique n-dimensional $F/\mathbb{Q}$ Hermitian space of signature $(n-1, 1)$ which satisfies $\varepsilon(V_p) = 1$ for all primes p. Again, set $G := U(V)$, let $L \subseteq V$ be a full-rank self-dual lattice, and write $K_{L,f} \subseteq G(\mathbb{A}_f)$ for the adèlic stabilizer of L. For sufficiently small open compact $K_f \subseteq G(\mathbb{A}_f)$, there is a complex (analytic) Shimura variety

$$\mathrm{Sh}_{K_f, \mathbb{C}} = G(\mathbb{A}) \backslash (\mathcal{D} \times G(\mathbb{A}_f) / K_f) \tag{8.2.2}$$

of dimension $n-1$, where $\mathcal{D}$ is the Hermitian symmetric domain from [Che24b, Section 2.1] (parameterizing maximal negative definite subspaces of $V_{\mathbb{R}}$); the V of loc. cit. is our $V_{\mathbb{R}}$, with $\mathbb{C} = F \otimes_{\mathbb{Q}} \mathbb{R}$-action. The metrized tautological bundle $\widehat{\mathcal{E}}^{\vee}$ of loc. cit. descends to $\mathrm{Sh}_{K_f, \mathbb{C}}$. For any open compact $K_f' \subseteq G(\mathbb{A}_f)$ and any sufficiently small $K_f \subseteq K_f'$, we set

$$\mathrm{vol}(\mathrm{Sh}_{K_f, \mathbb{C}}) := \int_{\mathrm{Sh}_{K_f, \mathbb{C}}} c_1(\widehat{\mathcal{E}})^{n-1} \qquad \mathrm{vol}(\mathrm{Sh}_{K_f', \mathbb{C}}) := \frac{1}{[K_f' : K_f]} \mathrm{vol}(\mathrm{Sh}_{K_f, \mathbb{C}}). \tag{8.2.3}$$

If $K_{L',f} \subseteq G(\mathbb{A}_f)$ is the adèlic stabilizer of a full-rank lattice $L' \subseteq V$ which is self-dual for the Hermitian pairing, the quantity $\mathrm{vol}(\mathrm{Sh}_{K_{L',f}, \mathbb{C}})$ was computed explicitly in [BH21, Theorem A]. We show that the level $K_{L,f}$ (self-dual for the *trace pairing*) removes the additional factors at ramified primes in loc. cit., and that the resulting complex volume agrees with $2\Lambda_n(0)_n^{\circ}$ exactly.

The volume identity should also follow from [LL21, Footnote 11] (or possibly other geometric Siegel–Weil results). We instead compute $\mathrm{vol}(\mathrm{Sh}_{K_{L,f}, \mathbb{C}})$ using [BH21, Theorem A] by calculating the "change of level" via the following lemma.

Lemma 8.2.2. *Let E_v^+ be a non-Archimedean local field of odd residue cardinality q_v, and let E_v / E_v^+ be a ramified quadratic extension with involution $a \mapsto a^{\sigma}$.*

Let W be a rank $2d$ non-degenerate E_v / E_v^+ Hermitian space, and assume W contains a full-rank lattice $M \subseteq W$ which is self-dual (for the trace pairing). Let $M' \subseteq W$ be any full-rank lattice which is self-dual for the Hermitian pairing.

If $K, K' \subseteq U(W)$ are the stabilizers of M and M' respectively, we have

$$\frac{\mathrm{vol}(K)}{\mathrm{vol}(K')} = 2^{-1}(1 + q_v^d) \tag{8.2.4}$$

for any Haar measure on $U(W)$.

Proof. We know that any two full-rank lattices in W which are self-dual (resp. self-dual for the Hermitian form) are isomorphic [Jac62, Proposition 8.1] (false if E_v^+ is allowed to have residue characteristic 2). Hence $\mathrm{vol}(K)/\mathrm{vol}(K')$ does not depend on the choice of M and M' (nor the choice of Haar measure).

Let ϖ be a uniformizer of E_v, and assume $\varpi^{\sigma} = -\varpi$. The lattices M and M' admit bases with Gram matrices

$$\begin{pmatrix} 0 & \varpi^{-1} \\ -\varpi^{-1} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.2.5}$$

respectively. Choose a basis $e_1, \ldots, e_{2d}$ for M with Gram matrix as above. We may assume that M' is the lattice with basis $e_1, \ldots, e_d, \varpi e_{d+1}, \ldots, \varpi e_{2d}$. Let $\overline{W}$ (resp. $\overline{W}'$) be the $2d$-dimensional

vector space over $\mathbb{F}_{q_v}$ with symplectic pairing (resp. bilinear pairing) given by the block matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{resp.} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.2.6}$$

If $P_W \subseteq \mathrm{Sp}(\overline{W})$ and $P_{\overline{W}'} \subseteq \mathrm{O}(\overline{W})$ are the subgroups upper triangular matrices (in $d \times d$ blocks), we have

$$\#(K/(K \cap K')) = \#(\mathrm{Sp}(\overline{W})(\mathbb{F}_{q_v})/P_{\overline{W}}(\mathbb{F}_{q_v})) \tag{8.2.7}$$

$$\#(K'/(K \cap K')) = \#(\mathrm{O}(\overline{W}')(\mathbb{F}_{q_v})/P_{\overline{W}'}(\mathbb{F}_{q_v})). \tag{8.2.8}$$

The lemma now follows from the formulas

$$\#\mathrm{Sp}(\overline{W})(\mathbb{F}_{q_v}) = q_v^{d^2} \prod_{i=1}^{d}(q_v^{2i} - 1) \qquad \#\mathrm{O}(\overline{W}')(\mathbb{F}_{q_v}) = 2q_v^{d(d-1)}(q_v^d + 1)^{-1} \prod_{i=1}^{d}(q_v^{2i} - 1)$$

$$\#P_{\overline{W}}(\mathbb{F}_{q_v}) = q_v^{d(d+1)/2} \prod_{i=1}^{d}(q_v^d - q_v^{i-1}) \qquad \#P_{\overline{W}'}(\mathbb{F}_{q_v}) = q_v^{d(d-1)/2} \prod_{i=1}^{d}(q_v^d - q_v^{i-1}). \qquad \square$$

We return to the global situation with $F/\mathbb{Q}$ as above and $L \subseteq V$ a self-dual lattice.

Proposition 8.2.3. *We have*

$$\mathrm{vol}(\mathrm{Sh}_{K_{L,f},\mathbb{C}}) = 2\Lambda_n(0)_n^\circ. \tag{8.2.9}$$

Proof. If $K_{L',f} \subseteq G(\mathbb{A}_f)$ is the adèlic stabilizer of a full-rank lattice $L' \subseteq V$ which is self-dual for the Hermitian pairing, the result [BH21, Theorem A] (see also [BH21, Theorem 5.5.1] to compare $c_1(\widehat{\mathcal{E}})$ with the Chern form of the metrized Hodge bundle; note our $\widehat{\mathcal{E}}$ is $\widehat{\mathcal{L}}$ in loc. cit. (up to restricting)) gives

$$\mathrm{vol}(\mathrm{Sh}_{K_{L',f},\mathbb{C}}) = \left[2^{1-o(\Delta)} \prod_{\ell \mid \Delta}(1 + \varepsilon(V_\ell)\ell^{-n/2}) \prod_{j=1}^{n} \frac{\Delta^{j/2}\Gamma(s+j)L(2s+j,\eta^j)}{2^j \pi^{s+j}} \right]_{s=0} \tag{8.2.10}$$

where $o(\Delta)$ is the number primes dividing Δ. We assumed $\varepsilon(V_\ell) = 1$ for all ℓ, and a direct computation shows

$$\Lambda_n(s)_n^\circ = \Delta^{n/2(s-1)} \prod_{j=1}^{n} \frac{\Delta^{j/2}\Gamma(s+j)L(2s+j,\eta^j)}{2^j \pi^{s+j}} \tag{8.2.11}$$

(using even-ness of n). The claim now follows from the computation of $\mathrm{vol}(K_{L,f})/\mathrm{vol}(K_{L',f})$ (for any Haar measure on $G(\mathbb{A}_f)$) from Lemma 8.2.2. Note that the only discrepancy between $\mathrm{vol}(K_{L,f})$ and $\mathrm{vol}(K_{L',f})$ is at ramified primes, since self-dual lattices for the Hermitian pairing are the same as self-dual lattices at unramified primes. $\square$

9. Arithmetic Siegel–Weil

9.1. **Main theorems.** This section contains the statements and proofs of our main global results (Theorem 9.1.1 and the secondary Theorem 9.1.6). Theorem 9.1.1 relies on essentially all preceding results in our four-part sequence of papers. We necessarily heavily cite our companion papers [Che24a; Che24b; Che24c]. In the proof, we explain how to combine our local main results (proved in [Che24a, Section 9] [Che24b, Section 4]) and a (new) "local diagonalization" argument to deal

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with singular T (including those which are not-necessarily $\mathrm{GL}_n(\mathcal{O}_F)$-conjugate to a block diagonal matrix with nonsingular diagonal blocks).

Assume $2 \nmid \Delta$, and let L be any non-degenerate self-dual Hermitian $\mathcal{O}_F$-lattice of signature $(n-1,1)$. Set $n := \mathrm{rank}\, L$, and note $n \equiv 2 \pmod 4$ (by the global product formula for local invariants of Hermitian spaces; note $\varepsilon(L_p) = 1$ for all primes p).

Form the associated (smooth) moduli stack $\mathcal{M} \to \mathrm{Spec}\, \mathcal{O}_F$ ([Che24c, Section 2.1] and [Che24a, Section 3.1]). We are imposing "no level structure" on $\mathcal{M}$ (i.e. $K_{0,f} \times K_f = K_{L_0,f} \times K_{L,f}$ in the notation of [Che24c, Section 2.2]).

For any m, given $T \in \mathrm{Herm}_m(\mathbb{Q})$ (with F-coefficients), and given $y \in \mathrm{Herm}_m(\mathbb{R})_{>0}$ (with $\mathbb{C}$-coefficients), recall that there is a arithmetic special cycle class $[\widehat{\mathcal{Z}}(T)] \in \widehat{\mathrm{Ch}}^m(\mathcal{M})_{\mathbb{Q}}$ [Che24c, Section 3] and a normalized T-th Fourier coefficient $E_T^*(y,s)_n^\circ$ Section 6.1 of a $U(m,m)$ Eisenstein series. If $\mathrm{rank}(T) \geq n-1$ or if T is nonsingular and not positive definite, we are using the current $g_{T,y}$ from [Che24c, Section 5.4] (constructed from the local analogue in [Che24b, Section 2.4]). The class $[\widehat{\mathcal{Z}}(T)]$ thus implicitly depends on y.

For special cycles $\mathcal{Z}(T)$ which are proper over $\mathrm{Spec}\, \mathcal{O}_F$, recall that we have defined certain arithmetic degrees without boundary contributions (1.3.5). These are the arithmetic degrees appearing in our main theorem below.

For use below, we record the expression

$$\frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ} = -\frac{1}{2}L(2s+1,\eta)\Gamma(s+1)|\Delta|^{s+1/2}\pi^{-s-1} \tag{9.1.1}$$

which follows from our formula for the normalizing factor $\Lambda_m(s)_n^\circ$ (6.1.2). We thus have

$$\frac{\Lambda_n(0)_n^\circ}{\Lambda_{n-1}(1/2)_n^\circ} = -\frac{h_F}{w_F} \qquad \frac{d}{ds}\bigg|_{s=0}\left(\frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ}\right) = 2\frac{h_F}{w_F}h_{\widehat{\mathcal{E}}^\vee}^{\mathrm{CM}} \tag{9.1.2}$$

where the left expression follows from the analytic class number formula, and $h_{\widehat{\mathcal{E}}^\vee}^{\mathrm{CM}}$ is the height constant from [Che24c, (2.1.13)], arising from Faltings heights of elliptic curves with complex multiplication by $\mathcal{O}_F$ (for the purpose of the present paper, one could also take (9.1.2) as the definition of $h_{\widehat{\mathcal{E}}^\vee}^{\mathrm{CM}}$).

Theorem 9.1.1 (Corank 1 arithmetic Siegel–Weil)**.** *Assume the prime* 2 *splits in* $\mathcal{O}_F$.

(1) For any $T \in \mathrm{Herm}_n(\mathbb{Q})$ *with* $\mathrm{rank}(T) = n-1$ *and any* $y \in \mathrm{Herm}_n(\mathbb{R})_{>0}$, *we have*

$$\widehat{\mathrm{deg}}([\widehat{\mathcal{Z}}(T)]) = \frac{h_F}{w_F}\frac{d}{ds}\bigg|_{s=0}E_T^*(y,s)_n^\circ. \tag{9.1.3}$$

(2) For any $T^\flat \in \mathrm{Herm}_{n-1}(\mathbb{Q})$ *with* $\det T^\flat \neq 0$ *and any* $y^\flat \in \mathrm{Herm}_{n-1}(\mathbb{R})_{>0}$, *we have*

$$\widehat{\mathrm{deg}}([\widehat{\mathcal{Z}}(T^\flat) \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)]) = 2\frac{h_F}{w_F}\frac{d}{ds}\bigg|_{s=0}\left(\frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ}E_{T^\flat}^*(y^\flat,s+1/2)_n^\circ\right). \tag{9.1.4}$$

Proof. In the theorem statement, $[\widehat{\mathcal{Z}}(T)]$ and $[\widehat{\mathcal{Z}}(T^\flat)]$ are implicitly formed with respect to y and $y^\flat$, respectively. Note that $E_T^*(y,s)_n^\circ$ is a normalized Fourier coefficient for a $U(n,n)$ Eisenstein series, while $E_{T^\flat}^*(y^\flat,s)_n^\circ$ is a normalized Fourier coefficient for a $U(n-1,n-1)$ Eisenstein series. In the theorem statement, note that $\mathcal{Z}(T) \to \mathrm{Spec}\, \mathcal{O}_F$ and $\mathcal{Z}(T^\flat) \to \mathrm{Spec}\, \mathcal{O}_F$ are both proper [Che24c, Lemma 3.3.5], so we may use (1.3.5) to define arithmetic degrees without boundary contributions.

Note that Theorem 9.1.1(2) is the special case of Theorem 9.1.1(1) when $T = \mathrm{diag}(0, T^\flat)$ and $y = \mathrm{diag}(1, y^\flat)$. This follows from the unfolding of Fourier coefficients in Corollary 6.2.2 (also the functional equation in Lemma 6.1.1) and from the definition of arithmetic degrees in (1.3.5).

Fix T and y as in the statement of part (1) (not necessarily block diagonal). Fix any prime p. It is enough to show that (9.1.3) holds modulo $\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell$ (i.e. as elements of the additive quotient $\mathbb{R}/(\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell)$), where the sum runs over primes $\ell \neq p$. Varying the prime p removes this discrepancy (giving an equality as elements of $\mathbb{R}$) because the real numbers $\log \ell$ (ranging over all primes ℓ in $\mathbb{Z}$) form a $\mathbb{Q}$-linearly independent set.

(*Step 1: Diagonalize*) For convenience, we fix an embedding $F \to \mathbb{C}$. Pick any $b \in \mathrm{GL}_m(F)$ such that ${}^t\bar{b}^{-1} T b^{-1} = \mathrm{diag}(0, T^\flat)$ for some $T^\flat \in \mathrm{Herm}_{n-1}(\mathbb{Q})$ with $\det T^\flat \neq 0$. We may (and do) assume $b \in \mathrm{GL}_n(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$ as well. The proof below will show that the theorem holds modulo $\mathbb{Q} \cdot \log \ell$ for primes ℓ such that $b \notin \mathrm{GL}_n(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)})$.

For each place v of $\mathbb{Q}$, select any $b_v^{\#} \in \mathrm{GL}_1(F_v)$ and $b_v^\flat \in \mathrm{GL}_{n-1}(F_v)$ associated to an Iwasawa decomposition of $b_v \in \mathrm{GL}_n(F_v)$, as in (7.4.1) (where b_v denotes the image of image of b). Also consider the (unique) decomposition

$$by^t\bar{b} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\#} & 0 \\ 0 & y^\flat \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {}^t\bar{c} & 1 \end{pmatrix} \tag{9.1.5}$$

as in [Che24b, (2.4.2)], where $c \in M_{1,n-1}(\mathbb{C})$, $y^{\#} \in \mathbb{R}_{>0}$, and $y^\flat \in \mathrm{Herm}_{n-1}(\mathbb{R})_{>0}$. Pick any $a_\infty^{\#} \in \mathrm{GL}_1(\mathbb{C})$ and $a_\infty^\flat \in \mathrm{GL}_{n-1}(\mathbb{C})$ such that $a_\infty^{\#}{}^t\bar{a}_\infty^{\#} = y^{\#}$ and $a_\infty^\flat{}^t\bar{a}_\infty^\flat = y^\flat$.

Let $a^{\#} \in \mathrm{GL}_1(\mathbb{A}_F)$ be the element with component $a_v^{\#} := b_v^{\#}$ for places $v < \infty$ and $a_v^{\#} := a_\infty^{\#}$ for the place $v = \infty$. Similarly define $a^\flat \in \mathrm{GL}_{n-1}(\mathbb{A}_F)$, and set $a := \mathrm{diag}(a^{\#}, a^\flat) \in \mathrm{GL}_n(\mathbb{A}_F)$.

By unfolding for corank 1 Fourier coefficients (Corollary 6.2.2) and Fourier coefficient invariance properties (see (2.3.3), (2.3.4), (4.2.4), and (4.3.4) for $U(m)$ invariance when $v \mid \infty$ and $\mathrm{GL}_m(\mathcal{O}_{F_v^+})$ invariance when $v < \infty$), we find

$$E_T^*(y, s)_n^\circ = \chi_\infty(d)^{-1} \det(y)^{-n/2} E_{T'}^*(m(a), s)_n^\circ = \tilde{E}_{T'}^*(a, s)_n^\circ$$

$$= |a^{\#}|_F^s \frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s + 1/2)_n^\circ} \tilde{E}_{T^\flat}^*(a^\flat, s + 1/2)_n^\circ$$

$$- |a^{\#}|_F^{-s} \frac{\Lambda_n(-s)_n^\circ}{\Lambda_{n-1}(-s + 1/2)_n^\circ} \tilde{E}_{T^\flat}^*(a^\flat, s - 1/2)_n^\circ$$

where $T' := \mathrm{diag}(0, T^\flat)$. We remind the reader that the notation $E_T^*(-, s)_n^\circ$ is overloaded (Section 6.1, also end of Section 2.2) and has slightly different meaning when "$-$" is $y \in \mathrm{Herm}_m(\mathbb{R})_{>0}$ versus $h \in U(m,m)(\mathbb{A})$ (e.g. $h = m(a)$).

(*Step 2: Leibniz rule*) Since $n \equiv 2 \pmod 4$, the functional equation for $\tilde{E}_{T^\flat}^*(a^\flat, s)_n^\circ$ (Lemma 6.1.1) implies

$$\left.\frac{d}{ds}\right|_{s=0} E_T^*(y, s)_n^\circ \tag{9.1.6}$$

$$= 2\left.\frac{d}{ds}\right|_{s=0} \left(|a^{\#}|_F^s \frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s + 1/2)_n^\circ} \tilde{E}_{T^\flat}^*(a^\flat, s + 1/2)_n^\circ \right).$$

Since $\det T^\flat \neq 0$, we may factorize $\tilde{E}^*_{T^\flat}(a^\flat, s+1/2)^\circ_n$ into a product of (variants of) normalized local Whittaker functions (6.1.8). Also recall the formulas in (9.1.2). We have $|a^\#_\ell|_\ell = 1$ (ℓ-adic norm of $a^\#_\ell$) for any prime ℓ such that $b \in \mathrm{GL}_n(\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_{(\ell)})$ (by construction, this includes $\ell = p$). By the Leibniz rule, we thus find

$$\left(\frac{2h_F}{w_F}\right)^{-1} \frac{d}{ds}\bigg|_{s=0} E^*_T(y, s)^\circ_n \tag{9.1.7}$$

$$= 2h^{\mathrm{CM}}_{\widehat{\mathcal{E}}^\vee} \tilde{E}^*_{T^\flat}(a^\flat, 1/2)^\circ_n \tag{9.1.8}$$

$$- \left(\frac{d}{ds}\bigg|_{s=1/2} \left(|a^\#_\infty|^s_\infty \tilde{W}^*_{T^\flat,\infty}(a^\flat_\infty, s)^\circ_n\right)\right) \prod_\ell \tilde{W}^*_{T^\flat,\ell}(a^\flat_\ell, 1/2)^\circ_n \tag{9.1.9}$$

$$- \left(\frac{d}{ds}\bigg|_{s=1/2} \tilde{W}^*_{T^\flat,p}(a^\flat_p, s)^\circ_n\right) \prod_{v \neq p} \tilde{W}^*_{T^\flat,v}(a^\flat_v, 1/2)^\circ_n \tag{9.1.10}$$

$$- \sum_{\ell \neq p} \left(\frac{d}{ds}\bigg|_{s=1/2} |a^\#_\ell|^s_\ell \tilde{W}^*_{T^\flat,\ell}(a^\flat_\ell, s)^\circ_n\right) \prod_{v \neq \ell} \tilde{W}^*_{T^\flat,v}(a^\flat_v, 1/2)^\circ_n. \tag{9.1.11}$$

The product in (9.1.9) runs over all primes ℓ (not including the Archimedean place ∞). The products in (9.1.10) and (9.1.11) run over all places v of $\mathbb{Q}$ (with $v \neq p$ or $v \neq \ell$ as indicated), including $v = \infty$. The sum in (9.1.11) runs over all primes $\ell \neq p$. We remind the reader that $|a^\#_\infty|_\infty = \bar{a}^\#_\infty a^\#_\infty \in \mathbb{R}_{>0}$, by definition.

For all but finitely many primes ℓ, the Hermitian matrix ${}^t\bar{a}^\flat_\ell T^\flat a^\flat_\ell \in \mathrm{Herm}_{n-1}(\mathbb{Q}_\ell)$ defines a (non-degenerate) self-dual Hermitian $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_\ell$-lattice. For such ℓ, we have $\tilde{W}^*_{T^\flat,\ell}(a^\flat_\ell, s)^\circ_n$ identically equal to 1 (as a function in the s-variable). This follows from (4.5.7) and an invariance property for local Whittaker functions (4.3.4). In particular, the sums and products are finite in the right-hand side of (9.1.7).

For every prime ℓ, we have $\tilde{W}^*_{T^\flat,\ell}(a^\flat_\ell, s + 1/2)^\circ_n \in \mathbb{Z}[\ell^{-1}, \ell^{-s}, \ell^s]$ (see (4.5.6), and again the invariance property in (4.3.4)). We also have $\tilde{W}^*_{T^\flat,v}(a^\flat_v, 1/2)^\circ_n \in \mathbb{Q}$ for all place v of $\mathbb{Q}$ (if $v \mid \infty$, this quantity is 1 if $T^\flat$ is positive definite and 0 otherwise by (4.2.6)). The quantity in (9.1.10) thus lies in $\mathbb{Q} \cdot \log p$, and the quantity in (9.1.11) thus lies in $\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell$.

As we explain below, every quantity on the right-hand side of (9.1.7) has geometric meaning via our main local results, at least modulo $\mathbb{Q} \cdot \log \ell$ for primes ℓ such that $b \notin \mathrm{GL}_n(\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_{(\ell)})$.

(*Step 3a: Local geometric interpretation: complex degree*) Set $\mathcal{Z}(T)_\mathbb{C} = (\mathcal{Z}(T) \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathrm{Spec}\,\mathbb{C})$ for the embedding $F \to \mathbb{C}$ fixed above. We have $\deg \mathcal{Z}(T)_\mathbb{C} = (\deg_F \mathcal{Z}(T) \times_{\mathrm{Spec}\,\mathcal{O}_F} \mathrm{Spec}\,F) = 2 \deg_\mathbb{Q}(\mathcal{Z}(T) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Q}) =: \deg_\mathbb{Z} \mathcal{Z}(T)_\mathscr{H}$. Here $\deg_F$ and $\deg_\mathbb{Q}$ denote stacky degrees over $\mathrm{Spec}\,F$ and $\mathrm{Spec}\,\mathbb{Q}$, respectively, as defined at the end of [Che24a, Appendix A.1].

By the geometric Siegel–Weil formula for Kudla–Rapoport 0-cycles over $\mathbb{C}$ (Proposition 8.1.1, also Remark 8.1.2), we conclude

$$\deg_\mathbb{Z} \mathcal{Z}(T)_\mathscr{H} = 2 \deg \mathcal{Z}(T)_\mathbb{C} = \frac{4h^2_F}{w^2_F} \tilde{E}^*_{T^\flat}(a^\flat, 1/2)^\circ_n. \tag{9.1.12}$$

This gives a geometric interpretation of (9.1.8).

(*Step 3b: Local geometric interpretation: at* ∞) We claim that

$$\mathrm{Int}_\infty(T,y) = -\frac{d}{ds}\Big|_{s=1/2}\left(|a_\infty^\#|_\infty^s \tilde{W}^*_{T^\flat,\infty}(a_\infty^\flat,s)_n^\circ\right) \quad \mathrm{mod} \sum_{\substack{\ell \text{ such that} \\ b\notin\mathrm{GL}_n(\mathcal{O}_F\otimes_\mathbb{Z}\mathbb{Z}_{(\ell)})}} \mathbb{Q}\cdot\log\ell \qquad (9.1.13)$$

where $\mathrm{Int}_\infty(T,y)$ is the geometric quantity defined in [Che24c, (5.4.7)].

Indeed, [Che24b, (2.4.6)] implies

$$\mathrm{Int}_\infty(T,y) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9.1.14)$$

$$\begin{cases} \mathrm{Int}_\infty(T^\flat, a_\infty^\flat \,{}^t\overline{a}_\infty^\flat) - \log(|a_\infty^\#|_\infty) \mod \sum_{\substack{\ell \text{ such that} \\ b\notin\mathrm{GL}_n(\mathcal{O}_F\otimes_\mathbb{Z}\mathbb{Z}_{(\ell)})}} \mathbb{Q}\cdot\log\ell & \text{if } T^\flat > 0 \\[2em] \mathrm{Int}_\infty(T^\flat, a_\infty^\flat \,{}^t\overline{a}_\infty^\flat) \mod \sum_{\substack{\ell \text{ such that} \\ b\notin\mathrm{GL}_n(\mathcal{O}_F\otimes_\mathbb{Z}\mathbb{Z}_{(\ell)})}} \mathbb{Q}\cdot\log\ell & \text{if } T^\flat \not> 0. \end{cases} \qquad (9.1.15)$$

The notation $T^\flat > 0$ (resp. $T^\flat \not> 0$) means that $T^\flat$ is positive definite (resp. not positive definite). We have $\mathrm{Int}_\infty(T^\flat, a_\infty^\flat \,{}^t\overline{a}_\infty^\flat) = \mathrm{Int}_\infty({}^t\overline{a}_\infty^\flat T^\flat a_\infty^\flat, 1)$ [Che24b, (2.4.1)]. By our main Archimedean local identity ([Che24b, Theorem 4.1.1]), we have $\mathrm{Int}_\infty({}^t\overline{a}_\infty^\flat T^\flat a_\infty^\flat, 1) = \frac{d}{ds}\big|_{s=-1/2} W^*_{{}^t\overline{a}_\infty^\flat T a_\infty^\flat,\infty}(s)_n^\circ$.

The Whittaker function invariance property (4.2.4) implies $W^*_{{}^t\overline{a}_\infty^\flat T^\flat a_\infty^\flat}(s)_n^\circ = \tilde{W}^*_{T^\flat}(a_\infty^\flat,s)_n^\circ$. By the Archimedean local functional equation (5.2.1) we have $\frac{d}{ds}\big|_{s=-1/2}\tilde{W}^*_{T^\flat,\infty}(a_\infty^\flat,s)_n^\circ = -\frac{d}{ds}\big|_{s=1/2}\tilde{W}^*_{T^\flat,\infty}(a_\infty^\flat,s)_n^\circ$. This is still true when $T^\flat$ has signature $(n-1-r,r)$ for $r \geq 2$, as both sides are zero in this case (by definition for the geometric side, and by [Che24b, (4.1.4)] for the local Whittaker function). As already mentioned, recall that $\tilde{W}^*_{T^\flat}(a^\flat,1/2)_n^\circ$ is 1 if $T^\flat$ is positive definite, and is 0 is $T^\flat$ is not positive definite (4.2.6). Now (9.1.13) follows from what we have just discussed.

Next, recall the global Archimedean intersection number $\mathrm{Int}_{\infty,\mathrm{global}}(T,y) = \int_{\mathcal{M}_\mathbb{C}} g_{T,y}$ (where $g_{T,y}$ is a current associated with T and y) as in [Che24c, (5.4.5)]. Recall the relation [Che24c, (5.4.6)]

$$\mathrm{Int}_{\infty,\mathrm{global}}(T,y) = \frac{h_F}{w_F}\mathrm{Int}_\infty(T,y)\cdot\deg\left[U(V)(\mathbb{Q})\backslash \coprod_{\substack{\underline{x}\in V^n \\ (\underline{x},\underline{x})=T}}\mathcal{D}(\underline{x}_f)\right] \qquad (9.1.16)$$

where $V := L\otimes_{\mathcal{O}_F} F$ and $\mathcal{D}(\underline{x}_f)$ is a certain "away-from-∞" local special cycle (it is a discrete set), defined in [Che24c, Section 5.1]. The displayed groupoid cardinality $\deg[\cdots]$ describes certain "complex uniformization degrees" [Che24c, (5.4.5)]. If there exists $\underline{x}\in V^n$ with $(\underline{x},\underline{x})=T$, the groupoid cardinality is

$$\deg\left[U(V)(\mathbb{Q})\backslash \coprod_{\substack{\underline{x}\in V^n \\ (\underline{x},\underline{x})=T}}\mathcal{D}(\underline{x}_f)\right] = \frac{2h_F}{w_F}\prod_\ell \tilde{W}^*_{T^\flat,\ell}(a_\ell^\flat,1/2)_n^\circ \qquad (9.1.17)$$

by local Siegel–Weil as in Lemma 7.4.1 (with $v_0 = \infty$ in the notation of loc. cit.). If there does not exist such $\underline{x}$, then the Hasse principle implies that $T^\flat$ has signature $(n-1-r,r)$ for some $r \geq 2$ (compare the proof of Proposition 8.1.1). In this case, we have $\frac{d}{ds}\big|_{s=1/2}\tilde{W}^*_{T^\flat,\infty}(a_\infty^\flat,s)_n^\circ = 0$ [Che24b, (4.1.4)]. In all cases, we thus have

$$\mathrm{Int}_{\infty,\mathrm{global}}(T,y) = -\frac{2h_F^2}{w_F^2}\left(\frac{d}{ds}\Big|_{s=1/2}\left(|a_\infty^\#|_\infty^s \tilde{W}^*_{T^\flat,\infty}(a_\infty^\flat,s)_n^\circ\right)\right)\prod_\ell \tilde{W}^*_{T^\flat,\ell}(a_\ell^\flat,1/2)_n^\circ. \qquad (9.1.18)$$

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modulo $\sum_{\ell} \mathbb{Q} \cdot \log \ell$ for primes ℓ such that $b \notin \mathrm{GL}_n(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)})$. This give a geometric interpretation of (9.1.9).

(*Step 3c: Local geometric interpretation: at p*) Recall $\mathrm{Int}_p(T) := \mathrm{Int}_{\mathscr{H},p}(T) + \mathrm{Int}_{\mathscr{V},p}(T)$ [Che24c, (4.9.9)], where $\mathrm{Int}_{\mathscr{H},p}(T)$ is a "horizontal local intersection number" [Che24c, (4.9.1)] and $\mathrm{Int}_{\mathscr{V},p}(T)$ is a "vertical local intersection number" [Che24c, (4.8.1)] associated with T. The former describes "local change of tautological (or Faltings) height" and the latter describes degrees for "components in positive characteristic" in terms of local special cycles on Rapoport–Zink spaces.

We claim that

$$\mathrm{Int}_p(T) = -e_p \frac{d}{ds}\Big|_{s=1/2} \tilde{W}^*_{T^\flat,p}(a_p^\flat, s)_n^\circ \tag{9.1.19}$$

where $e_p = 1$ if p is unramified (resp. $e_p = 2$ if p is ramified).

First note that the functional equation (5.1.4) implies $-\frac{d}{ds}\big|_{s=1/2} \tilde{W}^*_{T^\flat,p}(a_p^\flat, s)_n^\circ = \frac{d}{ds}\big|_{s=-1/2} \tilde{W}^*_{T^\flat,p}(a_p^\flat, s)_n^\circ$. The invariance property for Whittaker functions (4.3.4) implies $\tilde{W}^*_{T^\flat,p}(a_p^\flat, s)_n^\circ = \tilde{W}^*_{{}^t\bar{a}_p^\flat T^\flat a_p^\flat,p}(s)_n^\circ$.

Form the positive definite $F/\mathbb{Q}$ Hermitian spaces $\mathbf{W} \subseteq \mathbf{V}$ as in [Che24c, Section 4] (recall $\varepsilon(\mathbf{V}_p) = -1$ and $\varepsilon(\mathbf{V}_\ell) = \varepsilon(V_\ell)$ for all $\ell \neq p$). Set $\mathcal{O}_{F,p} := \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For any $\underline{\mathbf{x}}_p \in \mathbf{W}_p^n$ with Gram matrix T (such $\underline{\mathbf{x}}_p$ exists because $\mathrm{rank}(T) \leq n - 1$; recall $\mathbf{W}$ has rank n if p is nonsplit and rank $n - 1$ if p is split), there exists a basis of $L_p^\flat := \mathrm{span}_{\mathcal{O}_{F,p}}(\underline{x}_p)$ with Gram matrix ${}^t\bar{a}_p^\flat T^\flat a_p^\flat$. Indeed, we have $a_p \in \mathrm{GL}_n(\mathcal{O}_{F_p})$ and $a_p^\flat \in \mathrm{GL}_{n-1}(\mathcal{O}_{F_p})$ by construction (and recall ${}^t\bar{a}_p^{-1} T a_p^{-1} = \mathrm{diag}(0, T^\flat)$ by definition). We remind the reader that (4.5.6) may be used to pass between (normalized) local densities and local Whittaker functions. We also pass between the notation $\mathrm{Den}^*(X, L_p^\flat)_n = \mathrm{Den}^*(X, {}^t\bar{a}_p^\flat T^\flat a_p^\flat)_n$ as explained in Section 4.5. Now (9.1.19) follows from our main non-Archimedean local identity [Che24a, Theorem 9.1.2].

Next, recall the horizontal and vertical global intersection numbers $\mathrm{Int}_{\mathscr{H},p,\mathrm{global}}(T)$ and $\mathrm{Int}_{\mathscr{V},p,\mathrm{global}}(T)$ at p, associated with T (see [Che24c, (4.9.7)] and [Che24c, (4.8.3)]). These are elements of $\mathbb{Q} \cdot \log p$. Recall the $F/\mathbb{Q}$ Hermitian space $\mathbf{W}^\perp$ defined in [Che24c, Section 4.3], which satisfies $\mathbf{V} = \mathbf{W} \oplus \mathbf{W}^\perp$ (orthogonal direct sum). In particular, $\mathbf{W}^\perp = 0$ if p is nonsplit and $\dim_F \mathbf{W}^\perp = 1$ if p is split.

By [Che24c, (4.9.7)] and [Che24c, (4.8.3)] (and in the notation of loc. cit.), we have

$$\mathrm{Int}_{p,\mathrm{global}}(T) = \frac{h_F}{w_F} \mathrm{Int}_p(T) \cdot \deg\left[I_1(\mathbb{Q}) \backslash \left(\coprod_{\substack{\underline{\mathbf{x}} \in \mathbf{W}^n \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} U(\mathbf{W}_p^\perp)/K_{1, \mathbf{L}_p^\perp} \times \mathcal{Z}(\underline{\mathbf{x}}^p) \right) \right]. \tag{9.1.20}$$

The notation $\mathcal{Z}(\underline{\mathbf{x}}^p)$ means a certain "away-from-p" local special cycle (a discrete set), defined in [Che24c, Section 4.2]. Recall that $K_{1, \mathbf{L}_p^\perp} \subseteq U(\mathbf{W}_p^\perp)$ is the unique maximal open compact subgroup and $I_1 = U(\mathbf{W}) \times U(\mathbf{W}^\perp)$ as algebraic groups over $\mathbb{Q}$ [Che24c, Section 4.5]. The displayed groupoid cardinality $\deg[\cdots]$ encodes certain "Rapoport–Zink non-Archimedean uniformization degrees".

If there exists $\underline{\mathbf{x}} \in \mathbf{W}^n$ with Gram matrix T, then local Siegel–Weil (Lemma 7.4.1) implies

$$\deg\left[I_1(\mathbb{Q}) \backslash \left(\coprod_{\substack{\underline{\mathbf{x}} \in \mathbf{W}^n \\ (\underline{\mathbf{x}}, \underline{\mathbf{x}}) = T}} U(\mathbf{W}_p^\perp)/K_{1, \mathbf{L}_p^\perp} \times \mathcal{Z}(\underline{\mathbf{x}}^p) \right) \right] = \frac{2 h_F}{e_p w_F} \prod_{v \neq p} \tilde{W}^*_{T^\flat,v}(a_v^\flat, 1/2)_n^\circ. \tag{9.1.21}$$

(in the notation of Lemma 7.4.1, take $v_0 = p$ and use the hermitian space $\mathbf{V}$ for the V in loc. cit.).

Set $\Omega_T(R) := \{\mathbf{x} \in (\mathbf{W} \otimes_{\mathbb{Q}} R)^n : (\mathbf{x}, \mathbf{x}) = T\}$ for $\mathbb{Q}$-algebras R. If $\Omega_T(\mathbb{Q}) = \emptyset$, then the Hasse principle implies $\Omega_T(\mathbb{Q}_v) = \emptyset$ for some place v of $\mathbb{Q}$. We have $\Omega_T(\mathbb{Q}_p) \neq \emptyset$ (either p is nonsplit and $\mathbf{W} = \mathbf{V}$ and the claim follows because $\operatorname{rank} T < \operatorname{rank} \mathbf{W}$ (compare the proof of Proposition 8.1.1), or p is split and $\Omega_T(\mathbb{Q}_p) \neq \emptyset$ automatically). For all places v, we have $\Omega_T(\mathbb{Q}_v) = \emptyset$ if and only if $\Omega_{{}^t\bar{a}^\flat_v T^\flat a^\flat_v}(\mathbb{Q}_v) = \emptyset$ (where $\Omega_{{}^t\bar{a}^\flat_v T^\flat a^\flat_v}$ is defined like Ω_T but for $(n-1)$-tuples); this follows from our diagonalization of T (e.g. ${}^t\bar{a}_v^{-1} T a_v^{-1} = \operatorname{diag}(0, T^\flat)$ for all $v < \infty$).

If $\Omega_T(\mathbb{Q}_v) = \emptyset$, we thus conclude $\tilde{W}^*_{T^\flat,v}(a^\flat_v, 1/2)^\circ_n = \tilde{W}^*_{{}^t\bar{a}^\flat_v T^\flat a^\flat_v, v}(1/2)^\circ_n = 0$ by the invariance property for local Whittaker functions (see (4.2.4) and (4.3.4)) and by local Siegel–Weil (7.2.2). Hence (9.1.21) holds even if there is no $\mathbf{x} \in \mathbf{W}^n$ such that $(\mathbf{x}, \mathbf{x}) = T$ (both sides are 0 in this case).

We have shown

$$\operatorname{Int}_{p,\mathrm{global}}(T) = -\frac{2h_F^2}{w_F^2}\left(\frac{d}{ds}\bigg|_{s=1/2} \tilde{W}^*_{T^\flat,p}(a^\flat_p, s)^\circ_n\right) \prod_{v \neq p} \tilde{W}^*_{T^\flat, v}(a^\flat_v, 1/2)^\circ_n. \qquad (9.1.22)$$

This gives a geometric interpretation for (9.1.10).

($Step\ 4:\ Finish$) Recall the definition of arithmetic degree without boundary contributions $\widehat{\operatorname{deg}}([\widehat{\mathcal{Z}}(T)])$ (1.3.5). In our current situation, this is

$$\widehat{\operatorname{deg}}([\widehat{\mathcal{Z}}(T)]) := \left(\int_{\mathcal{M}_{\mathbb{C}}} g_{T,y}\right) + \widehat{\operatorname{deg}}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)_{\mathscr{H}}}) + \sum_{\ell} \operatorname{deg}_{\mathbb{F}_\ell}({}^{\mathbb{L}}\mathcal{Z}(T)_{\mathscr{V},\ell}) \log \ell.$$

where the sum runs over all primes ℓ. By definition, we have

$$\int_{\mathcal{M}_{\mathbb{C}}} g_{T,y} = \operatorname{Int}_{\infty,\mathrm{global}}(T,y) \qquad \operatorname{deg}_{\mathbb{F}_\ell}({}^{\mathbb{L}}\mathcal{Z}(T)_{\mathscr{V},\ell}) \log \ell = \operatorname{Int}_{\mathscr{V},\ell,\mathrm{global}}(T)$$
$$\widehat{\operatorname{deg}}(\widehat{\mathcal{E}}^\vee|_{\mathcal{Z}(T)_{\mathscr{H}}}) = (\operatorname{deg}_{\mathbb{Z}} \mathcal{Z}(T)_{\mathscr{H}}) \cdot h^{\mathrm{CM}}_{\widehat{\mathcal{E}}^\vee} + \sum_{\ell} \operatorname{Int}_{\mathscr{H},\ell,\mathrm{global}}(T) \qquad (9.1.23)$$

where $h^{\mathrm{CM}}_{\widehat{\mathcal{E}}^\vee}$ is the height constant from (9.1.2). See [Che24c, (5.4.5)] (Archimedean), [Che24c, (4.8.3)] (vertical), and [Che24c, (4.9.8)] (horizontal). For all primes ℓ, we have $\operatorname{Int}_{\mathscr{V},\ell,\mathrm{global}}(T) \in \mathbb{Q} \cdot \log \ell$ and $\operatorname{Int}_{\mathscr{H},\ell,\mathrm{global}}(T) \in \mathbb{Q} \cdot \log \ell$. These quantities are 0 for all but finitely many ℓ.

After multiplying both sides of (9.1.7) by $2(h_F/w_F)^2$, we apply the results of Steps 3a, 3b, and 3c above (see (9.1.12), (9.1.18), and (9.1.22)) to find

$$\frac{h_F}{w_F}\frac{d}{ds}\bigg|_{s=0} E^*_T(y,s)^\circ_n = \widehat{\operatorname{deg}}([\widehat{\mathcal{Z}}(T)]) \qquad (9.1.24)$$

as elements of $\mathbb{R}/(\sum_{\ell \neq p} \mathbb{Q} \cdot \log \ell)$. As we already discussed, varying p shows that this identity holds as an equality of real numbers. $\square$

Remark 9.1.2 (Nonsingular central-point arithmetic Siegel–Weil)**.** In the setup above (in particular, $n \equiv 2 \pmod 4$), consider any $T \in \operatorname{Herm}_n(\mathbb{Q})$ with $\det T \neq 0$ and any $y \in \operatorname{Herm}_n(\mathbb{R})_{>0}$. Assuming the prime 2 is split in $\mathcal{O}_F$, we still have

$$\widehat{\operatorname{deg}}([\widehat{\mathcal{Z}}(T)]) = \frac{h_F}{w_F}\frac{d}{ds}\bigg|_{s=0} E^*_T(y,s)^\circ_n. \qquad (9.1.25)$$

where the Green current for $[\widehat{\mathcal{Z}}(T)]$ is formed with respect to y, and where $\widehat{\deg}([\widehat{\mathcal{Z}}(T)])$ again denotes the arithmetic degree without boundary contributions as in (1.3.5). This should be compared with our preceding main theorem for singular T of corank 1 (Theorem 9.1.1).

Using the local theorems of Liu, Li–Zhang, and Li–Liu (cited below), one can prove (9.1.25) by a local decomposition as in the proof of Theorem 9.1.1 (no diagonalization procedure is necessary here) using the volume constant calculated in Lemma 8.1.1. This is possibly considered known to experts up to a volume constant by the cited local theorems. Nevertheless, the global statement is not available in the literature, so we have stated it. A sketch is provided below.

Decomposing $E_T^*(y, s)_n^\circ$ into a product of local Whittaker functions (Section 6.1), we find

$$\frac{d}{ds}\bigg|_{s=0} E_T^*(y, s)_n^\circ = \left(\frac{d}{ds}\bigg|_{s=0} W_{T,\infty}^*(y, s)_n^\circ\right) \prod_\ell W_{T,\ell}^*(0)_n^\circ \tag{9.1.26}$$

$$+ \sum_p \left(\frac{d}{ds}\bigg|_{s=0} W_{T,p}^*(s)_n^\circ\right) W_{T,\infty}^*(y, 0)_n^\circ \prod_{\ell \neq p} W_{T,\ell}^*(0)_n^\circ \tag{9.1.27}$$

$$\widehat{\deg}([\widehat{\mathcal{Z}}(T)]) = \mathrm{Int}_{\infty,\mathrm{global}}(T, y) + \sum_p \mathrm{Int}_{p,\mathrm{global}}(T). \tag{9.1.28}$$

At most one of the summands is nonzero (see below), and all but finitely many $W_{T,\ell}^*(s)_n^\circ$ are identically equal to 1 as functions of s. In contrast with our main theorem, these intersection numbers $\mathrm{Int}_{p,\mathrm{global}}(T)$ are "purely vertical", without a mixed characteristic contribution.

In this setup, the local Archimedean theorem [Liu11, Theorem 4.1.7] (restated in our notation in [Che24b, Theorem 4.1.1]) and the local Kudla–Rapoport theorems [LZ22, Theorem 1.2.1] (inert) and [LL22, Theorem 2.7] (ramified, exotic smooth, even n) take the place of our main local identities (which were for corank 1 singular T). In combination with local Siegel–Weil with explicit constants (Lemma 7.4.1(1)), the cited local theorems imply

$$\mathrm{Int}_{\infty,\mathrm{global}}(T, y) = \left(\frac{d}{ds}\bigg|_{s=0} W_{T,\infty}^*(y, s)_n^\circ\right) \prod_p W_{T,p}^*(s)_n^\circ \tag{9.1.29}$$

$$\mathrm{Int}_{p,\mathrm{global}}(T) = \left(\frac{d}{ds}\bigg|_{s=0} W_{T,p}^*(s)_n^\circ\right) W_{T,\infty}^*(y, 0)_n^\circ \prod_{\ell \neq p} W_{T,\ell}^*(0)_n^\circ \tag{9.1.30}$$

in our notation (end of Sections [Che24c, Section 5.4] and [Che24c, Section 4.8] respectively).

To apply local Siegel–Weil in the preceding discussion, we have in mind a (presumably routine) Hasse principle argument (compare [KR14, §9]). We briefly sketch this argument in our setup. For any prime p, set $\varepsilon_p(T) := \eta_p((-1)^{n(n-1)/2} \det T)$ (the usual local invariant, by our conventions in [Che24a, Section 2.2]), where $\eta_p \colon \mathbb{Q}_p^\times \to \{\pm 1\}$ is the local quadratic character associated to $F/\mathbb{Q}$.

We have $\mathrm{Int}_{\infty,\mathrm{global}}(T, y) = 0$ unless T has signature $(n - 1, 1)$ and $\varepsilon_p(T) = 1$ for all p. For such T, the special cycle $\mathcal{Z}(T)$ is empty (but may have a nontrivial Green current). We have $\mathrm{Int}_{p,\mathrm{global}}(T) = 0$ unless T is positive definite, $\varepsilon_p(T) = -1$, and $\varepsilon_\ell(T) = 1$ for all primes $\ell \neq p$. For such T, the special cycle $\mathcal{Z}(T)$ is supported in characteristic p (or empty). For all other T, the special cycle $\mathcal{Z}(T)$ is empty with Green current 0. These claims follow from e.g. uniformization of

special cycles (e.g. Sections [Che24c, Section 5.4] (Archimedean) and [Che24c, Section 4.8] (non-Archimedean)) and the Hasse principle (e.g. applied to $\mathbf{V}$ from loc. cit. in the non-Archimedean case). In particular, $\mathrm{Int}_{p,\mathrm{global}}(T) = 0$ if p is split in $\mathcal{O}_F$, and $\mathcal{Z}(T)$ is empty over any split p.

On the analytic side, we have $W_{T,p}^*(0)_n^\circ = 0$ if $\varepsilon_p(T) = -1$ (by local Siegel–Weil (7.2.2), or the functional equation (5.1.4)) and $W_{T,\infty}^*(y,0)_n^\circ = 0$ if T is not positive definite (local Siegel–Weil again, or (4.2.6)). If T has signature $(n-r, r)$ for $r \geq 2$, we have $\frac{d}{ds}\big|_{s=0} W_{T,\infty}^*(y,s)_n^\circ = 0$ [Che24b, (4.1.4)].

For the analogous global result (still $\det T \neq 0$ and $T \in \mathrm{Herm}_n$, central derivative) for an unramified CM extension of number fields F/F^+ where all 2-adic places are split (forcing $F^+ \neq \mathbb{Q}$) and a lattice L which is self-dual for the Hermitian pairing, see [LZ22, Theorem 15.5.1] (at least up to a volume constant). For the analogous global result (still $\det T \neq 0$ and $T \in \mathrm{Herm}_n$, central derivative) for possibly ramified F/F^+ where all 2-adic places are split, on Krämer integral models (semistable reduction at ramified primes), and again L self-dual for the Hermitian pairing, see [HLSY23, Theorem 10.1] (at least up to a volume constant). For the result on Krämer models, one needs to correct the Eisenstein series derivative by special values of other Eisenstein series.

Remark 9.1.3. When $n \equiv 0 \pmod 4$, there is no non-degenerate self-dual signature $(n-1, 1)$ Hermitian $\mathcal{O}_F$-lattice. In this case, Theorem 9.1.1(1) still holds in the sense that $\frac{d}{ds}\big|_{s=0} E_T^*(y,s)_n^\circ = 0$ (by the functional equation, Lemma 6.1.1).

Remark 9.1.4. We explain how Theorem 9.1.1 may be reformulated in terms of Faltings heights. Assume 2 is split in $\mathcal{O}_F$. Let $\widehat{\omega}$ be the metrized Hodge bundle on $\mathcal{M}$ as defined in [Che24c, Section 2.1] (also [Che24a, Section 3.5]). Take $T \in \mathrm{Herm}_n(\mathbb{Q})$ with $\mathrm{rank}(T) = n - 1$. By [Che24c, (4.9.10)], we have

$$\widehat{\deg}(\widehat{\omega}|_{\mathcal{Z}(T)_{\mathscr{H}}}) = (\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathscr{H}}) \cdot n \cdot h_{\mathrm{Fal}}^{\mathrm{CM}} - 2 \sum_p \mathrm{Int}_{\mathscr{H},p,\mathrm{global}}(T) \tag{9.1.31}$$

where $h_{\mathrm{Fal}}^{\mathrm{CM}}$ is the Faltings height of any elliptic curve with CM by $\mathcal{O}_F$ (as in [Che24c, (2.1.12)]). By definition of Faltings height, we have

$$\widehat{\deg}(\widehat{\omega}|_{\mathcal{Z}(T)_{\mathscr{H}}}) = 2 \sum_{\alpha' \in \mathcal{Z}(T)(\mathbb{C})} |\mathrm{Aut}(\alpha')|^{-1} h_{\mathrm{Fal}}(A) \tag{9.1.32}$$

where $\alpha' = (A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{Z}(T)(\mathbb{C})$ (choose $F \to \mathbb{C}$), and where $h_{\mathrm{Fal}}(A)$ is the Faltings height of A (as in [Che24a, Section 10.1]) after descent to any number field, with metric normalized as in [Che24a, (3.5.1)]. Alternatively, we could consider morphisms $\mathrm{Spec}\,\mathbb{C} \to \mathcal{M}$ over $\mathrm{Spec}\,\mathbb{Z}$, which would remove the factor of 2 in the previous formula.

Our main theorem (Theorem 9.1.1) admits the equivalent formulation

$$\mathrm{Int}_{\infty,\mathrm{global}}(T,y) - \frac{1}{2}\widehat{\deg}(\widehat{\omega}|_{\mathcal{Z}(T)_{\mathscr{H}}}) + (\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathscr{H}}) \cdot (h_{\widehat{\mathcal{E}}^\vee}^{\mathrm{CM}} + \frac{n}{2} \cdot h_{\mathrm{Fal}}^{\mathrm{CM}}) + \sum_p \mathrm{Int}_{\mathscr{V},p,\mathrm{global}}(T)$$

$$= \frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=0} E_T^*(y,s)_n^\circ \tag{9.1.33}$$

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via the decomposition in (9.1.23). We remind the reader that $\deg_{\mathbb{Z}} \mathcal{Z}(T)_{\mathscr{H}}$ is essentially a special value of a $U(n-1, n-1)$ Eisenstein series (9.1.12). For further discussion of the special case $n = 2$, see Section 9.2.

In the rest of Section 9.1, we discuss some results which are applicable even if L is not self-dual.

Allow possibly $2 \mid \Delta$, and let L be any non-degenerate Hermitian $\mathcal{O}_F$-lattice of signature $(n-1, 1)$ (with n not necessarily even). Select any character $\chi \colon F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{A}^\times} = \eta^n$, where η is the quadratic character associated with $F/\mathbb{Q}$. Set $V = L \otimes_{\mathcal{O}_F} F$, with associated local Hermitian space V_v for each place v of $\mathbb{Q}$. Suppose $m^\flat \geq 0$ is an integer. For each prime p, let $\varphi_p^\flat = \mathbf{1}_{L_p}^{m^\flat} \in \mathcal{S}(V_p^{m^\flat})$, form the local Siegel–Weil standard section $\Phi_{\varphi_v^\flat} \in I(\chi_v, s)$, and set

$$\Phi_L := \Phi_\infty^{(n)} \bigotimes_p \Phi_{\varphi_p^\flat} \in I(\chi, s) \tag{9.1.34}$$

where the Archimedean component $\Phi_\infty^{(n)}$ is the standard (normalized) scalar weight section from Section 2.2. Form the associated classical $U(m^\flat, m^\flat)$ Eisenstein series $E(z^\flat, s, \Phi_L)_n$ for $z^\flat \in \mathcal{H}_{m^\flat}$, and consider the normalized Eisenstein series Fourier coefficients

$$E_{T^\flat}^*(y^\flat, s, \Phi_L)_n := \left(\prod_p \gamma_{\psi_p}(V_p)^{m^\flat} \operatorname{vol}(L_p)^{-m^\flat} \right) \Lambda_{m^\flat}(s)_n^\circ E_{T^\flat}(y^\flat, s, \Phi_L)_n \tag{9.1.35}$$

for $T^\flat \in \operatorname{Herm}_{m^\flat}(\mathbb{Q})$. We are not sure whether this is a "good" normalization if L is not self-dual, so the preceding notation appears nowhere else in this work. As in Section 4.3, $\gamma_{\psi_p}(V_p)$ is a Weil index and $\operatorname{vol}(L_p)$ is the volume of L_p with respect to a certain self-dual Haar measure on V_p (these factors are 1 for all but finitely many p).

Form the moduli stack $\mathcal{M} \to \operatorname{Spec} \mathcal{O}_F[1/d_L]$ associated with L as in [Che24c, Section 2.1].

Remark 9.1.5. Since the proof of Theorem 9.1.1 is local in nature, it is possible to use our local main theorems to prove variants for non self-dual L, up to discarding finitely many primes.

Set $m^\flat = n - 1$. Consider $T^\flat \in \operatorname{Herm}_{n-1}(\mathbb{Q})$ with $\det T^\flat \neq 0$. Let $C \in \mathbb{Q}_{>0}$ be the volume constant from Lemma 7.4.1(3), for the Hermitian space V and with $v_0 = \infty$ etc. in the notation of loc. cit.. Consider $y^\flat \in \operatorname{Herm}_{n-1}(\mathbb{R})_{>0}$. Form $[\widehat{\mathcal{Z}}(T^\flat)]$ with Green current with respect to $y^\flat$. Arguing as in the proof of our main theorem (Theorem 9.1.1) gives

$$\widehat{\deg}([\widehat{\mathcal{Z}}(T^\flat)] \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)) = C \cdot \frac{d}{ds}\bigg|_{s=0} \left(\frac{\Lambda_n(s)_n^\circ}{\Lambda_{n-1}(s+1/2)_n^\circ} E_{T^\flat}^*(y^\flat, s+1/2, \Phi_L)_n \right) \quad \mod \sum_{p \mid 2d_L} \mathbb{Q} \cdot \log p. \tag{9.1.36}$$

For proving (9.1.36), the diagonalization argument (Step 1) in the proof of Theorem 9.1.1 can be skipped. If 2 is split in $\mathcal{O}_F$, the expression "$2d_L$" in (9.1.36) may be replaced by "d_L".

In the case $n = 1$, recall that $\mathcal{M}$ extends smoothly (and nontrivially) over all of $\operatorname{Spec} \mathcal{O}_F$ [Che24c, Remark 2.1.3]. In this case, we need not discard any primes in (9.1.36). As $m^\flat = 0$, the normalized $U(m^\flat, m^\flat)$ Eisenstein series E^* is the constant function 1 in this case.

Recall that our main Archimedean local result was valid in arbitrary "codimension" for empty local special cycles with possibly nontrivial Green current ("purely Archimedean intersection number"). This has the following global consequence.

Theorem 9.1.6. *Let $m^\flat$ be any integer with $1 \leq m^\flat \leq n$. Consider $T^\flat \in \mathrm{Herm}_{m^\flat}(\mathbb{Q})$ which is nonsingular and not positive definite. Let $C \in \mathbb{Q}_{>0}$ be the volume constant from Lemma 7.4.1(1), for the Hermitian space V, the lattice L, and $v_0 = \infty$ in the notation of loc. cit..*

For any $y^\flat \in \mathrm{Herm}_{m^\flat}(\mathbb{R})_{>0}$, we have an equality of real numbers

$$\widehat{\deg}([\widehat{\mathcal{Z}}(T^\flat)] \cdot \widehat{c}_1(\widehat{\mathcal{E}}^\vee)^{n-m^\flat}) := \int_{\mathcal{M}_\mathbb{C}} g_{T^\flat, y^\flat} \wedge c_1(\widehat{\mathcal{E}}_\mathbb{C}^\vee)^{n-m^\flat} = (-1)^{n-m^\flat} C \cdot \frac{h_F}{w_F} \frac{d}{ds}\bigg|_{s=s_0^\flat} E_{T^\flat}^*(y^\flat, s, \Phi_L)_n \tag{9.1.37}$$

where $s_0^\flat := (n - m^\flat)/2$.

Proof. In the theorem statement, we set $\mathcal{M}_\mathbb{C} := \mathcal{M} \times_{\mathrm{Spec}\, \mathcal{O}_F} \mathrm{Spec}\, \mathbb{C}$ for either choice of embedding $F \to \mathbb{C}$. Recall that the special cycle $\mathcal{Z}(T^\flat)$ is empty by the non-positive definite-ness [Che24c, Section 2.1]. The current $g_{T^\flat, y^\flat}$ associated with $[\widehat{\mathcal{Z}}(T^\flat)]$ is formed with respect to $y^\flat$, as usual.

Using our main Archimedean result [Che24b, Theorem 4.1.1] and local Siegel–Weil (Lemma 7.4.1) for uniformization degrees, the theorem follows as in the proof of Theorem 9.1.1, Step (3a). Since $\det T^\flat \neq 0$, the proof is simpler here as the diagonalization argument of loc. cit. plays no role. Recall $W_{T^\flat, \infty}^*(y^\flat, s_0^\flat)_n^\circ = 0$ (4.2.6), so the derivatives of non-Archimedean Whittaker functions play no role. If $T^\flat$ has signature $(m^\flat - r, r)$ for $r \geq 2$, then both sides of (9.1.37) are zero. The sign $(-1)^{n-m^\flat}$ comes from the Archimedean local functional equation (Lemma 5.2.1), since [Che24b, Theorem 4.1.1] was stated at $s = -s_0^\flat$. $\qquad\square$

When $m^\flat = n$, the preceding result is due to Liu (see [Liu11, Theorem 4.17, Proof of Theorem 4.20] and also [LZ22, Theorem 15.3.1]). We do not have a new proof of this case (we deduced our local result for arbitrary $m^\flat$ from Liu's result using our local limiting method).

9.2. Faltings heights of Hecke translates of CM elliptic curves.

Using the Serre tensor construction, we restate part of the simplest case ($n = 2$) of our main theorem (Theorem 9.1.1) in more elementary terms, via Faltings heights of Hecke translates of CM elliptic curves (Corollary 9.2.2).

We assume $2 \nmid \Delta$, but allow 2 inert or split in $\mathcal{O}_F$ for the moment. When $n = 2$ and L is a self-dual Hermitian $\mathcal{O}_F$-lattice of signature $(1,1)$, recall

$$\mathcal{M} = \mathcal{M}_0 \times_{\mathrm{Spec}\, \mathcal{O}_F} \mathcal{M}(1,1)^\circ \tag{9.2.1}$$

in the notation of [Che24a, Section 3.1]. Recall that $\mathcal{M}_0$ is the moduli stack parameterizing $(A_0, \iota_0, \lambda_0)$ where A_0 is an elliptic curve with signature $(1,0)$ action ι_0 by $\mathcal{O}_F$, and λ_0 the unique principal polarization. Recall that $\mathcal{M}(1,1)^\circ$ is the closure of the generic fiber in the moduli stack of signature $(1,1)$ Hermitian abelian schemes (A, ι, λ) where $|\Delta| \cdot \lambda$ is a polarization with $\ker(|\Delta| \cdot \lambda) = A[\sqrt{\Delta}]$.

For integers $j > 0$, we first recall how to relate the special cycles $\mathcal{Z}(j) \to \mathcal{M}$ to Hecke translates of CM elliptic curves, as explained in [KR14, §14]. Our $|\Delta| \cdot \lambda$ is their λ.

Write $\mathcal{M}_{\mathrm{ell}}$ for the moduli stack of elliptic curves base-changed to $\mathrm{Spec}\, \mathcal{O}_F$. If $\mathcal{O}_F^* := \mathrm{Hom}_\mathbb{Z}(\mathcal{O}_F, \mathbb{Z})$, we write $\lambda_{\mathrm{tr}} : \mathcal{O}_F \to \mathcal{O}_F^*$ for the σ-linear map corresponding to the symmetric $\mathbb{Z}$-bilinear pairing

$\text{tr}_{F/\mathbb{Q}}(a^\sigma b)$ on $\mathcal{O}_F$. As in [KR14, §14], there is a *Serre tensor* morphism

$$\mathscr{M}_{\text{ell}} \xrightarrow{i_{\text{Serre}}} \mathscr{M}(1,1)^\circ$$
$$E \longmapsto E \otimes_{\mathbb{Z}} \mathcal{O}_F$$

(9.2.2)

where $E \otimes_{\mathbb{Z}} \mathcal{O}_F$ is given the polarization $|\Delta|^{-1}(\lambda_E \otimes \lambda_{\text{tr}})\colon E \otimes_{\mathbb{Z}} \mathcal{O}_F \to E^\vee \otimes_{\mathbb{Z}} \mathcal{O}_F^*$. As we have seen previously, $E \otimes_{\mathbb{Z}} \mathcal{O}_F$ is (by definition) the functor given by $(E \otimes_{\mathbb{Z}} \mathcal{O}_F)(S') = E(S') \otimes_{\mathbb{Z}} \mathcal{O}_F$ for schemes S' (over the understood base for E).

For the rest of Section 9.2, we now assume $\mathcal{O}_F^\times = \{\pm 1\}$. In this case, the Serre tensor morphism is an open and closed immersion.[11] Indeed, i_{Serre} is proper (valuative criterion) and a monomorphism of algebraic stacks, hence a closed immersion of algebraic stacks. Since the source and target are Deligne–Mumford, smooth, finite type, and separated over $\operatorname{Spec} \mathcal{O}_F$ of the same relative dimension, this implies that i_{Serre} is also an open immersion.

The class group $\text{Cl}(\mathcal{O}_F)$ acts $\mathscr{M}(1,1)^\circ$ as follows. Given any fractional ideal $\mathfrak{a} \subseteq F$, set $\mathfrak{a}^\vee :=$ $\text{Hom}_{\mathcal{O}_F}(\mathfrak{a}, \mathcal{O}_F)$, and consider the σ-linear map $\lambda_{\mathfrak{a}}\colon \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^\vee$ given by the perfect positive-definite Hermitian pairing $a, b \mapsto N(\mathfrak{a})^{-1} a^\sigma b$ on $\mathfrak{a}$. There is an induced automorphism of $\mathscr{M}(1,1)^\circ$ sending

$$(A, \iota, \lambda) \to (A \otimes_{\mathcal{O}_F} \mathfrak{a}, \iota, \lambda \otimes \lambda_{\mathfrak{a}}).$$

(9.2.3)

The action of $\text{Cl}(\mathcal{O}_F)$ on $\mathscr{M}(1,1)^\circ$ is simply transitive on the set of connected components (see the proof of [KR14, Proposition 14.4]). There is a similar action of $\text{Cl}(\mathcal{O}_F)$ on $\mathscr{M}_0$ which sends $(A_0, \iota_0, \lambda_0) \mapsto (A_0 \otimes_{\mathcal{O}_F} \mathfrak{a}, \iota_0, \lambda_0 \otimes \lambda_{\mathfrak{a}})$. Given a fractional ideal $\mathfrak{a} \subseteq F$, we write $f_{\mathfrak{a}}\colon \mathcal{M} \to \mathcal{M}$ for the induced automorphism just described.

Given any integer $j > 0$, the action of $\text{Cl}(\mathcal{O}_F)$ preserves $\mathcal{Z}(j)$, in the sense that there is a 2-Cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}(j) & \xrightarrow{\tilde{f}_{\mathfrak{a}}} & \mathcal{Z}(j) \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow{f_{\mathfrak{a}}} & \mathcal{M} \end{array}$$

(9.2.4)

for any fractional ideal $\mathfrak{a}$, where $\tilde{f}_{\mathfrak{a}}$ sends

$$(A_0, \iota_0, \lambda_0, A, \iota, \lambda, x) \mapsto (A_0 \otimes_{\mathcal{O}_F} \mathfrak{a}, \iota_0, \lambda_0 \otimes \lambda_{\mathfrak{a}}, A \otimes_{\mathcal{O}_F} \mathfrak{a}, \iota, \lambda \otimes \lambda_{\mathfrak{a}}, x \otimes 1)$$

(9.2.5)

for $x \in \text{Hom}_{\mathcal{O}_F}(A_0, A)$ satisfying $x^\dagger x = j$.

Consider the *j-th Hecke correspondence* $\mathcal{T}_j \to \mathscr{M}_0 \times_{\operatorname{Spec} \mathcal{O}_F} \mathscr{M}_{\text{ell}}$, where $\mathcal{T}_j$ is the stack parameterizing tuples $(E_0, \iota_0, \lambda_0, E, w)$ for $(E_0, \iota_0, \lambda_0) \in \mathscr{M}_0$, for $E \in \mathscr{M}_{\text{ell}}$, and $w\colon E \to E_0$ an isogeny of degree j.

---

[11]The hypothesis $\mathcal{O}_F^\times = \{\pm 1\}$ should be added in [KR14, Proposition 14.4], as otherwise $\text{Aut}(E) \neq \text{Aut}(E \otimes_{\mathbb{Z}} \mathcal{O}_F)$ (right-hand side means $\mathcal{O}_F$-linear automorphisms preserving the polarization) so $i_{\text{Serre}}\colon \mathscr{M}_{\text{ell}} \to \mathscr{M}(1,1)^\circ$ is not a monomorphism and hence cannot be a closed immersion in the sense of [SProject, Section 04YK]. The remaining arguments are the same at least if $2 \nmid \Delta$.

Consider the map $\mathscr{M}_0 \times \mathscr{M}_{\text{ell}} \to \mathcal{M}$ induced by i_{Serre} (and the identity on $\mathscr{M}_0$). The Kudla–Rapoport cycle $\mathcal{Z}(j)$ pulls back to the Hecke correspondence $\mathcal{T}_j$, i.e. there is a 2-Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{T}_j & \longrightarrow & \mathcal{Z}(j) \\
\downarrow & & \downarrow \\
\mathscr{M}_0 \times_{\operatorname{Spec} \mathcal{O}_F} \mathscr{M}_{\text{ell}} & \longrightarrow & \mathcal{M}
\end{array}
\tag{9.2.6}
$$

where $\mathcal{T}_j \to \mathcal{Z}(j)$ sends

$$
(E_0, \iota_0, \lambda_0, E, w) \mapsto (E_0, \iota_0, \lambda_0, E \otimes_{\mathbb{Z}} \mathcal{O}_F, \iota, \lambda_E \otimes \lambda_{\text{tr}}, x_w)
\tag{9.2.7}
$$

(with λ_E denoting the unique principal polarization of E) and where $x_w \colon E_0 \to E \otimes_{\mathbb{Z}} \mathcal{O}_F$ is the $\mathcal{O}_F$-linear map such that $\sqrt{\Delta} x_w^\dagger \in \operatorname{Hom}_{\mathcal{O}_F}(E \otimes_{\mathbb{Z}} \mathcal{O}_F, E_0)$ corresponds to w via the adjunction

$$
\operatorname{Hom}_{\mathcal{O}_F}(E \otimes_{\mathbb{Z}} \mathcal{O}_F, E_0) = \operatorname{Hom}(E, E_0).
\tag{9.2.8}
$$

Here, we are implicitly claiming $\deg(w) = x_w^\dagger x_w$. The fact that (9.2.6) is well-defined and 2-Cartesian is proved in [KR14, Proposition 14.5].

We next discuss the Eisenstein series of Theorem 9.1.1(2) in more elementary terms when $n = 2$. In this case, the $U(1,1)$ Eisenstein series $E^*(z,s)_2^\circ$ (with $m = 1$ in our usual notation, and normalized as in Section 6.1) admits the classical expression

$$
E^*(z,s)_2^\circ = -\frac{\pi^{-s+1/2}}{8\pi^2} \Gamma(s + 3/2) \zeta(2s+1) \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^{s-1/2}}{(cz+d)^2 |cz+d|^{2(s-1/2)}}
\tag{9.2.9}
$$

for $z = x + iy \in \mathcal{H}$, where $\mathcal{H} \subseteq \mathbb{C}$ is the usual upper-half space (here z corresponds to $z^\flat$ in Theorem 9.1.1(2)).

For nonzero $j \in \mathbb{Z}$, the (normalized) j-th Fourier coefficient of $E^*(z,s)_2^\circ$ factorizes into (normalized) local Whittaker functions

$$
E_j^*(y,s)_2^\circ = W_{j,\infty}^*(y,s)_2^\circ \prod_p W_{j,p}^*(s)_2^\circ
\tag{9.2.10}
$$

as in Section 6.1. We have the formulas

$$
W_{j,p}^*(s)_2^\circ = p^{v_p(j)(s+1/2)} \sigma_{-2s}(p^{v_p(j)}) \qquad \prod_p W_{j,p}^*(s)_2^\circ = |j|^{s+1/2} \sigma_{-2s}(|j|)
\tag{9.2.11}
$$

where $v_p(-)$ means p-adic valuation and

$$
\sigma_s(|j|) := \sum_{d \mid |j|} d^s
\tag{9.2.12}
$$

is the classical divisor function. These formulas for local Whittaker functions are likely classical, but they also follow from [Che24a, (9.2.8)] on local densities (translation to local Whittaker functions via (4.5.6)). A integral expression for $W_{j,\infty}^*(y,s)_2^\circ$ may be found in [Che24b, Section 4.2]. For $j > 0$, recall $W_{j,\infty}^*(y,1/2)_2^\circ = 1$ (4.2.6).

We require $j > 0$ for the rest of Section 9.2. Fix an embedding $F \to \mathbb{C}$. Given a CM elliptic curve $(E_0, \iota_0, \lambda_0) \in \mathscr{M}_0(\mathbb{C})$, we consider the set of j-th Hecke translates of E_0 given by

$$
\mathcal{T}_j(E_0) := \{(E_0, \iota_0, \lambda_0, E, w) \in \mathcal{T}_j(\mathbb{C})\}.
\tag{9.2.13}
$$

Phrased alternatively, the fiber of $\mathcal{T}_j \to \mathscr{M}_0$ over the point $\operatorname{Spec}\mathbb{C} \to \mathscr{M}_0$ corresponding to E_0 is a finite scheme over $\operatorname{Spec}\mathbb{C}$, and $\mathcal{T}_j(E_0)$ is its set of $\mathbb{C}$-points. We set

$$\deg\mathcal{T}_j(E_0) := |\mathcal{T}_j(E_0)| \qquad h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) := \sum_{E\in\mathcal{T}_j(E_0)} h_{\mathrm{Fal}}(E) \tag{9.2.14}$$

where $|-|$ denotes set cardinality, the sum runs over $(E_0, \iota_0, \lambda_0, E, w) \in \mathcal{T}_j(E_0)$, and $h_{\mathrm{Fal}}(E)$ denotes the Faltings height of E (with metric normalized as in [Che24a, (3.5.1)], see also [Che24a, Section 10.1]) after descending from $\mathbb{C}$ to any number field.

The following lemma states that the (total) Faltings height of j-th Hecke translates of a chosen elliptic curve with CM by $\mathcal{O}_F$ does not depend on the choice of CM elliptic curve. It should admit a general formulation in terms of Hecke correspondences over $\mathscr{M}_0$. We give a more elementary treatment in the spirit of this section.

Lemma 9.2.1. *Fix $j \in \mathbb{Z}_{>0}$. For any $(E_0, \iota_0, \lambda_0) \in \mathscr{M}_0(\mathbb{C})$ and $(E_0', \iota_0', \lambda_0') \in \mathscr{M}_0(\mathbb{C})$, we have*

$$\deg\mathcal{T}_j(E_0) = \deg\mathcal{T}_j(E_0') \qquad h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) = h_{\mathrm{Fal}}(\mathcal{T}_j(E_0')). \tag{9.2.15}$$

Proof. Given any $d \in \mathbb{Z}$, we claim that there exists an isogeny $\phi\colon E_0' \to E_0$ of degree prime to d. Consider

$$E_0(\mathbb{C}) = \mathbb{C}/\Lambda_0 \qquad E_0'(\mathbb{C}) = \mathbb{C}/\Lambda_0' \tag{9.2.16}$$

for lattices Λ_0 and Λ_0'. Without loss of generality, we may assume $\Lambda_0 = \mathcal{O}_F \subseteq \mathbb{C}$ and that $\Lambda_0' = \mathfrak{a}_0'$ for some fractional ideal $\mathfrak{a}_0' \subseteq \mathbb{C}$. By the Chinese remainder theorem, we can assume $\mathfrak{a}_0' \subseteq \mathcal{O}_F$ and that $\mathfrak{a}_0'$ has norm prime to d (without changing the ideal class of $\mathfrak{a}_0'$). The inclusion $\mathfrak{a}_0' \subseteq \mathcal{O}_F$ gives an isogeny $E_0' \to E_0$ of degree prime to d.

Let p be any prime. Let $\phi\colon E_0' \to E_0$ be an isogeny of degree prime to pj. As above, we view $\phi\colon E_0(\mathbb{C}) \to E_0'(\mathbb{C})$ as an inclusion of lattices $\Lambda_0' \to \Lambda_0$ of index prime to pj. There is an induced bijection

$$\begin{array}{ccc} \mathcal{T}_j(E_0) & \longrightarrow & \mathcal{T}_j(E_0') \\ & & \\ \Lambda & \longmapsto & \Lambda \cap \Lambda_0'. \end{array} \tag{9.2.17}$$

We are viewing Λ as the element $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda_0$ of $\mathcal{T}_j(E_0)$, and similarly for $\Lambda \cap \Lambda_0'$.

The isogeny $\mathbb{C}/(\Lambda \cap \Lambda_0') \to \mathbb{C}/\Lambda$ has degree $\deg\phi$, which is prime to p. As these elliptic curves are defined over $\overline{\mathbb{Q}}$, this isogeny also descends to $\overline{\mathbb{Q}}$. By the formula for change for Faltings height along an isogeny [Che24a, (10.2.4)], we conclude $h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) - h_{\mathrm{Fal}}(\mathcal{T}_j(E_0')) \in \sum_{\ell | \deg\phi} \mathbb{Q}\cdot\log\ell$. Varying p shows $h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) = h_{\mathrm{Fal}}(\mathcal{T}_j(E_0'))$, as the real numbers $\log p$ are $\mathbb{Q}$-linearly independent for varying p. $\qquad\square$

Consider any $(E_0, \iota_0, \lambda_0) \in \mathscr{M}_0(\mathbb{C})$. Using (9.2.6) (Kudla–Rapoport cycle pulls back to Hecke correspondence), the geometric Siegel–Weil statement in Remark 8.1.2 implies

$$\frac{h_F^2}{w_F}\deg\mathcal{T}_j(E_0) = 2\frac{h_F^2}{w_F^2}E_j^*(y, 1/2)_2^\circ \tag{9.2.18}$$

for any $y \in \mathbb{R}_{>0}$. On the left, one factor of h_F appears because the Serre tensor morphism $i_{\mathrm{Serre}}\colon \mathscr{M}_{\mathrm{ell}} \to \mathscr{M}(1,1)^\circ$ is the inclusion of one connected component (and $\mathscr{M}(1,1)^\circ$ has h_F connected components, by the action of $\mathrm{Cl}(\mathcal{O}_F)$ discussed above; we discussed that this action is

compatible with Kudla–Rapoport cycles). On the left, the additional factor h_F/w_F appears via Lemma 9.2.1 (instead of summing over $\mathcal{M}_0(\mathbb{C})$, it is enough to consider a fixed E_0 and multiply by $h_F/w_F = \deg_{\mathbb{C}}(\mathcal{M}_0 \times_{\operatorname{Spec}\mathcal{O}_F} \operatorname{Spec}\mathbb{C}))$.

By the formulas in (9.2.11) and surrounding discussion, this recovers the well-known identity $\deg \mathcal{T}_j(E_0) = \sigma_1(j)$ for degrees of Hecke correspondences (recall our running assumption $|\mathcal{O}_F^\times| = \{\pm 1\}$ for most of Section 9.2, i.e. $w_F = 2$).

In the next lemma, $h_{\mathrm{Fal}}^{\mathrm{CM}} = h_{\mathrm{Fal}}(E_0)$ is the Faltings height of any elliptic curve with CM by $\mathcal{O}_F$, normalized as in [Che24c, (2.1.12)]. It is well known that this does not depend on the choice of CM elliptic curve (also follows from Lemma 9.2.1).

Corollary 9.2.2. *Suppose* 2 *is split in* $\mathcal{O}_F$. *For any integer* $j > 0$ *and any CM elliptic curve* $(E_0, \iota_0, \lambda_0) \in \mathcal{M}_0(\mathbb{C})$, *we have*

$$h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) - \sigma_1(j) \cdot h_{\mathrm{Fal}}^{\mathrm{CM}} = \frac{1}{2}\frac{d}{ds}\bigg|_{s=1/2} \left(j^{s+1/2}\sigma_{-2s}(j)\right). \tag{9.2.19}$$

Proof. Set $n = 2$ and consider the 2×2 matrix $T = \operatorname{diag}(0, j)$. Again using (9.2.6) to pull back Kudla–Rapoport cycles to Hecke correspondences, we have

$$2\frac{h_F^2}{w_F}\left(2h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) - 2(\deg \mathcal{T}_j(E_0)) \cdot h_{\mathrm{Fal}}^{\mathrm{CM}}\right) = -2\sum_p \operatorname{Int}_{\mathcal{H},p,\mathrm{global}}(T) \tag{9.2.20}$$

in our previous notation (Remark 9.1.4). On the left, the outer factor of 2 has the same explanation as in (9.1.32) (see following discussion). The factor h_F^2/w_F has the same explanation as in (9.2.18), via Lemma 9.2.1 on Faltings height. The factor of 2 in $2h_{\mathrm{Fal}}(\mathcal{T}_j(E_0))$ appears because $h_{\mathrm{Fal}}(E \otimes_{\mathbb{Z}} \mathcal{O}_F) = h_{\mathrm{Fal}}(E \times E) = 2h_{\mathrm{Fal}}(E)$. The factor of 2 in $2(\deg \mathcal{T}_j(E_0)) \cdot h_{\mathrm{Fal}}^{\mathrm{CM}}$ is the n in Remark 9.1.4.

In our previous notation, we have $\operatorname{Int}_{\mathcal{V},p,\mathrm{global}}(T) = 0$ for all primes p as the vertical special cycle class ${}^{\mathbb{L}}\mathcal{Z}(T)_{\mathcal{V},p}$ is 0 when $n = 2$ [Che24c, Lemma 4.7.6]. Hence $\operatorname{Int}_{p,\mathrm{global}}(T) = \operatorname{Int}_{\mathcal{H},p,\mathrm{global}}(T) + \operatorname{Int}_{\mathcal{V},\ell,\mathrm{global}}(T) = \operatorname{Int}_{\mathcal{H},p,\mathrm{global}}(T)$.

Then (9.1.22) ("horizontal local part" of our main result) implies

$$\operatorname{Int}_{p,\mathrm{global}}(T) = -\frac{2h_F^2}{w_F^2}\left(\frac{d}{ds}\bigg|_{s=1/2} W_{j,p}^*(s)_2^\circ\right)\prod_{\ell \neq p} W_{j,\ell}^*(1/2)_2^\circ \tag{9.2.21}$$

for all p (in the notation of loc. cit., take $T^\flat = j$, $a_v^\flat = 1$ for all $v < \infty$, and recall our notation $\tilde{W}_{T^\flat,v}^*(1,s)_n^\circ = W_{T^\flat,v}^*(1,s)_n^\circ =: W_{T^\flat,v}^*(s)_n^\circ$). Since $j > 0$, we have used $W_{j,\infty}^*(1/2)_2^\circ = 1$ (4.2.6) as recalled above.

Combining (9.2.21) and (9.2.20) along with the formula $\deg \mathcal{T}_j(E_0) = \sigma_1(j)$, we obtain

$$h_{\mathrm{Fal}}(\mathcal{T}_j(E_0)) - \sigma_1(j) \cdot h_{\mathrm{Fal}}^{\mathrm{CM}} = \frac{1}{2}\frac{d}{ds}\bigg|_{s=1/2}\left(\prod_p W_{j,p}^*(1/2)_2^\circ\right) \tag{9.2.22}$$

where the product runs over all primes (not including the Archimedean place). The corollary now follows from the formulas in (9.2.11). □

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